Computable functions of reals *

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Abstract

We introduce a new notion of computable function on $\mathbb{R}^N$ and prove some basic properties. We give two applications, first a short proof of Yoshinaga's theorem that periods are elementary (they are actually lower elementary). We also show that the lower elementary complex numbers form an algebraically closed field closed under exponentiation and some other special functions.

1 Introduction

We here develop a notion of computable functions on the reals along the lines of the *bit-model* as described in [2]. In contrast to the algebraic approach towards computation over the reals developed in [1], our approach goes back to Grzegorczyk and the hierarchy of elementary functions and real numbers developed by him and Mazur (see [4] footnote p. 201). We thank Dimiter Skordev for a careful reading of an earlier version of this note. We also thank the referee for pointing out several flaws in the first version.

2 Good classes of functions

A class $\mathcal{F}$ of functions $\mathbb{N}^n \to \mathbb{N}$ is called *good* if it contains the constant functions, the projection functions, the successor function, the modified difference function $x \dot{-} y = \max\{0, x-y\}$, and is closed under composition and *bounded summation*

$$f(\bar{x}, y) = \sum_{i=0}^{y} g(\bar{x}, i).$$

The class of *lower elementary* functions is the smallest good class. The smallest good class which is also closed under *bounded product*

$$f(\bar{x}, y) = \prod_{i=0}^{y} g(\bar{x}, i),$$

*v5-2-ge9a838d, Thu Nov 11 13:41:27 2010 +0100 or – equivalently – the smallest good class which contains $n \mapsto 2^n$, is the class of *elementary* functions The elementary functions are the third class ϵ^3 of the Grzegorczyk hierarchy. The lower elementary functions belong to ϵ^2. It is not known whether all functions in ϵ^2 are lower elementary.

A function $f : \mathbb{N}^n \to \mathbb{N}^m$ is an $\mathcal{F}$–function if its components $f_i : \mathbb{N}^n \to \mathbb{N}$, $i = 1, \ldots, m$, are in $\mathcal{F}$. A relation $R \subset \mathbb{N}^n$ is called an $\mathcal{F}$–relation if its characteristic function belongs to $\mathcal{F}$. Note that a good class is closed under the bounded μ–operator: if R belongs to $\mathcal{F}$, then so does the function

$$f(\bar{x}, y) = \min\{i \mid R(\bar{x}, i) \vee i = y\}.$$

As a special case we see that $\lfloor \frac{x}{y} \rfloor$ is lower elementary. The $\mathcal{F}$–relations are closed under Boolean combinations and bounded quantification:

$$S(x, y) \;\Leftrightarrow\; \exists\, i \leq y\, R(x, i).$$

It follows for example that for any f in $\mathcal{F}$ the maximum function

$$\max_{j \leq y} f(\bar{x}, j) = \min\{\, i \mid \forall j \leq y\ f(\bar{x}, j) \leq i\,\}$$

is in $\mathcal{F}$ since it is bounded by $\sum_{i=0}^{y} f(\bar{x}, i)$.

We call a set X an $\mathcal{F}$–retract (of $\mathbb{N}^n$) if there are functions $\iota : X \to \mathbb{N}^n$ and $\pi : \mathbb{N}^n \to X$ given with $\pi \circ \iota = \mathrm{id}$ and $\iota \circ \pi \in \mathcal{F}$. Note that the product $X \times X'$ of two $\mathcal{F}$–retracts X and X' is again an $\mathcal{F}$–retract (of $\mathbb{N}^{n+n'}$) in a natural way. We define a function $f : X \to X'$ to be in $\mathcal{F}$ if $\iota' \circ f \circ \pi : \mathbb{N}^n \to \mathbb{N}^{n'}$ is in $\mathcal{F}$. By this definition the two maps $\iota : X \to \mathbb{N}^n$ and $\pi : \mathbb{N}^n \to X$ belong to $\mathcal{F}$. For an $\mathcal{F}$–retract X, a subset of X is in $\mathcal{F}$ if its characteristic function is. It now makes sense to say that a set Y (together with $\iota : Y \to X$ and $\pi : X \to Y$) is a retract of the retract X. Clearly Y is again a retract in a natural way.

Easily, $\mathbb{N}_{>0}$ is a lower elementary retract of $\mathbb{N}$ and $\mathbb{Z}$ is a lower elementary retract of $\mathbb{N}^2$ via $\iota(z) = (\max(z, 0), -\min(0, z))$ and $\pi(n, m) = n - m$. We turn $\mathbb{Q}$ into a lower elementary retract of $\mathbb{Z} \times \mathbb{N}$ by setting $\iota(r) = (z, n)$, where $\frac{z}{n}$ is the unique representation of r with $n > 0$ and $(z, n) = 1$. Define $\pi(z, n)$ as $\frac{z}{n}$ if $n > 0$ and as 0 otherwise.

For the remainder of this note we will consider $\mathbb{Z}$ and $\mathbb{Q}$ as *fixed lower elementary* retracts of $\mathbb{N}$, using the maps defined in the last paragraph.

The following lemma is clear.

Lemma 2.1. *Let X, X' and X'' be $\mathcal{F}$–retracts, and $f : X \to X'$ and $f' : X' \to X''$ in $\mathcal{F}$. Then $f' \circ f$ also belongs to $\mathcal{F}$.* □

The *height* $\mathrm{h}(a)$ of a rational number $a = \frac{z}{n}$, for relatively prime z and n ($n > 0$), is the maximum of $|z|$ and n. The height of a tuple of rational numbers is the maximum height of its elements.

Lemma 2.2. *If $f : \mathbb{Q}^N \to \mathbb{Q}$ is in $\mathcal{F}$, there is an $\mathcal{F}$–function $\beta_f : \mathbb{N} \to \mathbb{N}$ such that*

$$\mathrm{h}(a) \leq e \ \Rightarrow \ |f(a)| \leq \beta_f(e).$$

Proof. Write $f(\frac{z_1}{n_1}, \ldots, \frac{z_N}{n_N}) = g(z_1, \ldots, n_N)$ for an $\mathcal{F}$–function $g : (\mathbb{Z} \times \mathbb{N})^N \to \mathbb{Q}$. The function $\lceil x \rceil : \mathbb{Q} \to \mathbb{N}$ is lower elementary. So we can define $\beta_f(e)$ to be the maximum of all $\lceil |g(x_1, \ldots, x_{2N})| \rceil$, where $|x_i| \leq e$. β_f is easily seen to be in $\mathcal{F}$. □

Lemma 2.3. *Let $g : X \times \mathbb{N} \to \mathbb{Q}$ be in $\mathcal{F}$ for some $\mathcal{F}$-retract X. Then there is an $\mathcal{F}$–function $f : X \times \mathbb{N} \times \mathbb{N}_{>0} \to \mathbb{Q}$ such that*

$$\Big| f(x, y, k) - \sum_{i=0}^{y} g(x, i) \Big| < \frac{1}{k} \text{ for all } x \in X, y \in \mathbb{N} \text{ and } k \in \mathbb{N}_{>0}.$$

Proof. We note first that for every $\mathcal{F}$–function $t : X \times \mathbb{N} \to \mathbb{Z}$ the function $\sum_{i=0}^{y} t(x, i)$ belongs to $\mathcal{F}$. It is also easy to see that there is a lower elementary function $h : \mathbb{Q} \times \mathbb{N} \to \mathbb{Z}$ such that

$$\Big| \frac{h(r, k)}{k} - r \Big| < \frac{1}{k}.$$

Now define f by

$$f(x, y, k) = \frac{\sum_{i=0}^{y} h(g(x, i), (y+1)k)}{(y+1)k}$$

□

Definition 2.4. *A real number x is an $\mathcal{F}$–real if for some $\mathcal{F}$–function $a : \mathbb{N} \to \mathbb{Q}$*

$$|x - a(k)| < \frac{1}{k}$$

Lemma 2.5. *x is an $\mathcal{F}$–real if and only if there is an $\mathcal{F}$–function $n : \mathbb{N} \to \mathbb{Z}$ with $\frac{n(k)-1}{k} < x < \frac{n(k)+1}{k}$.* □

Remark 2.6. *Let $x = \sum_{i=0}^{\infty} a_i b^{-i}$ for integers $b \geq 2$ and $a_i \in \{0, 1, \ldots, b-2\}$. Then x is an $\mathcal{F}$-real if and only if*

$$a_i = f(b^i)$$

for some $\mathcal{F}$–function f.

In [8] Skordev has shown among other things that π is lower elementary and that e is in ϵ^2. Weiermann [11] proved that e is lower elementary. He used the representation $\mathrm{e} = \sum \frac{1}{n!}$ and a theorem of d'Aquino [3], which states that the graph of the factorial is Δ_0–definable and therefore lower elementary.[1]

We will show in Section 6 that volumes of bounded 0–definable semialgebraic sets are lower elementary, and so is π as the volume of the unit circle. In Section 7 we show that the exponential function maps lower elementary reals into lower elementary reals.

[1] D. Skordev and also the referee have informed us that lowness of the graph of the factorial follows from the fact that the class of functions with lower elementary graph is closed under bounded multiplication, which is not hard to prove.

