Sharply 2-transitive groups

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Abstract

We give an explicit construction of sharply 2-transitive groups with fixed point free involutions and without nontrivial abelian normal subgroup.

1 Introduction

The finite sharply 2-transitive groups were classified by Zassenhaus [Z] in the 1930's. They were shown to always contain a regular abelian normal subgroup. It remained an open question whether the same holds for infinite sharply 2-transitive groups. The first examples of sharply 2-transitive groups without abelian normal subgroup were recently constructed in [RST]. In these examples involutions have no fixed points. We here give an alternative approach to such a construction by using partially defined group actions as also suggested by Rips. See [RST] for more background on the problem.

2 The construction

Theorem 2.1. *Let G_0 be a group containing an involution t. Suppose that G_0 acts on a set X and satisfies the following:*

1. *no nontrivial element of G_0 fixes more than one element of X (we say that G_0 is 2-sharp);*
2. *all involutions are conjugate to t;*
3. *t does not fix any element of X.*

Let κ be an infinite cardinal $\geq |X|$. Then we can extend G_0 to a sharply 2-transitive action of

$$G = \big(G_0 *_{\langle t\rangle} (\langle t\rangle \times F(S))\big) * F(R)$$

on a suitable set $Y \supset X$, where $F(R), F(S)$ are free groups on disjoint sets R, S of size κ.

Hence we obtain:

Corollary 2.2. *Any group can be extended to a group acting sharply* 2*-transitively on some appropriate set without nontrivial abelian normal subgroup.*

Proof. By adding a direct factor of order 2 if necessary and iterated HNN-extensions any group can be extended to a group with a unique nontrivial conjugacy class of involutions. Letting this group act regularly on itself by right translation all assumptions of Theorem 2.1 are satisfied. Finally we note that the existence of normal forms in free products easily implies that if H is nontrivial and K contains a nontrivial element k of order different from 2, then $G = H * K$ cannot have a nontrivial abelian normal subgroup: indeed, consider a nontrivial $g \in G$. If g belongs to K, it does not commute with g^h for any $h \in H \setminus 1$. Otherwise, g does not commute with g^k. $\square$

Definition 2.3. *A partial action of G on a set X consists of an action of G_0 on X and (injective) partial actions of the generators in $S \cup R$ such that for $s \in S, x \in X$ if xs is defined, then so is $(xt)s$ and we have $(xt)s = (xs)t$.*

Any element of G can be written as a reduced word in elements of

$$\mathcal{P} = (G_0 \setminus 1) \cup R \cup R^{-1} \cup S \cup S^{-1},$$

where we say that a word is *reduced* if there are no subwords of the form $g_1 g_2$, $r^\epsilon r^{-\epsilon}$, $s^\epsilon s^{-\epsilon}$, $ts_1^{\pm 1} \cdots s_n^{\pm 1} t$ or $s^\epsilon t s^{-\epsilon}$ for $g_i \in G_0 \setminus 1$, $r \in R$, $s, s_i \in S$, $\epsilon \in \{1, -1\}$. It is easy to see that two reduced words represent the same element of G if and only if they can be transformed into each other by swapping adjacent letters t and s^ϵ.

If $w = p_1 \cdots p_n$ is a word and x and element X we say that xw is *defined* if for all initial segments of w the action on x is defined, i.e. all xp_1, $(xp_1)p_2, \ldots$, $(\ldots(xp_1)\ldots)p_n$ are defined and we set $xw = (\ldots(xp_1)\ldots)p_n$. Notice that for elements from G_0 the action on X is defined everywhere. If xw is defined and w' is a reduced word which represents the same element of G as w, then xw' is also defined and we have $xw = xw'$. Thus the expression $xg = y$ makes sense for $g \in G, x, y \in X$. Furthermore X becomes a gruppoid with $\hom(x, y) = \{g \in G \mid xg = y\}$ under the natural map $\hom(x, y) \times \hom(y, z) \to \hom(x, z)$.

If G acts partially on X, then there is a canonical partial action on the set of *pairs*

$$(X)^2 = \{(x, y) \in X^2 \mid x \neq y\}.$$

Notice that since t does not fix a point, we have $(x, xt) \in (X)^2$ for all $x \in X$. For $a = (x, y)$ we denote by $\overline{a}$ the *flip* (y, x) of a. If ag is defined, then so is $\overline{a}g = \overline{ag}$.

