On the isometry group of the Urysohn space

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Abstract

We give a general criterion for the (bounded) simplicity of the automorphism groups of certain countable structures and apply it to show that the isometry group of the Urysohn space modulo the normal subgroup of bounded isometries is a simple group.

1 Introduction

Many very homogeneous mathematical structures are known to have simple, or at least *essentially simple* automorphism groups. This is true for the complex numbers [3], for irreducible Riemannian symmetric spaces of non-compact type, and also as shown in [4] for structures arising as a Fraïssé limit of a free amalgamation class. The subject of the present paper is another very homogeneous structure, Urysohn's metric space $\mathbb{U}$, which is the unique complete homogeneous separable metric space which embeds every finite metric space. It is easy to construct $\mathbb{U}$: it arises as the completion of the rational metric space obtained as the Fraïssé limit $\mathbb{QU}$ of the class of finite metric spaces with rational distances. $\mathbb{QU}$ is called the *rational* Urysohn space and the usual Urysohn space is sometimes called the *complete* Urysohn space.

Let G denote the isometry group of the (complete) Urysohn space $\mathbb{U}$ and B the normal subgroup of all isometries having bounded displacement. We will show that the quotient G/B is a simple group.

This will follow from a more general result on automorphism groups of countable structures with a certain independence relation. As another application of this general result we will give another proof that for classes with free amalgamation in relational languages the automorphism group is a simple group unless the Fraïssé limit is an indiscernible set.

Our formal framework will be introduced in Section 2 with the main theorem proved in Section 3. A detailed analysis of unbounded isometries in the Urysohn space in Section 4 then allows us to apply our main result to the Urysohn space. As another application we recover and sharpen the results of [4] in Section 5.

2 Terminology and notation

Let $\mathcal{M}$ be a structure and $\mathcal{G}$ its automorphism group. Using model theoretic language, for a tuple $\bar{a}$ and a finite set B we say that the tuple $\bar{a}'$ *realises the type* $p = \mathrm{tp}(\bar{a}/B)$ if there is an automorphism of $\mathcal{M}$ which maps $\bar{a}$ to $\bar{a}'$ and fixes B pointwise.[1]

Let $A \mathop{\downarrow}_B C$ be a ternary relation between finite subsets of $\mathcal{M}$, pronounced A and C are independent over B.

Definition 2.1. $\downarrow$ *is a stationary independence relation if the following axioms are satisfied.*

1. (Invariance) *A and B being independent over C depends only on the type of ABC. (We choose enumerations for A, B and C and consider them as tuples. Note that we write AB for the union $A \cup B$.)*

2. (Monotonicity)
$$A \mathop{\downarrow}_{B} CD \quad \textit{implies} \quad A \mathop{\downarrow}_{B} C \textit{ and } A \mathop{\downarrow}_{BC} D.$$

3. (Transitivity)
$$A \mathop{\downarrow}_{B} C \textit{ and } A \mathop{\downarrow}_{BC} D \quad \textit{implies} \quad A \mathop{\downarrow}_{B} D.$$

4. (Symmetry)
$$A \mathop{\downarrow}_{B} C \quad \textit{implies} \quad C \mathop{\downarrow}_{B} A,$$

[1] If $\mathcal{M}$ is countable and ω-saturated, the types so defined correspond exactly to types in the model theoretic sense. If $\mathcal{M}$ is only ω–homogeneous, they correspond to *realised* types. And if $\mathcal{M}$ is a Fraïssé limit (see below), they correspond to realised quantifier free types.

5. (Existence) *Let p be a type over B and C a finite set. Then p has a realisation which is independent from C over B.*

6. (Stationarity) *If $\bar{a}$ and $\bar{a}'$ have the same type over B and are both independent from C over B, then $\bar{a}$ and $\bar{a}'$ have the same type over BC.*

If $A \mathop{\smile\!\!\!\!\!|}_B C$ is only defined for non-empty B, we say that $\mathop{\smile\!\!\!\!\!|}$ is a local independence relation on $\mathcal{M}$.

It is easy to see that the axioms imply

$$A \mathop{\smile\!\!\!\!\!|}_B C \ \Leftrightarrow\ AB \mathop{\smile\!\!\!\!\!|}_B C \ \Leftrightarrow\ A \mathop{\smile\!\!\!\!\!|}_B BC.$$

Also, on the basis of the other axioms Stationarity follows from the following special case for single elements:

> (Stationarity') *If a and a' have the same type over B and are both independent from c over B, then a and a' have the same type over Bc.*

Examples 2.2.

1. By a well-known construction of Fraïssé, a countable class $\mathcal{C}$ of finitely generated structures, closed under finitely generated substructures and satisfying the amalgamation and joint embedding properties has a Fraïssé limit: this is a countable structure $\mathcal{M}$ whose finitely generated substructures are—up to isomorphism—exactly the elements of $\mathcal{C}$ and which has the property that any isomorphism between finitely generated substructures extends to a global automorphism of $\mathcal{M}$ (see [5], Ch. 4.4 for more details).

   In many cases the amalgamation property of $\mathcal{C}$ is verified by the existence of a "canonical" amalgam $X \otimes_Y Z$ of X and Z over the common substructure Y which is functorial in the sense that automorphisms of the factors X and Z fixing Y elementwise will extend to the amalgam. This can then be used to define two finite subsets A and C of $\mathcal{M}$ to be independent over B if $\langle A \cup B \cup C\rangle$ is isomorphic to $\langle A \cup B\rangle \otimes_{\langle B\rangle} \langle B \cup C\rangle$ under an isomorphism commuting with the embeddings, where $\langle S\rangle$ denotes the substructure generated by S. At this level of generality, the independence notion satisfies only Existence, Invariance and Stationarity. In the following cases it defines either a stationary independence relation, or a local stationary independence relation, and in the local case it suffices to have $A \mathop{\smile\!\!\!\!\!\!\;\mid}_B C$ defined for B non-empty:

(a) The class $\mathcal{C}$ of finite metric spaces with distances in a countable additive subsemigroup R of the positive reals has canonical amalgamation over a nonempty base: If B is non-empty and A and C are two extensions of B which intersect exactly in B, we can put $A \otimes_B C = A \cup C$ with the metric defined by

$$d(a,c) = \min\{d(a,b) + d(b,c) \colon b \in B\}$$

if $a \in B, c \in C$. The Fraïssé limit is the R-valued Urysohn space $R\mathbb{U}$. Then $A \mathop{\smile\!\!\!\!\!\!\;\mid}_B C$ if and only if for all $a \in A, c \in C$ there is some $b \in B$ with $d(a,c) = d(a,b) + d(b,c)$. Note that independence over the empty set is not defined. The complete Urysohn space $\mathbb{U}$ is the completion of $\mathbb{Q}\mathbb{U}$.