3 Functions on $\mathbb{R}^N$

Let O be an open subset of $\mathbb{R}^N$ where we allow $N = 0$. For $e \in \mathbb{N}_{>0}$ put

$$O \restriction e = \left\{ x \in O \;\middle|\; |x| \leq e \right\}$$

and

$$O_e = \left\{ x \in O \restriction e \;\middle|\; \operatorname{dist}(x, \mathbb{R}^N \setminus O) \geq \frac{1}{e} \right\}.$$

(We use the maximum norm on $\mathbb{R}^N$.) Notice:

1. O_e is compact.
2. $U \subset O \;\Rightarrow\; U_e \subset O_e$.
3. $e < e' \;\Rightarrow\; O_e \subset O_{e'}$.
4. $O = \bigcup_{e \in \mathbb{N}} O_e$
5. $O_e \subset (O_{2e})^\circ_{2e}$

Definition 3.1. *Let $\mathcal{F}$ be a good class. A function $F : O \to \mathbb{R}$ is in $\mathcal{F}$ if there are $\mathcal{F}$–functions $d : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{Q}^N \times \mathbb{N} \to \mathbb{Q}$ such that for all $e \in \mathbb{N}_{>0}$ and all $a \in \mathbb{Q}^N$ and $x \in O_e$*

$$|x - a| < \frac{1}{d(e)} \to |F(x) - f(a, e)| < \frac{1}{e}. \tag{1}$$

If (1) holds for all $x \in O \restriction e$, we call F uniformly $\mathcal{F}$.[2]

We will always assume that $d : N \to N$ is strictly increasing.

Remark 3.2. *Clearly, a function is uniformly $\mathcal{F}$ on O if it can be extended to an $\mathcal{F}$–function on some ϵ-neighborhood of O.*

This definition easily extends to $f : O \to \mathbb{R}^M$ (again under the maximum norm). Then f is in $\mathcal{F}$ if and only if all f_i, $i = 1, \ldots, M$, are in $\mathcal{F}$.

Lemma 3.3. *$\mathcal{F}$–functions map $\mathcal{F}$–reals to $\mathcal{F}$–reals. A constant function on $\mathbb{R}^N$ is uniformly in $\mathcal{F}$ if and only if its value is an $\mathcal{F}$–real.* □

Definition 3.4. *We call a function $F : O \to \mathbb{R}^M$ (uniformly) $\mathcal{F}$-bounded, if there is an $\mathcal{F}$–function $\beta_F : \mathbb{N} \to \mathbb{Q}$ in $\mathcal{F}$ such that $|F(x)| \leq \beta_F(e)$ for all $x \in O_e$ ($x \in O \restriction e$, respectively).*

Note that $\exp : \mathbb{R} \to \mathbb{R}$ is not lower elementary bounded, but elementary bounded.

[2] As pointed out by the referee even in the case when $\mathcal{F}$ is the class of lower elementary functions, the $\mathcal{F}$–functions on $\mathbb{R}$ are not necessarily computable in the sense of [12].

Lemma 3.5. *If $F : O \to \mathbb{R}^M$ is (uniformly) in $\mathcal{F}$, then F is (uniformly) $\mathcal{F}$-bounded.*

Proof. Assume the $\mathcal{F}$–functions d and f satisfy (1). Fix a number e and consider an $x \in O_e$. Choose $z \in \mathbb{Z}^N$ such that the distance between x and $a = \frac{1}{d(e)} z$ is less than $\frac{1}{d(e)}$. Since $|a| < e + 1$, the height of a is smaller than $(e+1)d(e)$. By Lemma 2.2 we have $|f(a, e)| \leq \beta_f((e + 1)d(e))$, and therefore

$$|F(x)| \leq \beta_f((e + 1)d(e)) + 1 = \beta_F(e).$$

$\square$

Lemma 3.6. *$\mathcal{F}$–functions $O \to \mathbb{R}$ are continuous.*

Proof. We show that F is uniformly continuous on every O_e. Assume that d and f satisfy (1). It suffices to show that for all $x, x' \in O_e$

$$|x - x'| < \frac{2}{d(e)} \to |F(x) - F(x')| < \frac{2}{e}.$$

Assume $|x - x'| < \frac{2}{d(e)}$. Choose $a \in \mathbb{Q}^N$ such that $|x - a| < \frac{1}{d(e)}$ and $|x' - a| < \frac{1}{d(e)}$. Then both $F(x)$ and $F(x')$ differ from $f(a, e)$ by less than $\frac{1}{e}$. Whence $|F(x) - F(x')| < \frac{2}{e}$. $\square$

Lemma 3.7. *If $F : O \to \mathbb{R}^M$ is in $\mathcal{F}$, $U \subset \mathbb{R}^M$ open and $G : U \to \mathbb{R}$ uniformly in $\mathcal{F}$, then $G \circ F : F^{-1}U \cap O \to \mathbb{R}$ is in $\mathcal{F}$. If F is uniformly in $\mathcal{F}$, then so is $G \circ F$.*

Proof. Assume that F satisfies (1) with the $\mathcal{F}$–functions d and f and assume that G satisfies (1) with d' and g.

Let $\beta = \beta_F$ be as in 3.5 and set $V = F^{-1}(U) \cap O$. Clearly we may assume $\beta(e) \geq e$ for all $e \in \mathbb{N}$. So if $x \in V_e \subseteq O_e$, then $F(x) \in U \restriction \beta(e)$. Thus for all $e \in \mathbb{N}$, $a \in \mathbb{Q}^N$ and $x \in V_e$ we have

$$\begin{aligned} |x - a| < \frac{1}{d(d'(\beta(e)))} &\Rightarrow |F(x) - f(a, d'(\beta(e)))| < \frac{1}{d'(\beta(e))} \\ &\Rightarrow \Big|G \circ F(x) - g\Big(f\big(a, d'(\beta(e))\big), \beta(e)\Big)\Big| < \frac{1}{\beta(e)} \leq \frac{1}{e} \end{aligned}$$

This shows also the second part of the theorem, only replace V_e and O_e by $V \restriction e$ and $O \restriction e$. $\square$

Definition 3.8. *A function $F : O \to U$ is called $\mathcal{F}$-compact if there is an $\mathcal{F}$–function $\beta : \mathbb{N} \to \mathbb{N}$ such that $F(O_e) \subseteq U_{\beta(e)}$ for all $e \in \mathbb{N}_{>0}$.*

Note that $F(x) = \frac{1}{x} : (0, \infty) \to (0, \infty)$ and $\ln(x) : (0, \infty) \to (-\infty, \infty)$ are lower elementary compact.

Corollary 3.9. *Let O, U and V be open sets in Euclidean space. If $F : O \to U$ is in $\mathcal{F}$ and $\mathcal{F}$–compact, and if $G : U \to V$ is in $\mathcal{F}$, then $G \circ F$ is in $\mathcal{F}$. If G is $\mathcal{F}$–compact, then so is $G \circ F : O \to V$.*

Proof. By the proof of Lemma 3.7. □

If F is defined on the union of two open sets U and V, and if F is in $\mathcal{F}$ restricted to U and restricted to V, it is not clear that F is in $\mathcal{F}$, without additional assumptions.

Remark 3.10. *Let F be defined on the union of U and V and assume that $F \restriction U$ and $F \restriction V$ are in $\mathcal{F}$. Assume also*

1. *that for some $\mathcal{F}$–function $u : \mathbb{N} \to \mathbb{N}$ $(U \cup V)_e \subset U_{u(e)} \cup V_{u(e)}$*

2. *that U and V are $\mathcal{F}$–approximable in the sense of Definition 7.1 below,*

then F is in $\mathcal{F}$. □

The two conditions are satisfied if U and V are open intervals with $\mathcal{F}$–computable endpoints.

4 Semialgebraic functions

In this article semialgebraic functions (relations) are functions (relations) definable *without parameters* in $\mathbb{R}$. The *trace* of a relation $R \subseteq \mathbb{R}^N$ on $\mathbb{Q}$ is $R \cap \mathbb{Q}^N$.

The following observation is due to Yoshinaga [13].

Lemma 4.1. *The trace of semialgebraic relations on $\mathbb{Q}$ is lower elementary.*

Proof. By quantifier elimination. □

Note that any semialgebraic function $g : \mathbb{R} \to \mathbb{R}$ is polynomially bounded, i.e. there is some $n \in \mathbb{N}$ with $|g(x)| \leq |x|^n$ for sufficiently large x.

Theorem 4.2. *Continuous semialgebraic functions $F : O \to \mathbb{R}$ are lower elementary for every open semialgebraic set O.*

Proof. Fix some large $e \in \mathbb{R}_{>0}$. Note that the O_e – for real positive e – are uniformly definable. Since F is uniformly continuous on O_{2e}, there is some positive real d such that for all $x, x' \in O_{2e}$

$$|x - x'| < \frac{1}{d} \;\Rightarrow\; |F(x) - F(x')| < \frac{1}{2e}.$$

Since the infimum of all such d is a semialgebraic function of e, there is some $n \in \mathbb{N}$ such that for all large $e \in \mathbb{N}$ and all $x, x' \in O_{2e}$ we have

$$|x - x'| < \frac{1}{e^n} \;\Rightarrow\; |F(x) - F(x')| < \frac{1}{2e}.$$

By a similar argument we obtain a polynomial bound e^m for $|F|$ on O_{2e}.