Definition 2.4. *We call a partial action of G on X* good *if for all pairs $a \in (X)^2$ and $g \in G$ the following holds:*

1. *$ag = a$ implies $g = 1$.*
2. *If $ag = \overline{a}$, then g is conjugate to t.*

3. t does not fix an element of X.

Consider the action of G_0 on X as a partial action of G on X. Then our assumptions on G_0 in Theorem 2.1 translate exactly into saying that G acts well on X.

A word in $\mathcal{P}$ is *cyclically* reduced if every cyclic permutation of w is reduced. If a word is cyclically reduced, then every reduced word which represents the same element of G is also cyclically reduced. Thus, to be cyclically reduced is a property of elements of G. Clearly every element of G is conjugate to a cyclically reduced one. This shows that in the definition of a good partial action we can restrict ourselves to cyclically reduced elements. Note that the cyclically reduced conjugates of t are the involutions of G_0.

Lemma 2.5 (Extending s)**.** *Assume that G acts well on X and that for some $x \in X, s \in S$ and $\epsilon \in \{1,-1\}$ the expression xs^ϵ is not defined (and hence neither is xts^ϵ). Let $x'G_0 = \{x'g_0 \mid g_0 \in G_0\}$ be a set of new elements on which G_0 acts regularly and extend the partial operation of G to $X' = X \cup x'G_0$ by putting $xs^\epsilon = x'$ and $(xt)s^\epsilon = x't$. Then G acts well on X'.*

Proof. Assume $\epsilon = 1$, the other case being entirely similar. Let w be cyclically reduced and $aw = a$ for some pair a in X'. Then the word w describes a cycle in $(X')^2$ containing a. If the cycle contains pairs from X only, we are done. If there are two neighbouring pairs in the cycle which do not belong to X, they must be connected by an element $g_0 \in G_0 \setminus 1$. Thus the cycle contains a segment b, c'_1, d or a segment b, c'_1, c'_2, d where $b, d \in X$ and $c'_i \notin X$. In the first case we have $bs = c'_1$, $c'_1 s^{-1} = d$ and in the second case $bs = c'_1$, $c'_1 t = c'_2$, $c_2 s^{-1} = d$. In the first case a cyclic permutation of w contains the subword $s \cdot s^{-1}$, in the second case $s \cdot t \cdot s^{-1}$. Thus w is not cyclically reduced, a contradiction.

The proof for $aw = \overline{a}$ is similar: instead of a cycle such an element w describes a Moebius strip and we have the additional possibility that $a = (x', x'i)$ and $w = i$ for an involution $i \in G$. $\square$

Lemma 2.6 (Extending r)**.** *Assume that G acts well on X and that for some $x \in X, r \in R$ and $\epsilon \in \{1,-1\}$ the expression xr^ϵ is not defined. Choose a set $x'G_0 = \{x'g_0 \mid g_0 \in G_0\}$ of new elements on which G_0 acts regularly. Extending the partial operation of G on $X' = X \cup x'G_0$ by putting $xr^\epsilon = x'$ yields again a good action of G on X'.*

Proof. Consider a non-trivial cycle (or Moebius strip) in $(X)^2$ described by a cyclically reduced word w. It is easy to see that the cycle (Moebius strip) must either be completely contained in $(x'G_0)^2$ or completely contained in $(X)^2$. In the first case we have a Moebius strip of the form $(x', x'i)i = (x'i, x')$ for an involution $i \in G_0$. The second case cannot occur since G acts well on X by assumption. $\square$

Lemma 2.7 (Joining t-pairs)**.** *Assume that G acts well on X and let $a = (x, xt)$ and $b = (y, yt)$ be pairs for which there is no $g \in G$ with $ag = b$. Let $s \in S$ be an element which does not yet act anywhere. Extend the action by setting $as = b$. Then this action of G on X is again good.*

Proof. Let w be a cyclically reduced word with $cw = c$ for some pair $c \in (X)^2$. If s does not occur in w, then we have $w = 1$ since the previous action on X was good. Hence we may assume that w contains s. By cyclically permuting w and taking inverses we may also assume that $w = s \cdot w'$ and $aw = a$ and thus $bw' = a$. By assumption on a, b the subword w' must contain s. Hence we may write $w' = u \cdot s^\epsilon v$ for some subword u not containing s. We distinguish two cases:

1. $\epsilon = 1$. Then we must have $bu = a$ or $bu = \overline{a}$ as s is only defined on these pairs. Since $bu = \overline{a}$ implies $b(ut) = a$ both cases contradict the assumption on a, b.