(b) The bounded Urysohn space $\mathbb{U}_1$ enjoys similar properties with respect to the class of finite metric spaces with diameter at most 1 and is constructed in a similar fashion, as the completion of a Fraisse limit. We let $A \otimes_B C$ denote the metric space such that for $a \in A, c \in C$ the distance of a and c is the minimum of

$$\{d(b,a) + d(b,c) \colon b \in B\} \cup \{1\}.$$

Here B may be empty.

(c) If $\mathcal{C}$ is a class of relational structures, we may put $A \otimes_B C$ as the *free amalgam*, i.e. the structure on the set $A \cup C$ with no new relations on $(A \setminus B) \cup (C \setminus B)$. Then A and C are independent over B if and only if whenever $R(d_1, \ldots, d_n)$ holds for elements $d_1, \ldots d_n$ of $B \cup A \cup C$ then either all d_i are in $B \cup A$ or all d_i are in $B \cup C$. The random graph and random hypergraphs, the K_n-free graphs and their hypergraph analogs arise in this way. Again B may be empty here.

2. Let T be a stable complete theory and $\mathcal{M}$ an ω-homogeneous countable model on T. Then forking–independence has all properties of Definition 2.1 except possibly Stationarity, see [5, Ch. 8.5]. For Stationarity

we have to assume that all 1–types are stationary, which implies that all types are stationary. (An example of such a theory is the theory of trees with infinite valency.)

To see that Transitivity holds in the Urysohn spaces assume $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$ and consider $a \in A$ and $d \in D$. By assumption there is some $x \in BC$ with $d(a,d) = d(a,x) + d(x,d)$. If $x \in C$, there is some $b \in B$ with $d(a,x) = d(a,b) + d(bx)$. This implies $d(a,d) = d(a,b) + d(b,d)$, as required. The rest is clear.

The independence relations in examples 1(a), 1(b) and in 1(c) for binary relations have stronger properties than forking-independence has in general, notably

$$A \underset{B}{\downarrow} C \text{ and } B \subset B' \text{ implies } A \underset{B'}{\downarrow} C.$$

However, our proofs do not make use of these additional properties.

Definition 2.3. *We say that a finite tuple $\bar{x}$ is* independent from a tuple $\bar{y}$ over $A; B$ if

$$\bar{x} \underset{A}{\downarrow} B\bar{y} \text{ and } \bar{x}A \underset{B}{\downarrow} \bar{y}.$$

Lemma 2.4. *Let $\downarrow$ be a stationary independence relation on $\mathcal{M}$. Then the following holds.*

1. *For $\bar{x}$ to be independent from $\bar{y}$ over $A; B$ it is enough to have $\bar{x} \downarrow_A B$ and $\bar{x}A \downarrow_B \bar{y}$.*

2. (Existence) *Let p be a type over A and q a type over B. Then there are realisations $\bar{x}$ of p and $\bar{y}$ of q such that $\bar{x}$ is independent from $\bar{y}$ over $A; B$. The type $\mathrm{tp}(\bar{x}\bar{y}/AB)$ is uniquely determined.*

3. (Transitivity) *If $\bar{x}$ is independent from $\bar{y}$ over $A; B$ and $\bar{x}'$ is independent from $\bar{y}'$ over $\bar{x}A, \bar{y}B$, then $\bar{x}\bar{x}'$ is independent from $\bar{y}\bar{y}'$ over $A; B$.*

4. (Symmetry) *If $\bar{x}$ is independent from $\bar{y}$ over A, B, then $\bar{y}$ is independent from $\bar{x}$ over $B; A$.*

Proof.
1. By Symmetry and Monotonicity $\bar{x}A \downarrow_B \bar{y}$ implies $\bar{x} \downarrow_{AB} \bar{y}$. By Transitivity and $\bar{x} \downarrow_A B$ this implies $\bar{x} \downarrow_A B\bar{y}$.

2. Choose $\bar{x}$ such that $\bar{x} \mathop{\smile\!\!\!\!\!\!|}_A B$ and then $\bar{y}$ such that $\bar{x}A \mathop{\smile\!\!\!\!\!\!|}_B \bar{y}$.

3. Note that $\bar{x}A \mathop{\smile\!\!\!\!\!\!|}_{B\bar{y}} \bar{y}'$ implies $\bar{x} \mathop{\smile\!\!\!\!\!\!|}_{AB\bar{y}} \bar{y}'$. From $\bar{x} \mathop{\smile\!\!\!\!\!\!|}_A B\bar{y}$ and Transitivity we get $\bar{x} \mathop{\smile\!\!\!\!\!\!|}_A B\bar{y}\bar{y}'$. This and $\bar{x}' \mathop{\smile\!\!\!\!\!\!|}_{\bar{x}A} B\bar{y}\bar{y}'$ imply $\bar{x}\bar{x}' \mathop{\smile\!\!\!\!\!\!|}_A B\bar{y}\bar{y}'$ by Transitivity. Similarly one proves $\bar{x}\bar{x}'A \mathop{\smile\!\!\!\!\!\!|}_B \bar{y}\bar{y}'$.

4. This follows directly from the symmetry of $\mathop{\smile\!\!\!\!\!\!|}$. □

Definition 2.5. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be a (local) independence relation on $\mathcal{M}$ and $g \in \mathcal{G}$. For a finite set X and p a type over X we say that g moves a realisation $\bar{x}$ of p maximally if $\bar{x}$ is independent from $g(\bar{x})$ over $X; g(X)$. We say that g moves maximally if for all (non-empty) finite sets X and all types p over X, g moves some realisation of p maximally.*

Note that part (4) of Lemma 2.4 implies that g moves maximally if and only if g^{-1} does.