Now define $f : \mathbb{Q}^N \times N \to \mathbb{Q}$ in the following way: If a does not belong to O_{2e}, set $f(a, e) = 0$. Otherwise let $f(a, e)$ be the unique

$$b \in \{-e^m, -e^m + \frac{1}{2e}, \dots, e^m - \frac{1}{2e}, e^m\}$$

such that $F(a) \in [b, b + \frac{1}{2e})$. Then f is lower elementary by Lemma 4.1. Now assume $e \in \mathbb{N}_{>0}, a \in \mathbb{Q}^N$ and $x \in O_e$ with $|x-a| < \frac{1}{e^n}$. We may assume $2e \leq e^n$. Then $a \in O_{2e}$ and therefore

$$|F(x) - f(a, e)| \leq |F(x) - F(a)| + |F(a) - f(a, e)| < \frac{1}{2e} + \frac{1}{2e} = \frac{1}{e}.$$

□

By Remark 3.2 this yields:

Corollary 4.3. *Let $F : O \to \mathbb{R}$ be semialgebraic. If there is some open semialgebraic set U containing an ϵ-neighborhood of O and such that F can be extended continuously and semialgebraically to U, then $F : O \to \mathbb{R}$ is uniformly lower elementary.* □

Remark 4.4. *It is easy to see that, if $F : O \to V$ is continuous and F, O, V are semialgebraic, then F is lower elementary compact.*

Corollary 4.5. *The set of $\mathcal{F}$-reals $\mathbb{R}_{\mathcal{F}}$ forms a real closed field.*

Proof. By Theorem 4.2, $\mathbb{R}_{\mathcal{F}}$ is a field. To see that $\mathbb{R}_{\mathcal{F}}$ is real closed consider for odd $n \in \mathbb{N}$ the semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$ where $f(a_0, \dots a_{n-1})$ is the minimal zero of the polynomial $\sum_{i=0}^{n-1} a_i X^i + X^n$. By semialgebraic cell decomposition (see [10], Ch. 3), $\mathbb{R}^n$ can be decomposed into finitely many semialgebraic cells on which f is continuous. Each cell is homeomorphic to an open subset of $\mathbb{R}^k$ for some $k \geq 0$ via the appropriate (semialgebraic) projection map. Thus, composing the inverse of such a projection π with f we obtain a semialgebraic map on an open subset of $\mathbb{R}^k$. Applying Theorem 4.2 and Lemma 3.3 first to π^{-1} and then to the composition, we see that a polynomial with coefficients in $\mathbb{R}_{\mathcal{F}}$ has a zero in $\mathbb{R}_{\mathcal{F}}$. □

Corollary 4.5 was first proved by Skordev (see [7]). For countable good classes $\mathcal{F}$, like the class of lower elementary functions, clearly $\mathbb{R}_{\mathcal{F}}$ is a countable subfield of $\mathbb{R}$.

5 Integration

Theorem 5.1. *Let $O \subset \mathbb{R}^N$ be open and $G, H : O \to \mathbb{R}$ in $\mathcal{F}$ such that $G < H$ on O. Put*

$$U = \{(x, y) \in \mathbb{R}^{N+1} \mid x \in O,\ G(x) < y < H(x)\}.$$

Assume further that $F : U \to \mathbb{R}$ is in $\mathcal{F}$ and that $|F(x,y)|$ is bounded by an $\mathcal{F}$–function $K(x)$. Then

$$I(x) = \int_{G(x)}^{H(x)} F(x,y)\,\mathrm{d}y \tag{2}$$

is an $\mathcal{F}$–function $O \to \mathbb{R}$.

Proof. We fix witnesses for F, G, H being in $\mathcal{F}$: let $d : \mathbb{N} \to \mathbb{N}$, $f : \mathbb{Q}^{N+1} \times \mathbb{N} \to \mathbb{Q}$, $g, h : \mathbb{Q}^N \times \mathbb{N} \to \mathbb{Q}$ be $\mathcal{F}$–functions such that for $x \in O_e$, $a \in \mathbb{Q}^N$, $b \in \mathbb{Q}$

$$|x-a| < \frac{1}{d(e)} \Rightarrow |G(x) - g(a,e)| < \frac{1}{e} \quad \text{and} \tag{3}$$

$$|H(x) - h(a,e)| < \frac{1}{e} \tag{4}$$

and, if $|y| \leq e$ and $y \in [G(x) + \frac{1}{e}, H(x) - \frac{1}{e}]$,

$$|x-a| < \frac{1}{d(e)} \wedge |y-b| < \frac{1}{d(e)} \Rightarrow |F(x,y) - f(a,b,e)| < \frac{1}{e}. \tag{5}$$

By Lemma 3.5 we get an $\mathcal{F}$–function $\kappa : \mathbb{N} \to \mathbb{N}$ with $|F(x,y)| \leq \kappa(e)$ for all $(x,y) \in U$ and $H(x) - G(x) \leq \kappa(e)$ for all $x \in O_e$. We assume that $\kappa \geq 2$.

Let $d'(e) = d(e'')$, where $e' = 12e\kappa(e)$ and $e'' = 2d(e')$. Define the function $j : \mathbb{Q}^N \times \mathbb{N} \to \mathbb{Q}$ as follows: if $h(a,e'') - g(a,e'') < \frac{4}{e'}$, set $j(a,e) = 0$. Otherwise set

$$j(a,e) = \sum_{s=0}^{S-1} f(a, g(a,e'') + \frac{1}{e'} + s\delta, e') \cdot \delta,$$

where $S = \kappa(e)e''$ and $\delta = \frac{1}{S}(h(a,e'') - g(a,e'') - \frac{2}{e'})$. By Lemma 2.3 there is a function $i'(a,S,2e) = i(a,e)$ in $\mathcal{F}$ with $|j(a,e) - i(a,e)| < \frac{1}{2e}$.

In order to show that for all $a \in \mathbb{Q}^N$ and $x \in O_e$

$$|x-a| < \frac{1}{d'(e)} \Rightarrow \big|I(x) - i(a,e)\big| < \frac{1}{e} \tag{6}$$

it suffices to show

$$|x-a| < \frac{1}{d'(e)} \Rightarrow \big|I(x) - j(a,e)\big| < \frac{1}{2e}.$$

Since[3] $e \leq e' \leq e''$, the hypothesis implies $x \in O_{e'} \subseteq O_{e''}$. We also have $|x-a| < \frac{1}{d(e'')}$ and therefore

$$|G(x) - g(a,e'')| < \frac{1}{e''} \tag{7}$$

$$|H(x) - h(a,e'')| < \frac{1}{e''}. \tag{8}$$

[3]Remember that d is strictly increasing by assumption.

Case 1: $h(a,e'') - g(a,e'') < \frac{4}{e'}$. Using (7) and (8) we have $|H(x) - G(x)| < \frac{2}{e''} + \frac{4}{e'} \le \frac{6}{e'}$. Therefore $\big|I(x)\big| < \frac{6}{e'}\kappa(e) = \frac{1}{2e}$, which was to be shown.

Case 2: $h(a,e'') - g(a,e'') \ge \frac{4}{e'}$. This implies $H(x) - G(x) \ge \frac{4}{e'} - \frac{2}{e''} \ge \frac{2}{e'}$. Thus $\Delta = \frac{1}{S}(H(x) - G(x) - \frac{2}{e'})$ is non-negative. Note that by (7) and (8)

$$|\Delta - \delta| < \frac{2}{Se''}. \tag{9}$$

We also have

$$\Delta < \frac{\kappa(e)}{S} = \frac{1}{e''}.$$

It is easy to see that for each $s \le S$, we have

$$\Big|\Big(G(x) + \frac{1}{e'} + s\Delta\Big) - \Big(g(a,e'') + \frac{1}{e'} + s\delta\Big)\Big| \le \max\Big(|G(x) - g(a,e'')|, |H(x) - h(a,e'')|\Big) < \frac{1}{e''}. \tag{10}$$

This implies for every $y \in [G(x) + \frac{1}{e'} + s\Delta, G(x) + \frac{1}{e'} + (s+1)\Delta]$ that

$$\Big|y - \Big(g(a,e'') + \frac{1}{e'} + s\delta\Big)\Big| < \frac{1}{e''} + \Delta \le \frac{2}{e''} = \frac{1}{d(e')}.$$

Since $|x - a| < \frac{1}{d(e')}$, we have therefore

$$\Big|F(x,y) - f\Big(a, g(a,e'') + \frac{1}{e'} + s\delta, e'\Big)\Big| < \frac{1}{e'}, \tag{11}$$

which implies

$$\Big| \int_{G(x)+\frac{1}{e'}}^{H(x)-\frac{1}{e'}} F(x,y)\,\mathrm{d}y \;-\; \sum_{s=0}^{S-1} f\Big(a, g(a,e'') + \frac{1}{e'} + s\delta, e'\Big) \cdot \Delta \;\Big| < \frac{\kappa(e)}{e'}. \tag{12}$$

By (11) we have that all $f\big(a, g(a,e'') + \frac{1}{e'} + s\delta, e'\big)$ are bounded by $\kappa(e) + 1$. It follows that

$$\Big| \sum_{s=0}^{S-1} f\Big(a, g(a,e'') + \frac{1}{e'} + s\delta, e'\Big) \cdot \Delta \;-\; j(a,e) \;\Big| \le S(\kappa(e)+1)|\Delta - \delta| \le \frac{2(\kappa(e)+1)}{e''}. \tag{13}$$

Finally the absolute values of $\int_{G(x)}^{G(x)+\frac{1}{e'}} F(x,y)\,\mathrm{d}y$ and $\int_{H(x)-\frac{1}{e'}}^{H(x)} F(x,y)\,\mathrm{d}y$ are bounded by $\frac{\kappa(e)}{e'}$. By (12) and (13), this yields

$$\big|I(x) - j(a,e)\big| < \frac{2(\kappa(e)+1)}{e''} + \frac{\kappa(e)}{e'} + \frac{2\kappa(e)}{e'} \le \frac{6\kappa(e)}{e'} = \frac{1}{2e}.$$

This proves (6). $\square$

An examination of the proof yields:

Corollary 5.2. *If F, G, H are as above and uniformly in $\mathcal{F}$, then the function*

$$x \mapsto \int_{G(x)}^{H(x)} F(x,y)\,\mathrm{d}y$$

is uniformly in $\mathcal{F}$.

