• Spectral structure

(H, A) : spectral structure

$:\Leftrightarrow \begin{cases} \cdot\ H : \text{Hilb. sp.} / \mathbb{C} \\ \cdot\ \mathcal{D}(A) \subset \mathcal{H} : \text{dense subsp.} \\ \quad A : \mathcal{D}(A) \longrightarrow H \ \text{linear,} \\ \quad\quad \text{self-adjoint, } \underline{\text{nonnegative}}. \end{cases}$

$\langle Au, u \rangle \geq 0 \ ({}^{\forall} u \in \mathcal{D}(A))$

<u>Ex.</u> (X, g) : Riemannian mfd.

$$H = L^2(X, \mu_g), \quad A = \Delta_g.$$

Kuwae-Shioya introduced the notion of

$\left\{ \begin{array}{l} \cdot \text{ compact} \\ \cdot \text{ strong} \end{array} \right\}$ conv. of $\{(H_i, A_i)\}_{i=1}^{\infty}$

- H_i : sequence of Hilbert sp.

$$H_i \xrightarrow{i\to\infty} H_\infty \;:\Longleftrightarrow\; \begin{cases} \exists\, \mathcal{C} \subset H_\infty : \text{dense subsp.} \\ \exists\, \Phi_i : \mathcal{C} \to H_i \ \text{linear.} \\ \forall u \in \mathcal{C} \ \lim_{i\to\infty} \|\Phi_i(u)\|_{H_i} = \|u\|_{H_\infty} \end{cases}$$

- Let $u_i \in H_i$, $H_i \xrightarrow{i\to\infty} H_\infty$

$$u_i \to u_\infty \text{ strongly } :\Longleftrightarrow \begin{cases} \exists \{\tilde{u}_\ell\}_{\ell=1}^{\infty} \in \mathcal{C} \text{ s.t. } \tilde{u}_\ell \xrightarrow{\ell\to\infty} u_\infty \in H_\infty \\ \text{and } \lim_{\ell\to\infty} \overline{\lim_{i\to\infty}} \|\Phi_i(\tilde{u}_\ell) - u_i\|_{H_i} = 0 \end{cases}$$

$$u_i \to u_\infty \text{ weakly } :\Longleftrightarrow \forall\, v_i \xrightarrow{\text{str}} v_\infty$$

$$\lim_{i\to\infty} \langle u_i, v_i \rangle_{H_i} = \langle u_\infty, v_\infty \rangle_{H_\infty}$$

Def (H_i, A_i) : sequence of spec. str.

Let $H_i \to H_\infty$. Put $\mathcal{E}_i(u) := \langle A_i u, u \rangle_{H_i}$.

(i) $(H_i, A_i) \to (H_\infty, A_\infty)$ strongly

$$:\Longleftrightarrow \begin{cases} \bullet\ {}^\forall u_i \xrightarrow{wk} u_\infty, \quad \mathcal{E}_\infty(u_\infty) \leq \varliminf_{i\to\infty} \mathcal{E}_i(u_i) \\ \bullet\ {}^\exists u_i \xrightarrow{str} u_\infty \text{ s.t. } \varlimsup_{i\to\infty} \mathcal{E}_i(u_i) \leq \mathcal{E}_\infty(u_\infty) \end{cases}$$

(ii) $(H_i, A_i) \to (H_\infty, A_\infty)$ compactly

$$:\Longleftrightarrow \begin{cases} \text{〃 strongly \&.} \\ \bullet\ \forall \{u_i\}_{i=1}^\infty \text{ with } \sup_{i\in\mathbb{N}} \left(\|u_i\|_{H_i}^2 + \mathcal{E}_i(u_i)\right) < +\infty \\ \quad {}^\exists u_\infty \in H_\infty\ {}^\exists \{u_{i_k}\} \subset \{u_i\} \text{ s.t.} \\ \quad u_{i_k} \xrightarrow{str} u_\infty \end{cases}$$