If $\mathcal{M}$ is the countable infinite set with no structure, A and C are independent over B if $A \cap C \subset B$. Hence a permutation of $\mathcal{M}$ moves maximally if and only it has infinite support. More generally if $\mathcal{M}$ is an ω–saturated countable strongly minimal structure in which algebraic and definable closure coincide and $\mathop{\smile\!\!\!\!\!\!|}$ is algebraic (i.e. forking-) independence, then g moves maximally if and only if g is unbounded in the sense of [3]. To see this note that an automorphism g of a strongly minimal structure $\mathcal{M}$ is bounded in the sense of [3] if and only if there is a finite set X such that for any $a \in \mathcal{M}$ we have $g(a) \in acl(aX)$.

Lemma 2.6. *For an automorphism to move maximally it suffices to move realisations of 1-types (i.e. types of single elements) maximally.*

Proof. This follows from Lemma 2.4(3). □

Here is our main result, which will be proved in Section 3:

Theorem 2.7. *Suppose that $\mathcal{M}$ is a countable structure with a local stationary independence relation and let $g \in \mathcal{G} = \mathrm{Aut}(\mathcal{M})$ move maximally. If $\mathcal{G}$ contains a dense conjugacy class, then any element of $\mathcal{G}$ is the product of eight conjugates of g.*

We note that for a structure with a stationary independence relation the assumption that $\mathcal{G}$ contains a dense conjugacy class is always satisfied:

Lemma 2.8. *The automorphism group of a countable structure $\mathcal{M}$ with a stationary independence relation has a dense conjugacy class. The same holds for a local stationary independence relation if for all finite tuples $\bar{a}$ and $\bar{b}$ in $\mathcal{M}$ realising the same type there is some element c with $\mathrm{tp}(\bar{a}/c) = \mathrm{tp}(\bar{b}/c)$.*

Proof. It is immediate that $\mathcal{G}$ contains a dense conjugacy class if and only the following is true: given finite tuples $\bar{x}, \bar{y}, \bar{a}, \bar{b}$ with $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ there are tuples $\bar{x}', \bar{y}'$ such that $\mathrm{tp}(\bar{x}'\bar{y}') = \mathrm{tp}(\bar{x}\bar{y})$ and $\mathrm{tp}(\bar{x}'\bar{a}) = \mathrm{tp}(\bar{y}'\bar{b})$. If $\mathcal{M}$ has a stationary independence relation, then we can choose $\bar{x}'y'$ realising $\mathrm{tp}(\bar{x}\bar{y})$ with $\bar{x}'y' \mathop{\smile\!\!\!\!\!\!|}\ \bar{a}\bar{b}$. By stationarity we then have $\mathrm{tp}(\bar{x}'\bar{a}) = \mathrm{tp}(\bar{y}'\bar{b})$.

If $\mathcal{M}$ has a local independence relation, let c be such that $\mathrm{tp}(\bar{a}/c) = \mathrm{tp}(\bar{b}/c)$ and work over c. □

Corollary 2.9. *The automorphism group of the Urysohn space has a dense conjugacy class.*

Proof. Just note that given $\bar{a}, \bar{b}$ satisfying the same type, we can find a point c at sufficiently large distance from $\bar{a}, \bar{b}$ such that $\mathrm{tp}(\bar{a}/c) = \mathrm{tp}(\bar{b}/c)$. □

Corollary 2.10. *Suppose that $\mathcal{M}$ is a countable structure with a stationary independence relation and let $g \in \mathcal{G}$ move maximally. Then any element of $\mathcal{G}$ is the product of eight conjugates of g.* □

The following example shows that in Theorem 2.7 the assumption that $\mathcal{G}$ contains a dense conjugacy class cannot be dispensed with:

Examples 2.11 (Cherlin)**.** Let $\mathcal{C}$ be the class of finite bipartite graphs in the language containing a binary relation presenting the edges and an equivalence relation with two classes presenting the bipartition. Then $\mathcal{C}$ has local stationary amalgamation, but for the Fraïssé limit $\mathcal{M}$ the automorphism group $\mathcal{G}$ contains no dense conjugacy class: the normal subgroup N of $\mathcal{G}$ consisting of the automorphisms preserving the equivalence classes is open. It is the automorphism group of an expansion of $\mathcal{M}$ by a predicate denoting one of the conjugacy classes. In this language $\mathcal{C}$ has stationary amalgamation. If $g \in \mathcal{G}$ moves maximally and preserves the equivalence classes, it is an automorphism of this expanded structure. By Corollary 2.10, every element of N is the product of eight conjugates of g. On the other hand, if $g \in \mathcal{G}$ does not preserve the equivalence classes, then any nontrivial commutator $[g, h]$ lies in N, showing that $\langle g \rangle^{\mathcal{G}} = \mathcal{G}$.

In Section 4 we will show that any isometry of the Urysohn space with unbounded displacement moves maximally (Proposition 4.1) and apply Theorem 2.7 to prove

Theorem 2.12. *For any unbounded isometry g of the Urysohn space the normal subgroup $\langle g\rangle^G$ is all of G. In fact, any element of G is the product of eight conjugates of g. Hence G/B is a simple group.*

We have not been able to establish the simplicity of the isometry group of the bounded Urysohn space with our methods.

The proof of Theorem 2.7 follows the general strategy of [3] and [4], using ideas of descriptive set theory. The main technical result is the following proposition, whose proof will be given in Section 3:

Proposition 2.13. *Under the assumptions and notations of Theorem 2.7, let*

$$\varphi : \mathcal{G}^4 \to \mathcal{G}, \varphi : (h_1, \dots h_4) \mapsto g^{h_1} \dots g^{h_4}.$$

Then for any open set $U \subseteq \mathcal{G}^4$ there is some open set $W \subseteq \mathcal{G}$ with $\varphi(U)$ dense in W. Equivalently, for any nowhere dense set X in $\mathcal{G}$, its preimage $\varphi^{-1}(X)$ is nowhere dense in $\mathcal{G}^4$

Proof of Theorem 2.7 from Proposition 2.13: By Proposition 2.13 the image of φ is not meagre, for if $\varphi(\mathcal{G}^4) = \bigcup X_i$ with X_i nowhere dense, we would have $\mathcal{G}^4 = \bigcup \varphi^{-1}(X_i)$ contradicting the Baire Category Theorem. Note that as the image under an analytic map, the set $\varphi(\mathcal{G}^4)$ has the Baire property and is invariant under conjugation. Since by assumption there is a dense conjugacy class, we conclude from [2, Theorem 8.46] (applied to $\mathcal{G}$ acting on itself by conjugation) that $\varphi(\mathcal{G}^4)$ is comeagre. Since g^{-1} moves maximally as well, the image of $\varphi' : (h_1, \dots h_4) \mapsto (g^{-h_1} \dots g^{-h_4})$ is also comeagre. So for any $f \in \mathcal{G}$ the translate $\varphi'(\mathcal{G}^4)f$ intersects $\varphi(\mathcal{G}^4)$, which is the claim of Theorem 2.7.