6 Periods

Kontsevich and Zagier [5] define a period as *a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.* The periods form a ring containing all algebraic reals. It is an open problem whether e is a period.

Yoshinaga [13] proved that periods are *elementary.* An analysis of his proof shows that he actually showed that periods are lower elementary. We here give a variant of his proof where part of his argument is replaced by an application of Theorem 5.1.

Definition 6.1. *A 1–dimensional bounded open cell is a bounded open interval with algebraic endpoints. An $n+1$–dimensional bounded open cell is of the form*

$$\big\{(x,y) \in \mathbb{R}^{N+1} \mid x \in O,\ G(x) < y < H(x)\big\}$$

for some n–dimensional bounded open cell O and bounded continuous semialgebraic functions $G < H$ from O to $\mathbb{R}$.

Thus our cells are semialgebraic.

Lemma 6.2. *Let C be an N–dimensional bounded open cell and $N = A + B$. Let O be the projection on the first A–coordinates and for $x \in O$ let C_x be the fiber over x. Then the map $x \mapsto \mathrm{vol}(C_x)$ is a bounded lower elementary function.*

Proof. By induction on B. Let U be the projection on the first $A+1$ coordinates. Then U is of the form

$$\big\{(x,y) \in \mathbb{R}^{A+1} \mid x \in O,\ G(x) < y < H(x)\big\}$$

for some bounded and continuous semialgebraic functions $G < H$. By induction hypothesis, $u \mapsto \mathrm{vol}(C_u)$ is a bounded lower elementary function $U \to \mathbb{R}$. By Fubini

$$\mathrm{vol}(C_x) = \int_{G(x)}^{H(x)} \mathrm{vol}(C_{x,y})\,\mathrm{d}y.$$

This is bounded and lower elementary by Theorem 5.1. $\square$

Corollary 6.3. *The volumes of bounded semialgebraic sets are lower elementary.*

Proof. Since every semialgebraic set is the disjoint union of semialgebraic cells, it is enough to know the claim for bounded semialgebraic cells, which is the $N = 0$ case of Lemma 3.3. □

Corollary 6.4. *Periods are lower elementary.*

Proof. By Lemma 24 of [13], periods are differences of sums of volumes of bounded open semialgebraic cells. □

7 The Inverse Function Theorem

We call a sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of subsets of $\mathbb{Q}^N$ an $\mathcal{F}$–*sequence*, if $\{(e, a) \mid a \in \mathcal{A}_e\}$ is an $\mathcal{F}$–subset of $\mathbb{N}_{>0} \times \mathbb{Q}^N$.

Definition 7.1. *An open set $O \subset \mathbb{R}^N$ is $\mathcal{F}$–approximable if there is an $\mathcal{F}$–sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of subsets of $\mathbb{Q}^N$ and an $\mathcal{F}$–function $\alpha : \mathbb{N} \to \mathbb{N}$ such that $O_e \cap \mathbb{Q}^N \subset \mathcal{A}_e \subset O_{\alpha(e)}$ for all $e \in \mathbb{N}_{>0}$.*

It follows from Lemma 4.1 that semialgebraic sets O are lower elementary approximable. We can simply set $\mathcal{A}_e = O_e \cap \mathbb{Q}^N$.

The following observations will not be used in the sequel. For the last part, we need Lemma 7.4.

Remark 7.2. *1. The intervals (x, ∞), $(-\infty, y)$ and (x, y) are $\mathcal{F}$–approximable if and only if x and y are $\mathcal{F}$–reals.*

2. If O is $\mathcal{F}$–approximable and $F, G : O \to \mathbb{R}$ are in $\mathcal{F}$, then $\{(x, y) \mid x \in O, F(x) < y < G(x)\}$ is $\mathcal{F}$–approximable.

3. If $F : O \to V$ is a homeomorphism and F and F^{-1} are uniformly in $\mathcal{F}$, then O is $\mathcal{F}$–approximable if and only if V is.

Theorem 7.3. *Let $F : O \to V$ be a bijection in $\mathcal{F}$ where O is $\mathcal{F}$–approximable and V open in $\mathbb{R}^N$. Assume that the inverse $G : V \to O$ satisfies:*

(i) There is an $\mathcal{F}$–function $d' : \mathbb{N} \to \mathbb{N}$ such that $|G(y) - G(y')| < \frac{1}{e}$ for all $y, y' \in V_e$ with $|y - y'| < \frac{1}{d'(e)}$.

(ii) G is $\mathcal{F}$–compact.

Then G is also in $\mathcal{F}$.

By the proof of Lemma 3.6 and Remark 7.5 below the conditions (i) and (ii) are necessary for the conclusion to hold.

Proof. As G is $\mathcal{F}$–compact, let $\gamma : \mathbb{N} \to \mathbb{N}$ be an $\mathcal{F}$–function such that $G(V_e) \subseteq O_{\gamma(e)}$.

Since F is in $\mathcal{F}$, we find $\mathcal{F}$–functions $d(e)$ and $f(a, e)$ such that for all $a \in \mathbb{Q}^N$ and $x \in O_e$

$$|x - a| < \frac{1}{d(e)} \to |F(x) - f(a, e)| < \frac{1}{e} \tag{14}$$

We also fix a function α and a sequence $\mathcal{A}_i, i \in \mathbb{N}_{>0}$ as in Definition 7.1.

We now construct two $\mathcal{F}$–functions $d'' : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{Q}^N \times \mathbb{N} \to \mathbb{Q}^N$ such that

$$|y - b| < \frac{1}{d''(e)} \to |G(y) - g(b,e)| < \frac{1}{e} \tag{15}$$

for all $e \in \mathbb{N}_{>0}$, $b \in \mathbb{Q}^N$ and $y \in V_e$.

Fix $e \in \mathbb{N}$ and $b \in \mathbb{Q}^N$. Set

$$d'' = d''(e) = \max\big(4d'(2e),\, 8e,\, \alpha(2\gamma(e)),\, 2\gamma(e)\big).$$

Also put $C = \max\big(2\gamma(e),\, d(d'')\big)$ and consider the set

$$\mathcal{A} = \mathcal{A}_{2\gamma(e)} \cap \big(\frac{1}{C}\ \mathbb{Z}\big)^N.$$

Since the elements of $\mathcal{A}$ are bounded by $\alpha(2\gamma(e))$, $\mathcal{A}$ is a finite set. If there is an $a \in \mathcal{A}$ such that

$$\big|b - f(a, d'')\big| < \frac{2}{d''}, \tag{16}$$

we choose such an a by an $\mathcal{F}$–function (!) $a = g(b,e)$. Otherwise put $g(b,e) = 0$.

Let us check that d'' and g satisfy (15). Start with e and b as above and consider an $y \in V_e$ with $|y - b| < \frac{1}{d''(e)}$.

We first show that $\mathcal{A}$ contains an element a' with $|b - f(a', d'')| < \frac{2}{d''}$. For this set $x = G(y) \in O_{\gamma(e)}$. Choose an $a' \in \big(\frac{1}{C}\ \mathbb{Z}\big)^N$ such that $|x - a'| < \frac{1}{C}$. Since $C \geq 2\gamma(e)$, we have $a' \in O_{2\gamma(e)}$ and therefore $a' \in \mathcal{A}$. Since $d'' \geq 2\gamma(e)$ and $d(d'') \leq C$ we have $|y - f(a', d'')| < \frac{1}{d''}$ by (14). This implies $|b - f(a', d'')| < \frac{2}{d''}$.

Now set $a = g(b,e)$. By the previous paragraph, we know that a is in $\mathcal{A}$ and satisfies (16). Since $a \in O_{\alpha(2\gamma(e))}$ and $d'' \geq \alpha(2\gamma(e))$, we have $\big|F(a) - f(a, d'')\big| < \frac{1}{d''}$. This implies $|y - F(a)| \leq |y - b| + |b - f(a, d'')| + |f(a, d'') - F(a)| < \frac{4}{d''}$. Since $\frac{d''}{4} \geq 2e$, this implies $F(a) \in V_{2e}$. Since $d'(2e) \leq \frac{d''}{4}$, we can use (i) to obtain $|x - a| < \frac{1}{e}$. □

Lemma 7.4. *Let $G : V \to O$ be open and continuous. Suppose*

1. *G is $\mathcal{F}$–bounded.*
2. *There is an $\mathcal{F}$–function $d' : \mathbb{N} \to \mathbb{N}$ such that*

$$|G(x) - G(x')| < \frac{1}{d'(e)} \ \to\ |x - x'| < \frac{1}{e}$$

whenever $G(x), G(x') \in O \restriction e$

Then G is $\mathcal{F}$–compact.

Remark 7.5. *By Lemma 3.5 and the proof of Lemma 3.6, respectively, the conditions of Lemma 7.4 are satisfied if G is a bijection in $\mathcal{F}$ and G^{-1} is uniformly in $\mathcal{F}$.*

Proof of 7.4. Choose an $\mathcal{F}$–function β such that $|G(x)| \leq \beta(e)$ for all $x \in V_e$, so $G(V_e) \subseteq O \restriction \beta(e)$. We may assume $\beta(e) \geq 2e$. Let $\gamma(e) = \max(2\beta(e), d'(2\beta(e)))$. We will show that $G(V_e) \subset O_{\gamma(e)}$ for all $e \in \mathbb{N}_{>0}$.