2. $\epsilon = -1$. Then we have $bu = b$ or $bu = \overline{b}$. If $bu = b$, then $u = 1$ and w is not reduced. If $bu = \overline{b} = bt$, then $u = t$ and w contains the subword $s \cdot t \cdot s^{-1}$, contradicting the assumption that w be reduced.

Next we assume that w is cyclically reduced with $cw = \overline{c}$ for some pair $c \in (X)^2$. If w does not contain s, then w is conjugate to t since the previous action on X was good. So we may assume that $w = s \cdot w'$ and $aw = \overline{a}$, i.e. $bw' = \overline{a}$. By choice of a, b we must have w' containing s and we see as before that this is impossible. $\square$

Lemma 2.8 (Joining other pairs)**.** *Assume that G acts well on X and let a and b be pairs in $(X)^2$ such that there is no $g \in G$ with $ag = b$ or $ag = \overline{b}$. Assume furthermore that there is no g in G flipping b and that the action of $r \in R$ is not yet defined anywhere. Extending the partial action by setting $ar = b$ yields again a good action of G on X.*

Note that a may or may not be a t-pair.

Proof. Let w be cyclically reduced and $cw = c$ for some pair $c \in (X)^2$. If r does not appear in w, then we have $w = 1$ since the previous action on X is good. Hence we may assume again as before that we have $w = r \cdot w'$ and $aw = a$. Hence $bw' = a$. By assumption on a, b, the word w' must contain r. Write $w' = u \cdot r^\epsilon v$ for some subword u not containing r. We distinguish two cases

1. $\epsilon = 1$. Then $bu = a$ or $bu = \overline{a}$ as r is only defined there. But this contradicts our choice of $a, b \in (X)^2$.

2. $\epsilon = -1$. Then we have $bu = b$ or $bu = \overline{b}$. If $bu = b$, then we have $u = 1$ by assumption on the previous action and w is not reduced. Hence $bu = \overline{b}$, contradicting the assumption that no element of G flips b.

Now assume that there is some pair c with $cw = \overline{c}$. If w does not contain r, then w is conjugate to t since the previous action is good. Hence we may again assume that we have $w = r \cdot w'$ and $aw = \overline{a}$, hence $bw' = \overline{a}$. By assumption on a and b, the word w' must contain r and as before we see that this is impossible. $\square$

Corollary 2.9. *Let X, G_0, t, κ, R, S and G as in Theorem 2.1. Assume furthermore that the action of G_0 has been extended to a good partial action of G on X and that both R and S contain κ-many elements whose action is still not defined anywhere. Then we can extend the partial action of G to a sharply 2-transitive action of G on some appropriate $Y \supset X$.*

Proof. Choose partitions $R = \bigcup_{i=0}^{\infty} R_i$ and $S = \bigcup_{i=0}^{\infty} S_i$ in disjoint sets of size κ, such that for $j > 0$, the elements of R_j and S_j are nowhere defined. Fix a t-pair a in X. We construct a sequence of sets $X = X_0 \subset X_1 \subset \cdots$ together with extensions of the good action of G, in such a way that for $j > i$, the elements of R_j and S_j are nowhere defined on X_i.

Assume that X_i is already defined. In a *first step* we use Lemmas 2.7 and 2.8 to define a partial action of the elements of S_{i+1} and R_{i+1} on X_i such that:

1. all t-pairs in X_i are connected to a;

2. any pair in X_i can be flipped by an element of G.

The last property can be achieved as follows: if b cannot be flipped before, Lemma 2.8 tells us how to connect a and b by an element of R_{i+1}. After this b can be flipped since a can. In a *second step* we us Lemmas 2.5 and 2.6 to extend the partial action of G to a superset X_{i+1} such that for all $j \leq i+1$, the elements of S_j and R_j are defined on the whole of X_i.

Let Y be the union of the X_i. Then G acts well and therefore 2-sharply on Y, and we have that all t-pairs in Y are connected to a and that all pairs can be flipped. This implies that the action of G on Y is 2-transitive: It is enough to show that all pairs are connected to a. Let b be a pair and $g \in G$ so that $bg = \overline{b}$. Then $t = hgh^{-1}$ for some $h \in G$. This implies $(bh)t = \overline{bh}$, so bh is a t-pair and whence connected to a □

This concludes the proof of Theorem 2.1 and its corollary. Note that our construction yields a group action for which no involution has a fixed point.

While the construction given in [RST] yields an explicit description of the point stabilizers, the construction described here can be extended to yield sharply 3-transitive groups, in which the point stabilizers - so sharply 2-transitive groups - have no abelian normal subgroups, see [T].

References

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