3 Proof of Proposition 2.13

We continue to work with the countable structure $\mathcal{M}$ with a local stationary independence relation. $\mathcal{G}$ is the automorphism group of $\mathcal{M}$. We write $\mathrm{Fix}(X)$ for the pointwise stabiliser of the set X in $\mathcal{G}$.

We start with a simple lemma.

Lemma 3.1.

1. *If $A \mathop{\smile\!\!\!\!\!\!\!\;|}_B C$ and D is arbitrary, then there is some D' such that $\mathrm{tp}(D'/BC) = \mathrm{tp}(D/BC)$ and $A \mathop{\smile\!\!\!\!\!\!\!\;|}_B CD'$*

2. *Let $A \mathop{\smile\!\!\!\!\!\!\!\;|}_B C$ and $g_1, \ldots, g_n \in \mathcal{G}$. Then there is $e \in \mathrm{Fix}(BC)$ with $A \mathop{\smile\!\!\!\!\!\!\!\;|}_B C g_1^e(C) \ldots g_n^e(C)$.*

Proof.
1. Choose D' of the right type with $A \mathop{\smile\!\!\!\!\!\!\!\;|}_{BC} D'$ and use Transitivity.

2. By (1) there are $C_1, \ldots, C_n$ with

$$\mathrm{tp}(C_1, \ldots, C_n/BC) = \mathrm{tp}(g_1(C), \ldots, g_n(C)/BC)$$

and

$$A \mathop{\smile\!\!\!\!\!\!\!\;|}_B CC_1 \ldots C_n.$$

Choose $e \in \mathrm{Fix}(BC)$ with $e(C_i) = g_i(C)$. Then we have $g_i^e(C) = C_i$. $\square$

Proposition 3.2. *Consider $g_1, \ldots, g_4 \in \mathcal{G}$ and finite sets $X_0, \ldots, X_4$ such that $g_i(X_{i-1}) = X_i$. Then for $i = 1, \ldots 4$ there are $a_i \in \mathrm{Fix}(X_{i-1}X_i)$ and extensions $X_i \subset Y_i$ such that*

1. *$g_i^{a_i}(Y_{i-1}) = Y_i$,*

2. *$Y_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_{Y_1} Y_2$ and $Y_2 \mathop{\smile\!\!\!\!\!\!\!\;|}_{Y_3} Y_4$.*

Proof.
Step 1. Choose a finite extension X_1' of X_1 such that $X_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_{X_1'} X_2X_3X_4$, for example $X_1' = X_0 \cup \ldots \cup X_4$.

Step 2. Apply 3.1(2) to $A = X_0$, $B = X_1'$, $C = X_1'X_2X_3X_4$ and the automorphisms g_2, g_3g_2 and $g_4g_3g_2$. We obtain $e \in \mathrm{Fix}(X_1'X_2X_3X_4)$ such that taking

$$X_2' = g_2^e(X_1'), X_3' = g_3^e(X_2'), X_4' = g_4^e(X_3'),$$

we have

$$X_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_{X_1'} X_2'X_3X_4'.$$

Step 3. The same argument as in Step 2 yields $f \in \mathrm{Fix}(X_0 X_1')$ such that taking

$$X_0' = (g_1^f)^{-1}(X_1')$$

we have

$$X_0' \underset{X_1'}{\downarrow} X_2' X_3' X_4'.$$

Set $h_1 = g_1^f$ and $h_i = g_i^e$ for $i = 2, 3, 4$.

Step 4. If we apply what we proved so far to the reversed sequence X_4', X_3', X_2', X_1', we obtain b_2, b_3, b_4 with $b_i \in \mathrm{Fix}(X_{i-1}' X_i')$ and extensions $X_i' \subset Y_i$ for $i = 1, \ldots, 4$ such that

$$h_i^{b_i}(Y_{i-1}) = Y_i \text{ and } Y_1 Y_2 \underset{Y_3}{\downarrow} Y_4.$$

Step 5. Lemma 3.1(1) shows that we may assume that

$$X_0' \underset{X_1'}{\downarrow} Y_1 Y_2 Y_3 Y_4.$$

By Monotonicity we conclude

$$X_0' \underset{Y_1}{\downarrow} Y_2 Y_3 Y_4.$$

Step 6. As in Step 3 we find some $b_1 \in \mathrm{Fix}(X_0' Y_1)$ such that with

$$Y_0 = (h_1^{b_1})^{-1}(Y_1)$$

we have

$$Y_0 \underset{Y_1}{\downarrow} Y_2 Y_3 Y_4.$$

□

Remark 3.3. In fact, the proof yields slightly more: we have

$$Y_1 Y_2 \underset{Y_3}{\downarrow} Y_4 \quad \text{and} \quad Y_0 \underset{Y_1}{\downarrow} Y_2 Y_3 Y_4$$

which together imply

$$Y_0 Y_1 Y_2 \underset{Y_3}{\downarrow} Y_4.$$

Note also that we may choose $a_2 = a_3$.