Let $x \in V_e$ and $y = G(x) \in O \restriction \beta(e) \subseteq O \restriction \gamma(e)$. So we have to show that $\operatorname{dist}(y, \mathbb{R}^N \setminus O) \geq \frac{1}{\gamma(e)}$. Let B be the open ball[4] around x with radius $\frac{1}{e+1}$. Then the closure $\overline{B} = B \cup \delta B$ is still a subset of V. Since G is continuous and open, $G(\overline{B})$ is compact and $G(B)$ is open in O and therefore in $\mathbb{R}^N$. Let y'' be any element in $\mathbb{R}^N \setminus O$. Look at the line segment L between y'' and y. L contains an element of $G(\delta B)$ since otherwise the traces of $G(B)$ and $\mathbb{R}^N \setminus G(\overline{B})$ on L were an open partition of L. So let $x' \in \delta B$ and $y' = G(x') \in L$. Then $|x - x'| = \frac{1}{e+1} \geq \frac{1}{2e}$.

By assumption

$$|y - y'| < \frac{1}{d'(2\beta(e))} \rightarrow |x - x'| < \frac{1}{2\beta(e)}$$

if $y, y' \in O \restriction 2\beta(e)$. So there are two cases. Either $|y - y'| \geq \frac{1}{d'(2\beta(e))} \geq \frac{1}{\gamma(e)}$ or $|y'| > 2\beta(e)$. But since $|y| \leq \beta(e)$, in this case we have $|y - y'| > 1$ and we are done either way. □

Corollary 7.6. *Let O be $\mathcal{F}$–approximable and V open and convex in $\mathbb{R}^N$ and $F : O \to V$ a bijection which is uniformly in $\mathcal{F}$. Assume that the inverse $G : V \to O$ is differentiable and that $|\operatorname{D}(G)|$ can be bounded by an $\mathcal{F}$–function $G' : V \to \mathbb{R}$. Then G belongs also to $\mathcal{F}$.*

Here the norm of a matrix $A = (a_{i,j})$ is $\max_i \sum_j |a_{i,j}|$.

Proof. It suffices to show that G satisfies (i) and (ii) of Theorem 7.3.
Proof of (i): By Lemma 3.5, there is an $\mathcal{F}$-function $\gamma : \mathbb{N} \to \mathbb{Q}$ such that $|G'(y)| \leq \gamma(e)$ for all $y \in V_e$. Assume that $y, y' \in V_e$ and $|y - y'| < \frac{1}{2e}$. Then the line segment between y and y' is contained in V_{2e} and it follows that $|G(y) - G(y')| \leq |y - y'|\gamma(2e)$ So we can set $d'(e) = \max\big(2e, \frac{e}{\gamma(2e)}\big)$.

Proof of (ii): We have to verify the two conditions of Lemma 7.4. Condition 2 follows from the assumption that F is uniformly in $\mathcal{F}$. It remains to show that G is $\mathcal{F}$-bounded.

Fix some $y_0 \in V$ and some e_0 with $y_0 \in V_{e_0}$. If $e \geq e_0$ and $y \in V_e$, then the line segment between y_0 and y lies in V_e. So

$$|G(y)| \leq |G(y) - G(y_0)| + |G(y_0)| \leq |y - y_0|\gamma(e) + |G(y_0)|.$$

We set $\beta(e) = 2e'\gamma(e') + \lceil |G(y_0)| \rceil$, where $e' = \max(e, e_0)$, and have $|G(y)| \leq \beta(e)$. □

The next proposition shows that for proper intervals we can weaken the assumptions:

[4] Actually this is a cube, since we use the maximum norm.

Proposition 7.7. *Let $F : O \to V \subseteq \mathbb{R}$ be a homeomorphism with inverse G where $O = (c_0, c_1)$ is a bounded open interval whose endpoints are $\mathcal{F}$–reals. Suppose that F belongs to $\mathcal{F}$ and that there is an $\mathcal{F}$–function $d' : \mathbb{N} \to \mathbb{N}$ such that $|G(y) - G(y')| < \frac{1}{e}$ for all $y, y' \in V_e$ with $|y - y'| < \frac{1}{d'(e)}$. Then G is in $\mathcal{F}$.*

Proof. Without loss of generality let us assume that F is increasing. For simplicity we also assume $c_0, c_1 \in \mathbb{Q}$. Since F is in $\mathcal{F}$, we find $\mathcal{F}$–functions $d(e)$ and $f(a, e)$ such that for all $a \in \mathbb{Q}$ and $x \in O_e$ we have

$$|x - a| < \frac{1}{d(e)} \to |F(x) - f(a, e)| < \frac{1}{e}$$

We may assume $d(e) \geq 4e$ and $d'(e) \geq e$ for all $e \in \mathbb{N}$. We will find an $\mathcal{F}$–function $g(b, e)$ such that for $y \in V_e$ we have

$$|y - b| < \frac{1}{d'(2d(e))} \to |G(y) - g(b, e)| < \frac{1}{e}$$

Let $e' = 2d'(2d(e))$. If $\frac{1}{2d(e')}\mathbb{Z} \cap O_{e'} = \emptyset$, put $g(b, e) = \frac{1}{2}(c_0 + c_1)$. If there is some $a \in \frac{1}{2d(e')}\mathbb{Z} \cap O_{e'}$ with $|b - f(a, e')| < \frac{1}{e'}$, put $g(b, e) = a$ with a minimal such. Otherwise, put $g(b, e) = c_1$ if $b - f(a, e') \geq \frac{1}{e'}$ for all $a \in \frac{1}{2d(e')}\mathbb{Z} \cap O_{e'}$ and c_0 if $f(a, e') - b \geq \frac{1}{e'}$ for all $a \in \frac{1}{2d(e')}\mathbb{Z} \cap O_{e'}$. Note that one of these cases occurs since for $a, a' \in \frac{1}{2d(e')}\mathbb{Z} \cap O_{e'}$ with $|a - a'| = \frac{1}{2d(e')}$ we have $|f(a, e') - f(a', e')| < \frac{2}{e'}$.

Now let $y \in V_e$ and $|y - b| < \frac{1}{d'(2d(e))}$. So $b, y \in V_{2d(e)}$ and $|G(y) - G(b)| < \frac{1}{2d(e)}$.
Case 1: Suppose that $x = G(y) \in O_e$. Note that $O_e \neq \emptyset$ implies that $\frac{1}{2d(e')}\mathbb{Z} \cap O_{e'} \neq \emptyset$. Since $|G(y) - G(b)| < \frac{1}{2d(e)}$, we have $G(b) \in O_{e'}$. Hence there is some smallest $a \in \frac{1}{2d(e')}\mathbb{Z} \cap O_{e'}$ with $|b - f(a, e')| < \frac{1}{e'}$. Then $g(b, e) = a$ and

$$|G(y) - g(b, e)| \leq |G(y) - G(b)| + |G(b) - a|.$$

Now

$$|F(a) - b| \leq |F(a) - f(a, e')| + |f(a, e') - b| \leq 2\frac{1}{e'} \leq \frac{1}{d'(2d(e))}.$$

Since $y \in V_e$ and $|y - F(a)| \leq |y - b| + |F(a) - b| < \frac{2}{d'(2d(e))}$ we have $F(a), b \in V_{2d(e)}$ and hence $|G(b) - a| \leq \frac{1}{2d(e)}$. Combining all this we see that

$$|G(y) - g(b, e)| < \frac{1}{d(e)} < \frac{1}{e}.$$

Case 2: Suppose that $x = G(y) \notin O_e$. Then if $g(b, e) = a \in O_{e'}$ the same argument as above works. Otherwise by construction of g, $g(b, e)$ equals either c_0 or c_1 and we again have $|G(y) - g(b, e)| < \frac{1}{e}$. □

8 Series of functions

Definition 8.1. *For an open set $O \subseteq R^N$ a sequence $F_1, F_2, \ldots$ of functions $O \to \mathbb{R}^M$ is in $\mathcal{F}$ if there are $\mathcal{F}$–functions $d : \mathbb{N}^2 \to \mathbb{N}$ and $f : \mathbb{Q}^N \times \mathbb{N}^2 \to \mathbb{Q}^M$ such that for all $i, e \in \mathbb{N}_{>0}$ and all $a \in \mathbb{Q}^N$ and $x \in O_e$*

$$|x - a| < \frac{1}{d(i,e)} \to |F_i(x) - f(a,i,e)| < \frac{1}{e}. \tag{17}$$

This definition is the $n = 1$ case of the obvious notion of an $\mathcal{F}$–function $F : \mathbb{N}^n \times O \to \mathbb{R}^M$. Note also that for $N = 0$ this defines $\mathcal{F}$–sequences of elements of $\mathbb{R}^M$.