Thm (Kuwae-Shioya)

$\{(H_i, A_i)\}_{i\in\mathbb{N}\cup\{\infty\}}$: spec. strs

(1). Assume ${}^\forall A_i$ has compact resolvent.

$$\Updownarrow$$

$$\left(\exists \zeta \in \mathbb{C} \text{ s.t. } (\zeta I - A_i)^{-1} \text{ is cpt op.} \right)$$

$$(H_i, A_i) \xrightarrow{\text{cpt.}} (H_\infty, A_\infty)$$

$(i \in \mathbb{N} \cup \{\infty\})$

$$\Rightarrow \exists u_{i,k} \in H_i \ (k \in \mathbb{N}), \ \exists \lambda_{i,k} \in \mathbb{R}_{\geq 0}$$

k-th eigenvalue

s.t.

- $A_i(u_{i,k}) = \lambda_{i,k} u_{i,k}$
- $\{u_{i,k}\}_{k \in \mathbb{N}}$ is an complete orth. system of H_i.
- $\lim_{i \to \infty} \lambda_{i,k} = \lambda_{\infty,k}$
- $\lim_{i \to \infty} u_{i,k} = u_{\infty,k}$ (strongly)

(2) $(H_i, A_i) \to (H_\infty, A_\infty)$ strongly.

$$\Rightarrow \forall \lambda \in \mathrm{Spec}(A_\infty) \ \exists \lambda_i \in \mathrm{Spec}(A_i)$$

$$\text{s.t. } \lim_{i \to \infty} \lambda_i = \lambda.$$

$$\left(\Leftrightarrow \mathrm{Spec}(A_\infty) \subset \lim_{i \to \infty} \mathrm{Spec}(A_i) \right)$$

Ex (strong conv., but not cpt converge.)

Let H, H' : Hilbert spaces.

$$H_i = H \oplus H'. \qquad \underset{\overset{\shortparallel}{\mathbb{C}}}{H_\infty} := H.$$

$$\Phi_i : \mathbb{C} \longrightarrow H_i \quad \rightsquigarrow H_i \to H_\infty.$$
$$x \mapsto (x, 0)$$

$A : \mathcal{D}(A) \to H, \quad A' : \mathcal{D}(A') \to H'$

self-ad. nonneg. op.

$$A_i := A \oplus A' \qquad A_\infty := A.$$

$$\Rightarrow \quad (H_i, A_i) \xrightarrow{\text{str}} (H_\infty, A_\infty)$$

not cpt. if $H' \neq 0$, $A' \neq 0$.

$\left(\because\ \forall x_i \in H' \subset H_i \quad (x_i)_{i=1}^\infty \text{ has no strong limit in } H_\infty. \right)$

The case of Riem. geom.

Let $(X_i, d_{g_i}, \mu_{g_i}) \xrightarrow{mGH} (X_\infty, d_\infty, \mu_\infty)$

Put $H_i = L^2(X_i, \mu_{g_i})$ then we have

$H_i \to H_\infty$ as follows.

- $\mathcal{C} := C_{cpt}(X_\infty) \subset H_\infty$: dense.
- $\Phi_i : \mathcal{C} \to H_i$ is given by the following.

(Let $\phi_i : X_i \to X_\infty$ be an approx. map associate with m-GH conv.)

$\Phi_i(f) := f \circ \phi_i$.

(if $X_i \xrightarrow{pmGH} X_\infty$ then ϕ_i is defined only on $B(p_i, R_i)$,

so we put $\Phi_i(f) = \begin{cases} f \circ \phi_i & \text{on } B(p_i, R_i) \\ 0 & \text{otherwise} \end{cases}$

- If $X_i \xrightarrow{\text{pmGH}} X_\infty$ and $\mathrm{Ric}_{g_i} \geq {}^{\exists}\kappa g_i$

then ${}^{\exists}\Delta_{d_\infty, \mu_\infty}$: Laplacian on $(X_\infty, d_\infty, \mu_\infty)$
(Cheeger-Colding)

Thm (Fukaya) (Cheeger-Colding)
(X_i^N, g_i) : closed Riem. mfd.