Proposition 3.4. *Let $g_1, \ldots, g_4 \in \mathcal{G}$ move maximally and let $Y_0, \ldots, Y_4$ be finite sets such that $g_i(Y_{i-1}) = Y_i$ for $i = 1, \ldots 4$. Assume also that $Y_0 \mathop{\smile\!\!\!\!\!\!\,|}_{Y_1} Y_2$ and $Y_2 \mathop{\smile\!\!\!\!\!\!\,|}_{Y_3} Y_4$. Let x_0 and x_4 be two tuples such that $g_4g_3g_2g_1$ maps $\mathrm{tp}(x_0/Y_0)$ to $\mathrm{tp}(x_4/Y_4)$. Then for $i = 1, \ldots 4$, there are $a_i \in \mathrm{Fix}(Y_{i-1}Y_i)$ such that*

$$g_4^{a_4} \ldots g_1^{a_1}(x_0) = x_4.$$

For the proof we need two lemmas:

Lemma 3.5. *Let $g \in \mathcal{G}$ move maximally, let X, Y, C be finite sets such that $g(X) = Y$ and $X \mathop{\smile\!\!\!\!\!\!\,|}_{Y} C$ and let x be a tuple. Then there is some $a \in \mathrm{Fix}(XY)$ such that*

$$g^a(x) \mathop{\smile\!\!\!\!\!\!\,|}_{Y} C.$$

Proof. Let x' be a realisation of $\mathrm{tp}(x/XY)$ moved maximally by g and let $a_1 \in \mathrm{Fix}(XY)$ be such that $a_1(x') = x$. Then g^{a_1} moves x maximally over XY. So we have

$$x \mathop{\smile\!\!\!\!\!\!\,|}_{XY} g^{a_1}(x).$$

Now let y be a realisation of $\mathrm{tp}(g^{a_1}(x)/XYx)$ with

$$y \mathop{\smile\!\!\!\!\!\!\,|}_{xXY} C.$$

We have then also $x \mathop{\smile\!\!\!\!\!\!\,|}_{XY} y$. By Transitivity, Symmetry and the assumption $X \mathop{\smile\!\!\!\!\!\!\,|}_{Y} C$ we conclude

$$y \mathop{\smile\!\!\!\!\!\!\,|}_{Y} C.$$

Finally choose $a_2 \in \mathrm{Fix}(xXY)$ with $a_2 g^{a_1}(x) = g^{a_1 a_2^{-1}}(x) = y$. □

Lemma 3.6. *Let $g \in \mathcal{G}$ move maximally and let X, Y be finite sets with $g(X) = Y$. Assume that x and y are tuples with x independent from y over $X; Y$ and such that $g(\mathrm{tp}(x/X)) = \mathrm{tp}(y/Y)$. Then there is some $a \in \mathrm{Fix}(XY)$ such that*

$$g^a(x) = y.$$

Proof. Let x' be a realisation of $\mathrm{tp}(x/X)$ which is moved maximally by g. Since $x' \mathop{\smile\!\!\!\!\!\!\,|}_{X} Y$, we have $\mathrm{tp}(x'/XY) = \mathrm{tp}(x/XY)$. Choose $a_1 \in \mathrm{Fix}(XY)$ with $a_1(x) = x'$. Then g^a moves x maximally over X. Set $y' = g^a(x)$. By Lemma 2.4.2 we have $\mathrm{tp}(xy'/XY) = \mathrm{tp}(xy/XY)$. Choose $a_2 \in \mathrm{Fix}(XY)$ with $a_2(xy) = a_2(xy')$. Then $g^{a_1 a_2}(x) = y$. □

Proof of Proposition 3.4. Note first that g_3^{-1} and g_4^{-1} also move maximally. Two applications of Lemma 3.5 yield $a_0 \in \mathrm{Fix}(Y_0Y_1)$ and $a_4 \in \mathrm{Fix}(Y_3Y_4)$ such that for

$$x_1 = g_1^{a_1}(x_0) \quad \text{and} \quad x_3 = (g_4^{-1})^{a_4}(x_4)$$

we have

$$x_1 \mathop{\downarrow}_{Y_1} Y_2 \quad \text{and} \quad Y_2 \mathop{\downarrow}_{Y_3} x_3.$$

Choose x_2 realising the type $g_2(\mathrm{tp}(x_1/Y_1)) = g_3^{-1}(\mathrm{tp}(x_3/Y_3))$ (over Y_2) and such that

$$x_2 \mathop{\downarrow}_{Y_2} x_1Y_1x_3Y_3.$$

Lemma 3.6 yields $a_2 \in \mathrm{Fix}(Y_1Y_2)$ and $a_3 \in \mathrm{Fix}(Y_2Y_3)$ such that $g_2^{a_2}(x_1) = x_2 = (g_3^{-1})^{a_3}(x_3)$. □

Proof of Proposition 2.13: We suppose g in $\mathcal{G} = \mathrm{Aut}(M)$ moves maximally and that U contained in $\mathcal{G}^4$ is open. We may assume that $U = U_1 \times \ldots U_4$, where each U_i is a basic open set $U_i = \mathcal{U}(u_i)$, with u_i a finite partial isomorphism and

$$\mathcal{U}(u) = \{g \in \mathcal{G} \mid u \subset g\}.$$

Extend each u_i to some $a_i \in \mathcal{G}$. Then choose finite sets $X_0, \ldots, X_4$ such that $\mathrm{im}(u_i) \subset X_i$ and $g^{a_i}(X_{i-1}) = X_i$ for $i = 1, \ldots, 4$. We apply Proposition 3.2 to this situation and obtain $b_i \in \mathrm{Fix}(X_{i-1}X_i)$ and extensions $X_i \subset Y_i$ with $g^{a_ib_i}(Y_{i-1}) = Y_i$ and such that

$$Y_0 \mathop{\downarrow}_{Y_1} Y_2 \text{ and } Y_2 \mathop{\downarrow}_{Y_3} Y_4.$$

Let w be the finite isomorphism $g^{a_4b_4} \ldots g^{a_1b_1} \restriction Y_0$. We set $W = \mathcal{U}(w)$.

In order to show that $\varphi(U)$ is dense in W we consider a basic open subset $\mathcal{U}(w')$ given by an extension $w \subset w'$. Let x be an enumeration of $\mathrm{dom}(w') \setminus Y_0$ and $y = w'(x)$. Proposition 3.4 gives us $c_i \in \mathrm{Fix}(Y_{i-1}Y_i)$ such that $g^{a_4b_4c_4} \ldots g^{a_1b_1c_1}(x) = y$. Since b_i and c_i both fix $\mathrm{im}(u_i)$ pointwise, we have $a_ib_ic_i \in U_i$. So the 4-tuple $(a_1b_1c_1, \ldots, a_4b_4c_4)$ belongs to U and is mapped by φ to $g^{a_4b_4c_4} \ldots g^{a_1b_1c_1}$, which belongs to W'. □

4 Application to the Urysohn space

We will now apply Theorem 2.7 to the complete Urysohn space. We extend our notion of independence to $\mathbb{U}$ in the obvious way: we write

$$A \underset{B}{\perp} C$$

if and only if for all $a \in A, c \in C$ there is some $b \in B$ with $d(a,c) = d(a,b) + d(b,c)$.