Lemma 8.2. *If $F : O \times \mathbb{R}_{>0} \to \mathbb{R}^M$ is in $\mathcal{F}$, then so is the sequence $F(-, i) : O \to \mathbb{R}^M$, $(i = 1, 2, \ldots)$.*

Proof. There are $\mathcal{F}$–functions $d' : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{Q}^{N+1} \times \mathbb{N} \to \mathbb{Q}^M$ such that for all $a \in \mathbb{Q}^N$, $x \in O_e$ and $b \in [\frac{1}{e}, e] \cap \mathbb{Q}$

$$|x - a| < \frac{1}{d'(e)} \to |F(x,b) - f(a,b,e)| < \frac{1}{e}.$$

So we can set $d(i,e) = d'(\max(i,e))$. □

Definition 8.3. *A sequence $F_1, F_2, \ldots$ of functions $O \to \mathbb{R}^M$ $\mathcal{F}$–converges against F, if there is an $\mathcal{F}$–function $m : \mathbb{N} \to \mathbb{N}$ such that $|F(x) - F_i(x)| < \frac{1}{e}$ for all $x \in O_e$ and $i \geq m(e)$.*

Lemma 8.4. *The $\mathcal{F}$–limit of an $\mathcal{F}$–sequence of functions is an $\mathcal{F}$–function.*

Proof. Let d and f be as in Definition 8.1 and m as in 8.3. Set $d'(e) = d(m(2e), 2e)$ and $f'(a,e) = f(a, m(2e), 2e)$. Consider $a \in \mathbb{Q}^N$, $x \in O_e$ and assume $|x - a| < \frac{1}{d'(e)}$. Then $|F(x) - F_{m(2e)}(x)| < \frac{1}{2e}$ and $|F_{m(2e)}(x) - f'(a,e)| < \frac{1}{2e}$. It follows that $|F(x) - f'(a,e)| < \frac{1}{e}$. □

Proposition 8.5. *Let $F_1, F_2, \ldots$ be an $\mathcal{F}$–sequence of functions $O \to \mathbb{R}^M$ such that the partial sums of the series $\sum_{i=1}^{\infty} F_i$ are $\mathcal{F}$–convergent. Then $\sum_{i=1}^{\infty} F_i : O \to \mathbb{R}^M$ is in $\mathcal{F}$.*

Proof. We have to show that the series of partial sums is an $\mathcal{F}$–series of functions. This can easily be done using Lemma 2.3. □

9 Examples

1. Inverse trigonometric functions

The function $\frac{x}{1+x^2t^2}$ is continuous and semialgebraic. So by Theorem 5.1

$$\arctan(x) = \int_0^1 \frac{x}{1 + x^2t^2} \,\mathrm{d}t$$

is lower elementary. The same argument shows that

$$\arcsin(x) = \int_0^1 \frac{x}{\sqrt{1 - x^2 t^2}} \,\mathrm{d}t$$

is lower elementary[5], as a function defined on $(-1, 1)$.

2. Logarithm

As a semialgebraic function $\frac{1}{x} : \mathbb{R}_{>0} \to \mathbb{R}$ is lower elementary. So by Theorem 5.1 $\ln(x) = \int_1^x \frac{1}{y} \,\mathrm{d}y$ is lower elementary, at least on $(0, 1)$ and $(1, \infty)$. But writing

$$\ln(x) = \int_0^1 \frac{x - 1}{1 + t(x - 1)} \,\mathrm{d}t,$$

we see that $\ln(x)$ is lower elementary on $(0, \infty)$.

The same formula defines also the main branch of the complex logarithm $\ln(z) : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$. Since the real and imaginary part of the integrand $\frac{z-1}{1+t(z-1)}$ are semialgebraic functions of $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and t, we conclude that $\ln(z)$ is lower elementary.[6] It is also easy to see that $\ln(z)$ is lower elementary compact as a function from $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ to $\{z \mid \mathrm{Im}(z) \in (-\pi, \pi)\}$.

3. Exponentiation

As $\exp(x)$ is bounded on every interval $(-\infty, r)$ we may apply Proposition 7.7 to $\ln : (0, 1) \to (-\infty, 0)$ to conclude that $\exp(x)$ is lower elementary on $(-\infty, 0)$ and – by translation – on every interval $(-\infty, r)$. $\exp(x)$ cannot be lower elementary on the whole real line since it grows too fast. Nevertheless the following is true.

Lemma 9.1. $G(x, y) = \exp(x)$ *is lower elementary on* $V = \{(x, y) \colon \exp(x) < y\}$.

Proof. Clearly, G is lower elementary on $V \cap (\mathbb{R}_{<1} \times \mathbb{R})$ and differentiable on V. So let us consider $V' = V \cap (\mathbb{R}_{>0} \times \mathbb{R})$, which is a convex open subset of $\mathbb{R}^2$. Let $O = \{(z, y) \colon 1 < z < y\}$, which is lower elementary approximable, let $F : O \longrightarrow V'$ map (z, y) to $(\ln z, y)$ and let H be the inverse of F. The norm of the differential of H is bounded by the lower elementary function $(x, y) \mapsto y$. Now F is uniformly low, which by Corollary 7.6 implies that H and therefore $G \restriction V'$ is lower elementary. Now G is lower elementary on V by Remark 3.10. □

The complex logarithm defines a homeomorphism

$$\ln : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \{z \mid \mathrm{Im}(z) \in (-\pi, \pi)\}$$

[5] We do not know whether arcsin is uniformly lower elementary.
[6] We identify $\mathbb{C}$ with $\mathbb{R}^2$.

and for all $r < s \in \mathbb{R}$ a uniformly lower elementary homeomorphism between $\{z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \mid \exp(r) < |z| < \exp(s)\}$ and $\{z \mid \mathrm{Im}(z) \in (-\pi, \pi), r < \mathrm{Re}(z) < s\}$. We can apply Corollary 7.6 to see that $\exp(z)$ is lower elementary on $\{z \mid \mathrm{Im}(z) \in (-\pi, \pi), r < \mathrm{Re}(z) < s\}$. Using the periodicity of $\exp(z)$ it is now easy to see that $\exp(z)$ is lower elementary on each strip $\{z \mid r < \mathrm{Re}(z) < s\}$. This implies that $\sin(x) : \mathbb{R} \to \mathbb{R}$ is lower elementary. Now also $\cos(x)$ is lower elementary and since $\exp(x + y\,\mathrm{i}) = \exp(x)(\cos(y) + \sin(y)\,\mathrm{i})$ we see that $\exp(z)$ is lower elementary on every half-space $\{z \mid \mathrm{Re}(z) < s\}$.

Remark 9.2. *It is easy to see that* $\exp(z)$ *is elementary on* $\mathbb{C}$*.*

4. x^y

Consider the function x^y, defined on $X = (\mathbb{C} \setminus \mathbb{R}_{\leq 0}) \times \mathbb{C}$ by

$$x^y = \exp(\ln(x) \cdot y).$$

Since $\ln(x)$ is lower elementary and $\exp(x)$ elementary on $\mathbb{C}$, it is clear that x^y is elementary.

Let us determine some subsets of X on which x^y is lower elementary. We use the notation $E = \{z \in \mathbb{C} \mid |z| < 1\}$ for the open unit disc.

Fact 4.1: x^y is lower elementary on $(\mathbb{E} \setminus \mathbb{R}_{\leq 0}) \times \mathbb{R}_{>0}$.

Proof: $\ln(x) \cdot y$ maps this set to $\{z \mid \mathrm{Re}(z) < 0\}$, on which exp is uniformly lower elementary.

Fact 4.2: x^y is lower elementary on $(0, 1) \times \{z \mid \mathrm{Re}(z) > 0\}$.

Proof: $\ln(x) \cdot y$ maps this set to $\{z \mid \mathrm{Re}(z) < 0\}$.

Fact 4.3: x^y is lower elementary on $\mathbb{R}_{>0} \times \{z \mid 0 < \mathrm{Re}(z) < r)$, for all positive r.

Proof: Since $x^{a+b\,\mathrm{i}} = x^a(\sin(\ln(x)b) + \cos(\ln(x)b)\,\mathrm{i})$, it suffices to consider x^y on $\mathbb{R}_{>0} \times (0, r)$. x^y is lower elementary on $(0, 2) \times (0, r)$, since $\ln(x)y$ maps $(0, 2) \times (0, r)$ to $(-\infty, \ln(2)r)$. Therefore we are left with $U = (1, \infty) \times (0, r)$. Let N be a natural number $\geq r + 1$. Then x^y is the composition of $F(x, y) = (\ln(x)y, x^N)$ and $G(z, w) = \exp(z)$. F maps U into $V = \{(z, w) \mid \exp(z) < w\}$, on which G is lower elementary by Lemma 9.1. We will show that $F : U \to V$ is lower elementary compact, so we can apply Corollary 3.9 to conclude that x^y is lower elementary on U. It is clear that F is lower elementary compact as a function from U to $\mathbb{R}^2$. So it suffices to show that for all $e \in \mathbb{N}_{>0}$ and $(x, y) \in [\frac{1}{e}, \infty) \times (0, r)$ we have $\mathrm{dist}(F(x, y), \mathbb{R}^2 \setminus V) \geq \frac{1}{4e}$. This amounts to

$\exp(\ln(x)y + \frac{1}{4e}) < x^N - \frac{1}{4e}$, which is easy to prove:

$$\begin{aligned}\exp\bigl(\ln(x)y + \tfrac{1}{4e}\bigr) &= x^y \exp\bigl(\tfrac{1}{4e}\bigr) \le x^y\bigl(1 + \tfrac{1}{2e}\bigr) < x^y\bigl(1 + \tfrac{1}{e}\bigr) - x^y\tfrac{1}{4e}\\ &\le x^N - x^y\tfrac{1}{4e} \le x^N - \tfrac{1}{4e}.\end{aligned}$$

5. Gamma function

We have for $\mathrm{Re}(x) > 1$

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{-1+x}\exp(-t)\,\mathrm{d}t\\ &= \int_0^1 t^{-1+x}\exp(-t)\,\mathrm{d}t + \int_1^\infty t^{-1+x}\exp(-t)\,\mathrm{d}t\\ &= \int_0^1 t^{-1+x}\exp(-t)\,\mathrm{d}t + \int_0^1 \frac{1}{t^{1+x}}\exp\bigl(\frac{-1}{t}\bigr)\,\mathrm{d}t.\end{aligned}$$

Let us check that for every bound $r > 1$ the two integrands are lower elementary on $X_r = \{(x,t) \mid \mathrm{Re}(x) \in (1,r), t \in (0,1)\}$: $\exp(-t)$ is lower elementary on $(0,1)$. And, since $\frac{1}{t} : (0,1) \to (1,\infty)$ is lower elementary compact (cf. Remark 4.4), the function $\exp(\frac{-1}{t})$ is lower elementary on $(0,1)$. t^{-1+x} and $\frac{1}{t^{1+x}} = \bigl(\frac{1}{t}\bigr)^{x+1}$ are lower elementary by Fact 4.3 above.