- $\left(X_i, d_{g_i}, \frac{\mu_{g_i}}{\mu_{g_i}(X_i)}\right) \xrightarrow[\text{mGH}]{} {}^{\exists}(X_\infty, d_\infty, \mu_\infty)$

- $\mathrm{diam}(X_i) \leq D$, $\underline{|\sec(X_i)| \leq K}$ $\mathrm{Ric}_{g_i} \geq \kappa g_i$.

$\Rightarrow \left(L^2(X_i), \Delta_{g_i}\right) \xrightarrow{i\to\infty} \left(L^2(X_\infty), \Delta_{d_\infty, \mu_\infty}\right)$
$\uparrow$
cpt.

Thm (Kuwae-Shioya)
- $(X_i, d_{g_i}, \mu_{g_i}, p_i) \xrightarrow[\text{pmGH}]{} (X_\infty, d_\infty, \mu_\infty, p_\infty)$

$\mathrm{Ric}_{g_i} \geq \kappa g_i$.

$$\Rightarrow \left(L^2(X_i), \Delta_{g_i}\right) \xrightarrow[\text{strongly}]{} \left(L^2(X_\infty), \Delta_{d_\infty, \mu_\infty}\right)$$

<u>Def.</u> $(\mathcal{S}_i, d_i, \mu_i, p_i)$: pointed metric measure sp.

$(\mathcal{S}_i, d_i, \mu_i) \curvearrowleft S^1$ preserving met & meas.

$$(\mathcal{S}_i, d_i, \mu_i, p_i) \xrightarrow[S^1\text{-pmGH}]{} (\mathcal{S}_\infty, d_\infty, \mu_\infty, p_\infty)$$

$:\Leftrightarrow \begin{cases} \text{pmGH convergence \&} \\ S^1\text{-equivariance of } \phi_i \end{cases}$

<u>Prop. 1.</u> $\left(\mathcal{S}_i^{\curvearrowleft S^1}, d_{\hat{g}_i}, \frac{\mu_{\hat{g}_i}}{C_i}, p_i\right) \xrightarrow[S^1\text{-pmGH}]{} (\mathcal{S}_\infty, \hat{d}_\infty, \hat{\mu}_\infty, p_\infty)$

$\exists \kappa \in \mathbb{R}$ s.t. $Ric_{\hat{g}_i} \geq \kappa \hat{g}_i$.

$$\Rightarrow \left((L^2(\mathcal{S}_i)\otimes\mathbb{C})^{\rho_k}, \Delta_{\hat{g}_i}^{\rho_k}\right) \xrightarrow[\text{strongly}]{} \left((L^2(\mathcal{S}_\infty)\otimes\mathbb{C})^{\rho_k}, \Delta_\infty^{\rho_k}\right)$$

$(\hat{g}_i = \underset{\sim\sim}{\hat{g}_{J_s}})$

<u>Rem</u> If $S = S(L, h)$, $\hat{g} = \hat{g}_J$: connection met. given by A and g_J.

$$\rightsquigarrow \quad \mathrm{Ric}_{\hat{g}_J} = \begin{array}{cl} \begin{pmatrix} \mathrm{Ric}_{g_J} - \frac{g_J}{2} & 0 \\ 0 & \frac{n}{2} \end{pmatrix} & \begin{array}{l} \} \text{ base horizontal} \\ \\ \} \text{ fiber.} \end{array} \end{array}$$

$$T_u S = H_u \oplus \underbrace{T_u(\text{fiber})}_{\cong \mathbb{R}.}$$

$\therefore$ lower bdd of $\mathrm{Ric}_{g_J} \Rightarrow$ " of $\mathrm{Ric}_{\hat{g}_J}$