We first establish the following proposition which may be of interest in its own right.

Proposition 4.1. *Any unbounded isometry of the Urysohn space moves maximally.*

It is easy to see that unbounded isometries exist, i.e. that B is a proper subgroup of G (see also [1], Prop.17). Just define an automorphism of g on $\mathbb{QU}$ by a back-and-forth construction. In the even steps ensure that g will be everywhere defined and surjective. In the odd steps ensure that there are points which g moves arbitrarily far. Then extend g to the completion.

An instructive variant goes as follows: apply Lemma 2.4(2) to $\mathbb{QU}$ to construct an automorphism g which moves maximally. Then observe that g is unbounded. Indeed, let a and x be two elements of distance N. Choose a realisation x' of $\mathrm{tp}(x/a)$ which is moved maximally by g. We have then $d(x', g(x')) = d(a, g(a)) + 2N$.

For the sake of readability we now write x^g for the image of a point x under an automorphism g.

We need some lemmas in preparation for the proof of Proposition 4.1.

Lemma 4.2 (Minimal distance amalgamation)**.** *Let (X,d) be a finite metric space and $(X \cup \{y\}, d_1)$, $(X \cup \{z\}, d_2)$ two extensions. Then there is a metric $\hat{d}$ on $X \cup \{y,z\}$ extending d_1, d_2 with*

$$\hat{d}(y,z) = \max_{x \in X} |d_1(y,x) - d_2(z,x)|,$$

where we identify y and z if $\hat{d}(y,z) = 0$.

Proof. This is easy to check. $\square$

We call a sequence $(x_0, x_1, \ldots, x_n)$ *geodesic* if

$$d(x_0, x_n) = d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{n-1}, x_n).$$

Note that (x_0, x_1, x_2) is geodesic if and only if x_0 is independent from x_2 over x_1. (x_0, x_1, x_2, x_3) is geodesic if and only x_0 is independent from x_3 over $x_1; x_2$. This shows that the next two lemmas are special cases of Proposition 4.1.

Lemma 4.3. *Let g be an unbounded isometry of $\mathbb{U}$. Then for any points $x_1, x_2 \in \mathbb{U}$ there are points $y \in \mathbb{U}$ with (y, x_1, x_2) geodesic and $d(y, y^g)$ arbitrarily large.*

Proof. First, observe that there are points $z \in \mathbb{U}$ with both $d(z, x_1)$ and $d(z, z^g)$ arbitrarily large. Indeed, once $d(z, z^g)$ is sufficiently large, one of z or z^g will do. In particular, we may take $z \in \mathbb{U}$ with $d(z, x_1) > d(x_1, x_2)$ and $d(z, z^g)$ arbitrarily large. Let $X = \{x_2, z\}$ and let $X \cup \{y\}$ be a metric extension of X with (y, x_2, z) isometric with (x_1, z, x_2). Applying Lemma 4.2 to $X \cup \{y\}$ and $X \cup \{x_1\}$, we get a pseudometric d on $X \cup \{x_1, y\}$ with

$$\begin{aligned} d(x_1, y) &= \max\big(|d(x_1, x_2) - d(y, x_2)|, |d(x_1, z) - d(y, z)|\big) \\ &= d(z, x_1) - d(x_1, x_2) \end{aligned}$$

Therefore we may take such a point $y \in \mathbb{U}$ and we see that (y, x_1, x_2) is geodesic. Furthermore $d(y^g, z^g) = d(y, z) = d(x_1, x_2)$ so as $d(z, z^g)$ goes to infinity, $d(y, y^g)$ goes to infinity as well. □

Lemma 4.4. *Let g be an unbounded isometry of $\mathbb{U}$, and $x \in \mathbb{U}$. Then there are points $z \in \mathbb{U}$ with $d(z, z^g)$ arbitrarily large, such that (z, x, x^g, z^g) is geodesic.*

Proof. Applying Lemma 4.3 to g^{-1} we find y with $d(y, y^g)$ arbitrarly large and (x, x^g, y^g) geodesic. The inequality

$$d(y, y^g) \leq d(y, x) + d(x, x^g) + d(x^g, y^g)$$

implies that $a = \frac{1}{2}(d(y, y^g) - d(x, x^g))$ is not larger than $b = d(y, x)$. We may assume that a is not negative.

Pick some point z such that (x, z, y) is geodesic with $d(z, x) = a$ and $d(z, y) = b - a$ and such that

$$z \mathop{\smile\!\!\!\!\!\!\!\;|}_{\{x,y\}} y^g.$$

The distance between z and y^g is the minimum of

$$d(z, x) + d(x, y^g) = a + d(x, y^g) = a + d(x, y) + b$$

and

$$d(z, y) + d(y, y^g) = (b - a) + d(y, y^g).$$

By the definition of a these two values are equal, implying that (z, x, y^g) and hence (z, x, x^g, z^g, y^g) are geodesic. Since

$$\begin{aligned} d(z, z^g) &\geq d(z, y^q) - d(y^g, z^g) \\ &= ((b - a) + d(y, y^g)) - (b - a) \\ &= d(y, y^g) \end{aligned}$$

we see that $d(z, z^g)$ can become arbitrarily large.

$\square$

For $p = \mathrm{tp}(a/X)$ we let $d(p, X) = \min\{d(a, x) \colon x \in X\}$.