If $r > 1$, the absolute values of the integrands are bounded by 1 and $(r+1)^{(r+1)}$, respectively. So by Theorem 5.1, Γ is lower elementary on every strip $\{z \mid 1 < \mathrm{Re}(z) < r\}$.

6. Zeta–function

Since x^y is lower elementary on $(0,1) \times \{z \mid \mathrm{Re}(z) > 0\}$ (Fact 4.2), the function $(\frac{1}{x})^y$ is lower elementary on $(1,\infty) \times \{z \mid \mathrm{Re}(z) > 0\}$. This implies that the sequence $\frac{1}{n^s}$, $(n = 1, 2, \ldots)$ is a lower elementary sequence of functions defined on $\{z \mid \mathrm{Re}(z) > 0\}$ by Lemma 8.2[7]. The series

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$$

converges whenever $t = \mathrm{Re}(z) > 1$ and we have the estimate

$$\Bigl|\zeta(z) - \sum_{n=1}^N \frac{1}{n^z}\Bigr| \;\le\; \int_N^\infty \frac{1}{x^t}\,\mathrm{d}x = \frac{1}{(t-1)N^{t-1}}.$$

So, if $\mathrm{Re}(z) \ge 1 + \frac{1}{k}$ and $N \ge (ke)^k$, we have $\Bigl|\zeta(z) - \sum_{n=1}^N \frac{1}{n^z}\Bigr| < \frac{1}{e}$. This shows that $\zeta(z)$ is lower elementary on every $\{z \mid \mathrm{Re}(z) > s\}$, $(s > 1)$ by Proposition 8.5.

[7]Strictly speaking, one applies 8.2 to $(\frac{1}{x+1})^y$ to get the lower elementary series $\frac{1}{2^s}, \frac{1}{3^s}, \ldots$.

10 Holomorphic functions

Lemma 10.1. *The sequence of functions z^n, $n = 0, 1, \ldots$ is a lower elementary sequence of functions on $\mathbb{E}$*

Proof. By Fact 4.1 and Lemma 8.2, the sequence z^1, z^2,... is lower elementary on $\mathbb{E} \setminus \mathbb{R}_{\leq 0}$. Since $(-z)^n = (-1)^n z^n$, it follows that z^1, z^2,... is lower elementary on $\mathbb{E} \setminus \{0\}$. It is now easy to see that z^1, z^2,... is actually lower elementary on $\mathbb{E}$. (Set $f(a, i, e) = 0$, if $|a| < \frac{1}{e}$.) $\square$

Lemma 10.2. *Let $F(z) = \sum_{i=0}^{\infty} a_i z^i$ be a complex power series with radius of convergence ρ. Let $0 < b < \rho$ be an $\mathcal{F}$–real such that $(a_i b^i)_{i \in \mathbb{N}}$ is an $\mathcal{F}$–sequence of complex numbers. Then F restricted to the open disc $\{z \colon |z| < b\}$ belongs to $\mathcal{F}$.*

Proof. By assumption and the last lemma the sequence $(a_i b^i z^i)$ is $\mathcal{F}$ on $\mathbb{E}$. If we plug in the $\mathcal{F}$–function $z \mapsto \frac{z}{b}$, we see that $(a_i z^i)$ is lower elementary on $\{z \colon |z| < b\}$. We are finished, if we can show that $\sum_{i=0}^{\infty} a_i z^i$ is $\mathcal{F}$–convergent on $\{z \colon |z| < b\}$.

For this we find a lower elementary function $m(e)$ such that $|\sum_{i=n}^{\infty} a_i z^i| < \frac{1}{e}$ for all $n \geq m(e)$ and $|z| \leq b - \frac{1}{e}$. Since $\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, there is an N such that $\sqrt[n]{|a_n|} < \frac{1}{b}$ for all $n > N$, or, $|a_n b^n|<1$ for all $n > N$.

We show that $\sum_{i=n}^{\infty} |a_i| x^i < \frac{1}{e}$ for all $n > \max(N, b^2 e^3)$ and $x \in [0, b - \frac{1}{e}]$:

$$\begin{aligned}
\sum_{i=n}^{\infty} |a_i| x^i &= \Big(\frac{x}{b}\Big)^n \sum_{i=0}^{\infty} |a_{n+i} b^{n+i}| \Big(\frac{x}{b}\Big)^i \leq \Big(\frac{x}{b}\Big)^n \sum_{i=0}^{\infty} \Big(\frac{x}{b}\Big)^i \\
&\leq \Big(1 - \frac{1}{be}\Big)^n \frac{1}{1 - \frac{x}{b}} \leq \Big(\frac{1}{1 + \frac{1}{be}}\Big)^n be \leq \frac{1}{1 + \frac{n}{be}} be \\
&\leq \frac{(be)^2}{n} < \frac{1}{e}
\end{aligned}$$

$\square$

Remark 10.3. *If $0 < b_0 < b_1$ are $\mathcal{F}$–reals and $(a_i b_1^i)$ is an $\mathcal{F}$–sequence, then also $(a_i b_0^i)$ is an $\mathcal{F}$–sequence.*

Proof. $\Big(\dfrac{b_0}{b_1}\Big)^i$ is an $\mathcal{F}$–sequence. $\square$

Definition 10.4. *Let A be a compact subset of $\mathbb{R}^N$. We call a function $F : A \to \mathbb{R}^M$ to be in $\mathcal{F}$ if there are $\mathcal{F}$–functions $d : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{Q}^N \times \mathbb{N} \to \mathbb{Q}^M$ such that for all $e \in \mathbb{N}_{>0}$ and all $a \in \mathbb{Q}^N$ and $x \in A$*

$$|x - a| < \frac{1}{d(e)} \to |F(x) - f(a, e)| < \frac{1}{e}.$$

Let $F : A \to \mathbb{R}^M$ be defined on the compact set A. The following is easy to see:

1. If F can be extended to an $\mathcal{F}$–function defined on an open set, then F is in $\mathcal{F}$.

2. If F is in $\mathcal{F}$, then all restrictions to open subsets of A belong to $\mathcal{F}$.

Lemma 10.5. *Let $F(z) = \sum_{i=0}^{\infty} a_i z^i$ be a complex power series with radius of convergence ρ. Assume that for some $\mathcal{F}$–real $b < \rho$, $F(z)$ restricted to $\{z\colon |z| \leq b\}$ is in $\mathcal{F}$. Then the sequence $(a_i b^i)$ belongs to $\mathcal{F}$.*

Proof. We have

$$\begin{aligned} a_n &= \frac{1}{2\pi\,\mathrm{i}} \int_{|z|=b} \frac{F(z)}{z^{n+1}}\,\mathrm{d}z = \frac{1}{2\pi\,\mathrm{i}} \int_{|z|=1} \frac{F(bz)}{(bz)^{n+1}}\,\mathrm{d}(bz) \\ &= \frac{1}{2\pi\,\mathrm{i}} \frac{1}{b^n} \int_{|z|=1} \frac{F(bz)}{z^{n+1}}\,\mathrm{d}z \end{aligned}$$

The integral can be computed as

$$\begin{aligned} &\int_0^{2\pi} \frac{F(b\exp(x\,\mathrm{i}))}{\exp(x\,\mathrm{i})^{n+1}} (\,\mathrm{i}\exp(x\,\mathrm{i}))\,\mathrm{d}x \\ &= \mathrm{i}\int_0^{2\pi} F(b\exp(x\,\mathrm{i}))\exp(-nx\,\mathrm{i})\,\mathrm{d}x \end{aligned}$$

An application of Theorem 5.1 yields that $y \mapsto \mathrm{i}\int_0^{2\pi} F(b\exp(x\,\mathrm{i}))\exp(-yx\,\mathrm{i})\,\mathrm{d}x$ is a lower elementary function from $\mathbb{R}$ to $\mathbb{C}$. The lemma follows from this by an application of Lemma 8.2. □

Lemma 10.6. *A sequence $(x_n) \in \mathbb{C}$ is in $\mathcal{F}$ if there is an $\mathcal{F}$–function $G : \mathbb{N}^2 \to \mathbb{Q}^2$ with $|x_n - G(n,e)| < 1/e$ for all $n, e \in \mathbb{N}_{>0}$.*

Proof. This is clear from the definitions. □

Lemma 10.7 (Speed-Up Lemma)**.** *Suppose $(a_n) \in \mathbb{C}$ is a bounded sequence and that $0 < b < 1$ is an $\mathcal{F}$–real. Then $(a_n b^n)$ is an $\mathcal{F}$–sequence if $(a_n b^{2n})$ is.*

Proof. We may assume that $|a_n| < 1$ for all n. Let $f : \mathbb{N}^2 \to \mathbb{Q}^2$ be in $\mathcal{F}$ such that $|a_n b^{2n} - f(n,e)| < \frac{1}{e}$.

$$y_e = \frac{\ln(e)}{\ln(b^{-1})}, \quad (e = 1, 2, \ldots)$$

is an $\mathcal{F}$–sequence of reals. So there is an $\mathcal{F}$–function $h : \mathbb{N} \to \mathbb{Q}$ such that $|y_e - h(e)| < 1$ for all $e \in \mathbb{N}_{>0}$. We fix also a natural number $B \geq b^{-1}$.

We want to define an $\mathcal{F}$–function $G : \mathbb{N}^2 \to \mathbb{Q}^2$ with

$$\big|a_n b^n - G(n,e)\big| < \frac{1}{e}$$

for all $n, e \in \mathbb{N}_{>0}$. Let n, e be given. We distinguish two cases:

Case 1: $h(e) < n - 1$. We have then $y_e < n$, which implies

$$|a_n b^n| < |b^n| < \frac{1}{e}.$$

So we set $G(n, e) = 0$.