Lemma 4.5. *Let g be an unbounded isometry of $\mathbb{U}$. Let X be a nonempty finite set. Then there is some $e = e(X) \geq 0$ such that every type p over X has some realisation y in $\mathbb{U}$ for which $d(y, y^g) \geq 2d(p, X) - e$.*

Proof. We will show that $e = 2\,\mathrm{diam}(X)$ suffices. Let $p = \mathrm{tp}(a/X)$, and fix $x_0 \in X$. Apply Lemma 4.4 to find a geodesic of the form (z, x_0, x_0^g, z^g) with

$$d(z, x_0) > (\max_{x \in X} d(a, X)) + \mathrm{diam}(X)$$

Then $d(z, x) > d(a, x)$ for all $x \in X$. Therefore, if we apply Lemma 4.2 to the metric spaces $X \cup \{z\}$ and $X \cup \{a\}$, we get a realisation y of p in $\mathbb{U}$ such that

$$\begin{aligned} d(y, z) &= \max_{x \in X} (d(z, x) - d(y, x)) \\ &= d(z, x_1) - d(y, x_1) \end{aligned}$$

for some $x_1 \in X$. We claim that

$$d(y, y^g) \geq 2d(p, X) - 2\,\mathrm{diam}(X).$$

Considering first the path (z, y, y^g, z^g) and then the path (z, x_0, x_0^g, z^g), we find

$$\begin{aligned} d(y, y^g) &\geq d(z, z^g) - 2d(y, z) \\ &= 2d(z, x_0) + d(x_0, x_0^g) - 2d(y, z) \\ &\geq 2[d(z, x_0) - d(y, z)] \end{aligned}$$

Then considering the triangle (z, x_0, x_1) we have

$$\begin{aligned} d(z, x_0) - d(y, z) &\geq [d(z, x_1) - d(x_0, x_1)] - [d(z, x_1) - d(y, x_1)] \\ &= d(y, x_1) - d(x_0, x_1) \\ &\geq d(p, X) - \mathrm{diam}(X) \end{aligned}$$

and thus $d(y, y^g) \geq 2[d(p, X) - \mathrm{diam}(X)]$, as claimed. □

Definition 4.6. *For a type $p = \mathrm{tp}(a/X)$ and $d \in \mathbb{R}_{\geq 0}$ we call the type $\mathrm{tp}(y/X) = \{d(y, x) = d(a, x) + d \colon x \in X\}$ the d-prolongation of p, and denote it by $p + d$ or $\mathrm{tp}(a/X) + d$.*

Proof of Proposition 4.1. Let g be an unbounded isometry of $\mathbb{U}$ and let p be a type over the nonempty finite set X. Let a be a realisation of p with

$$a \mathop{\downarrow}_{X} X^g X^{g^{-1}}$$

and let $q = \mathrm{tp}(a/X X^g X^{g^{-1}})$. We will show that we can find a realisation z of q for which

$$z \mathop{\downarrow}_{X^g} z^g \tag{1}$$

and we claim that g moves this realisation maximally.

We address the second point first. By Lemma 2.4 (1) it suffices to check that

$$z \mathop{\downarrow}_{X} X^g \quad \text{and} \quad zX \mathop{\downarrow}_{X^g} z^g.$$

As z is a realisation of q we find $z \downarrow_X X^g$ and $X \downarrow_{X^g} z^g$. Therefore the condition (1) suffices.

Now we take up the construction of the point z. Applying Lemma 4.5 to the set $X' = XX^gX^{g^{-1}}$ we find $e \geq 0$ such that for any d the prolongation $q+d$ has a realisation y satisfying

$$d(y, y^g) \geq 2[d(p, X') + d] - e.$$

In particular, if $d > e$ we have

$$d(y, y^g) > d.$$

Fix $d > e$ and a corresponding realisation y of $q + d$. An application of Lemma 4.2 yields a realisation z of p at distance d from y. We may suppose also that

$$z \underset{Xy}{\downarrow} y^g.$$

We claim that this point z has the required properties, if d is taken sufficiently large.

First we show

$$z \underset{X}{\downarrow} X^gX^{g^{-1}} \tag{2}$$

As $y \downarrow_X X^gX^{g^{-1}}$, for any $x' \in X^gX^{g^{-1}}$ we have some $x \in X$ so that (y, x, x') is geodesic. But (y, z, x) is also geodesic, so (z, x, x') is geodesic. Claim (2) follows.

Now we check

$$z \underset{X^g}{\downarrow} z^g.$$

We first examine $d(z, y^g)$. By the choice of z, this is the minimum of the values $d(z, u) + d(u, y^g)$ where u ranges over$X \cup \{y\}$. For $x \in X$ we have

$$\begin{aligned} d(z, x) + d(x, y^g) &= d(z, x) + d(x^{g^{-1}}, y) \\ &= d(z, x) + d(x^{g^{-1}}, z) + d \\ &\leq d + 2 \max_{x' \in X'} d(x', z) \end{aligned}$$

Compare this with

$$d(z, y) + d(y, y^g) = d + d(y, y^g).$$

We may take $d(y, y^g) > 2 \max_{x' \in X'} d(x', z)$, and then there will be some $x \in X$ for which

$$(z, x, y^g) \text{ is geodesic.}$$

As $x \mathop{\smile\!\!\!\!\!\!\;\!\downarrow}_{X^g} y^g$ we have $x' \in X^g$ such such that (x, x', y^g) is geodesic, and then (z, x', y^g) is geodesic, and our claim follows. $\square$

Proof of 2.12. Supose given an unbounded isometry g of $\mathbb{U}$ and an arbitrary isometry f of $\mathbb{U}$. By Proposition 4.1 g moves maximally. Consider

$$\mathcal{U} = (\mathbb{U}, f, g, d, \mathbb{R}, +, <)$$

as a 2-sorted structure, one sort given by the elements of $\mathbb{U}$ with isometries f and g and the other sort given by the reals, considered as a ordered abelian group with the distance function d. Fix a countable dense subset D of $\mathbb{U}$. By the Löwenheim-Skolem Theorem (see [5], Theorem 2.3.1), D can be extended to a countable elementary substructure $\mathcal{U}' = (\mathbb{U}', f', g', d', R, +, <)$ where R is a countable ordered abelian group. As an elementary substructure of $\mathcal{U}$, the R-metric space $\mathbb{U}'$ will be isometric to $R\mathbb{U}$. Also g' moves maximally.

In view of Corollary 2.9 we now may apply Theorem 2.7 to $R\mathbb{U}$ to conclude that there are $h'_1, \ldots, h'_8 \in \mathrm{Aut}(R\mathbb{U})$ such that $f \restriction_{R\mathbb{U}} = g'^{h'_1} \cdot \ldots g'^{h'_8}$. Since $R\mathbb{U}$ is dense in $\mathbb{U}$, there are extensions $h_1, \ldots, h_8$ of the $h'_1, \ldots, h'_8$ to isometries of $\mathbb{U}$ and we have $f = g^{h_1} \cdot \ldots g^{h_8}$ on $R\mathbb{U}$ and by density on all of $\mathbb{U}$. $\square$

5 Application to free amalgamation

In order to apply our main theorem to free amalgamation classes, we first prove a lemma in a more general context:

Lemma 5.1. *Suppose that $\mathcal{M}$ is a countable structure with a stationary independence relation. Assume the following additional hypothesis for finite subsets X, A, B of $\mathcal{M}$: If A and B are independent over X and X' is a subset of X with $(A \cup B) \cap X \subset X'$, then A and B are independent over X'.*

Suppose that $g \in \mathrm{Aut}(\mathcal{M})$ and there is no type p over a finite set whose set of realisations in M is infinite, and is fixed pointwise by g. Then there is some $h \in \mathrm{Aut}(\mathcal{M})$ such that the commutator $[g, h]$ moves maximally.

Proof. Let us first note two general facts which do not depend on the additional hypothesis.

1. Any 1-type over X has either exactly one realisation or infinitely many.

   Proof: Let p be a type over X and A a finite non-empty set of realisations. Consider a realisation a of p which is independent from A over X. Then all elements of A have the same type over Xa, which implies that either a does not belong to A or $A = \{a\}$.

2. If $\mathrm{tp}(a/X)$ has infinitely many realisations and a is independent from X' over X, then also $\mathrm{tp}(a/X')$ has infinitely many realisations.

   Proof: Let a_1 and a_2 by two different realisations of $\mathrm{tp}(a/X)$. Choose a realisation $a_1'a_2'$ of $\mathrm{tp}(a_1a_2/X)$ which is independent from X' over X. Then a_1' and a_2' are two different realisations of $\mathrm{tp}(a/X')$.

We build h by a 'back-and-forth' construction as the union of a chain of finite partial automorphisms. It is enough to show the following: Let h' be already defined on the finite set U and let p be a type over the finite set X. Then h' has an extension h such that $[g, h]$ moves p maximally.

If p has only one realisation a, then a is independent over X from every extension of X. So every automorphism moves p maximally. So by (1) we may assume that p has infinitely many realisations. By extending h' if necessary we may also assume that $[g, h']$ is defined on X and that

$$h'^{-1}gh'(X) \subset U. \tag{3}$$

Choose a realisation a of p which is independent from $X' = U \cup g(X) \cup [g, h'](X)$ over X. By (2) $\mathrm{tp}(a/X')$ has infinitely many realisations. So by the assumption on g we can find such a realisation a with $g(a) \neq a$. Put $V = h'(U)$ and let b realise $h'(\mathrm{tp}(a/U))$ in such a way that $b \mathop{\downarrow}_V g^{-1}(V)$. Since $\mathrm{tp}(b/V)$ has infinitely many realisations, we can again assume that $g(b) \neq b$. Extend h' to Ua by setting $h'(a) = b$. Finally realise $h'^{-1}(\mathrm{tp}(g(b)/Vb))$ by c such that

$$c \mathop{\downarrow}_{Ua} g(a)g(X) \tag{4}$$

and extend h' by setting $h'(c) = g(b)$. We then have $[g, h'](a) = g^{-1}h'^{-1}g(b) = g^{-1}(c)$.

Claim: a is independent from $[g, h'](a)$ over $X; [g, h'](X)$.

Proof: We know that $a \mathop{\smile\!\!\!\!\!\!\!\;\,|}_X [g,h'](X)$. So by Lemma 2.4(1) it remains to show that

$$[g,h'](a) \mathop{\downarrow}_{[g,h'](X)} aX. \qquad (5)$$

Since $g(b) \neq b$, c is different from a. Also a does not occur in $g(a)g(X)$. So by the additional hypothesis and (4) we have

$$c \mathop{\downarrow}_{U} g(a)g(X). \qquad (6)$$

$a \downarrow_X U$ implies $b \downarrow_{h'(X)} V$. This together with $b \downarrow_V g^{-1}(V)$ and $h'(X) \subset V$ gives

$$b \mathop{\downarrow}_{h'(X)} g^{-1}(V).$$

Since independence is invariant under automorphisms, application of $h'^{-1}g$ yields

$$c \mathop{\downarrow}_{h'^{-1}gh'(X)} U.$$

From this and (3), (6) we conclude

$$c \mathop{\downarrow}_{h'^{-1}gh'(X)} g(a)g(X).$$

An application of g^{-1} now yields (5). $\square$

Using a result from [4], we obtain the following, sharpening the main theorem there.

Corollary 5.2. *If $\mathcal{M}$ is the Fraïssé limit of a free amalgamation class in a relational language and not an indiscernible set, then if* $\mathrm{Aut}(\mathcal{M})$ *is transitive it is simple. For any nontrivial $g \in \mathrm{Aut}(\mathcal{M})$, every element can be written as a product of at most 16 conjugates of g and g^{-1}.*

Proof. It is easy to see that $\mathcal{M}$ satisfies the additional hypothesis of Lemma 5.1. Furthermore it was proved in [4], Corollary 2.10, that any element of $\mathrm{Aut}(\mathcal{M})$ satisfies the assumption on g. Hence the corollary follows directly from Lemma 5.1 via Corollary 2.10 and Example 2.2 (c). $\square$

A small change in the proof of Lemma 5.1 shows the following.

Lemma 5.3. *Suppose that $\mathcal{M}$ is a countable structure with a stationary independence relation. Assume that g moves* almost maximally *i.e. every* 1*-type over a finite set B has a realisation b which is independent from $g(b)$ over B. Then there is some $h \in \mathrm{Aut}(\mathcal{M})$ such that the commutator $[g,h]$ moves maximally.*

Proof. We can follow the proof of 5.1, but we need not concern ourselves with whether g fixes a or b. Instead we note that we can assume that $g(b)$ is independent from b over $Vg^{-1}(V)$. It follows that $g(b) \mathop{\smile\!\!\!\!\!\!\mid}_V b$, which implies $c \mathop{\smile\!\!\!\!\!\!\mid}_U a$. Now (4) implies (6) by transitivity. $\square$

This now implies:

Corollary 5.4. *Suppose that $\mathcal{M}$ is a countable structure with a stationary independence relation and let $g \in \mathcal{G}$ move* almost maximally*. Then any element of $\mathcal{G}$ is the product of sixteen conjugates of g.*

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