Case 2: $h(e) \geq n - 1$. Then $y_e \geq n - 2$, which means

$$b^{2-n} = \exp\bigl(\ln(b^{-1})(n-2)\bigr) \leq e.$$

We can now apply Lemma 9.1 and compute b^{2-n} as a lower elementary function of $\ln(b^{-1})(n-2)$ and e. It follows that we can compute $b^{-n} = b^{2-n}b^{-2}$ as an $\mathcal{F}$–function of n and e. So also $G(n, e) = f(n, B^2e^2)b^{-n}$ is an $\mathcal{F}$–function of n and e, and we have

$$\bigl|a_n b^n - G(n, e)\bigr| = \bigl|a_n b^{2n} - f(n, B^2e^2)\bigr| \cdot b^{-n} < \frac{b^{-n}}{B^2e^2} \leq \frac{b^{2-n}}{e^2} \leq \frac{1}{e}.$$

□

Corollary 10.8. *Let $F(z) = \sum_{i=0}^{\infty} a_i z^i$ be a complex power series with radius of convergence ρ. If F is in $\mathcal{F}$ on some closed subdisc of $\{z\colon |z| < \rho\}$, it is in $\mathcal{F}$ on all closed subdiscs.*

Proof. Let r be any positive rational number smaller than ρ. Choose a rational number s between r and ρ. We then have $|a_n s^n| < 1$ for almost all n. By assumption there is an N such that F is $\mathcal{F}$ on $\{z\colon |z| \leq s b_N\}$, where

$$b_N = \left(\frac{r}{s}\right)^{2^N}.$$

This implies by Lemma 10.5 that $((a_n s^n) b_N^n)$ is in $\mathcal{F}$. If $N > 0$, the Speed-Up Lemma shows that $((a_n s^n) b_{N-1}^n)$ is in $\mathcal{F}$. Continuing this way we conclude that $(a_n s^n b_0^n) = (a_n r^n)$ is in $\mathcal{F}$. So F is $\mathcal{F}$ on $\{z\colon |z| < r\}$ by Lemma 10.2. □

Theorem 10.9. *Let F be a holomorphic function, defined on an open domain $D \subset \mathbb{C}$. If F is in $\mathcal{F}$ on some non–empty open subset of D, it is in $\mathcal{F}$ on every compact subset of D.*

Proof. It is easy to see that one can connect any two rational[8] points a, b in D by a chain $a = a_0, \ldots, a_n = b$ of rational points such that for every $i < n$, some circle $O_i = \{z\colon |z - a_i| < r_i\}$ contains a_{i+1} and is itself contained in D.

If F is in $\mathcal{F}$ on some open neighborhood of a_0, Corollary 10.8 (applied to $F(z + a_0)$) shows that F is in $\mathcal{F}$ on any closed subdisc of O_0. So F is in $\mathcal{F}$ in some open neighborhood of a_1, etc. We conclude that all rational points of D

[8] i.e. in $\mathbb{Q}^2$

have an open neighborhood on which F is $\mathcal{F}$. So, again by Corollary 10.8, F is in $\mathcal{F}$ on all closed discs with rational center contained in D. Since we can cover any compact subset of D with a finite number of such discs, the theorem follows. □

We call all a holomorphic function which satisfies the condition of Theorem 10.9 to be *locally* in $\mathcal{F}$.

Corollary 10.10. *Let F be a holomorphic function, defined on an open domain $D \subset \mathbb{C}$. Let a be an $\mathcal{F}$–complex number in D and let b be a positive $\mathcal{F}$–real smaller than the radius of convergence of $F(a+z) = \sum_{i=0}^{\infty} a_n z^n$. Then F is locally in $\mathcal{F}$ if and only if $(a_n b^n)$ is an $\mathcal{F}$–sequence.* □

Corollary 10.11. *Let F be holomorphic on a punctured disk $D_\bullet = \{z \mid 0 < |z| < r\}$. Then the following holds:*

1. *If 0 is a pole of F and F is $\mathcal{F}$ on some non–empty open subset of $D_\bullet$, then F is $\mathcal{F}$ on every proper punctured subdisc $D'_\bullet = \{z \mid 0 < |z| < r'\}$.*
2. *If 0 is an essential singularity of F, F is not lower elementary on $D_\bullet$.*

Proof. 1: Let 0 be a pole of order k. Then $F(z)z^k$ is holomorphic on $D = \{z \, |z| < r\}$. By the theorem $F(z)z^k$ is $\mathcal{F}$ on any disc $D' = \{z : |z| < r'\}$, $r' < r$. Since z^{-k} is lower elementary on $D'_\bullet$, F is $\mathcal{F}$ on $D'_\bullet$.

2: If F would be lower elementary on $D_\bullet$, the absolute value of F on $\{z \mid 0 < |z| < \frac{1}{e}\}$ would be bounded by a polynomial in e (Lemma 3.5). So 0 would be a pole of F. □

Corollary 10.12. *Let $S = \{-n \mid n \in \mathbb{N}\}$ denote the set of poles of the Gamma function Γ. Γ is lower elementary on every set $\{z : |z| < r\} \setminus S$.*

Γ cannot be lower elementary on $\mathbb{C} \setminus S$ since $n!$ grows too fast. We believe that Γ is elementary on $\mathbb{C} \setminus S$.

Corollary 10.13. *The Zeta function $\zeta(z)$ is lower elementary on every punctured disk $\{z \mid 0 < |z-1| < r\}$.*

ζ cannot be lower elementary on $\mathbb{C} \setminus \{1\}$, since ∞ is an essential singularity. But ζ may be elementary on $\mathbb{C} \setminus \{1\}$.

Corollary 10.14. *The set $\mathbb{C}_\mathcal{F} = \mathbb{R}_\mathcal{F}[\mathrm{i}]$ of $\mathcal{F}$–complex numbers is algebraically closed and closed under $\ln(z)$, $\exp(z)$, $\Gamma(z)$ and $\zeta(z)$.*

Note that $\mathbb{R}_\mathcal{F}[\mathrm{i}]$ is algebraically closed since $\mathbb{R}_\mathcal{F}$ is real closed by Corollary 4.5.

If $a_0, a_1, \ldots$ are $\mathbb{Q}$–linearly independent algebraic numbers, the exponentials $\exp(a_0), \exp(a_1)\ldots.$ are lower elementary and algebraically independent by the Lindemann–Weierstraß Theorem. So the field of lower elementary complex numbers has infinite transcendence degree.

Remark 10.15. *If a holomorphic function belongs to $\mathcal{F}$, then also its derivative belongs to $\mathcal{F}$.*

Proof. This follows from the formula

$$F'(z_0) = \frac{1}{2\pi\,\mathrm{i}} \int_{|z-z_0|=r} \frac{F(z)}{(z-z_0)^2}\,\mathrm{d}z.$$

$\square$

Remark 10.16. *Let F be holomorphic function defined on an open domain, which maps some $\mathcal{F}$-complex number a to an $\mathcal{F}$–complex number $F(a)$. Then F is locally in $\mathcal{F}$ if and only if its derivative is.*

Proof. This follows from the fact that if a_0 and b are in $\mathcal{F}$, then $(a_n b^n)_{n\geq 0}$ is an $\mathcal{F}$–sequence if and only if $(na_n b^{n-1})_{n\geq 1}$ is an $\mathcal{F}$-sequence. $\square$

Pour-El and Richards ([6]) have shown that this is not true for functions of the reals: There is a recursive C^1–function with non–recursive derivative.

Remark 10.17. *Let F be a non-constant holomorphic function which belongs to $\mathcal{F}$. Then a is in $\mathcal{F}$ if and only if $F(a)$ is in $\mathcal{F}$.*

Proof. Assume first that $F'(a)$ is not zero. Then we can find a small open rectangle O with lower elementary endpoints which contains a and such that F defines a homeomorphism between U and $F(U)$. We can make U small enough such that $F \restriction U$ satisfies the conditions of Theorem 7.3 (see also Lemma 7.4). Then the inverse of $F \restriction U$ is in $\mathcal{F}$ and maps $F(a)$ to a.

If $F'(a) = 0$, let n be minimal such that $F^{(n+1)}(a)$ is not zero. Since $F^{(n)}$ is in $\mathcal{F}$ and $F^{(n)}(a) = 0$, the above shows that a is in $\mathcal{F}$. $\square$

We close with two more examples.

7.
The function $\exp(\frac{1}{z})$ is lower elementary on every annulus $\{z \mid r < |z|\}$, $r > 0$, but not on $\mathbb{C} \setminus \{0\}$.

8.
There is a lower elementary function $f : \mathbb{N} \to \{0, 1\}$ such that $n \mapsto f(2^n)$ is not lower elementary. Consider the series $F(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = f(2^n)$. F is holomorphic on $\mathbb{E}$ and lower elementary on every compact subset of $\mathbb{E}$, but the sequence (a_n) is not lower elementary.

Added in proof (26.9.2010)

The definition of a (primitive) recursive function $F : \mathbb{R} \to \mathbb{R}$ given by E. Specker in [9] is equivalent to the following: There are (primitive) recursive functions $d : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ such that for all $e \in \mathbb{N}_{>0}$ and all $a \in \mathbb{Q}$ and $x \in \mathbb{R}$

$$|x - a| < \frac{1}{d(e)} \to |F(x) - f(a, e)| < \frac{1}{2^e}.$$

Locally this agrees with our Definition 3.1.

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