On NIP and invariant measures

Ehud Hrushovski*
Hebrew University of Jerusalem

Anand Pillay†
University of Leeds

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Abstract

We study forking, Lascar strong types, Keisler measures and definable groups, under an assumption of NIP (not the independence property), continuing aspects of the paper [13]. Among key results are (i) if $p = tp(b/A)$ does not fork over A then the Lascar strong type of b over A coincides with the compact strong type of b over A and any global nonforking extension of p is Borel definable over $bdd(A)$, (ii) analogous statements for Keisler measures and definable groups, including the fact that $G^{000} = G^{00}$ for G definably amenable, (iii) definitions, characterizations and properties of "generically stable" types and groups, (iv) uniqueness of invariant (under the group action) Keisler measures on groups with finitely satisfiable generics, (v) a proof of the compact domination conjecture for (definably compact) commutative groups in o-minimal expansions of real closed fields.

1 Introduction and preliminaries

The general theme of this paper is to find and study stable-like behaviour in theories and definable groups without the independence property. This was a theme in the model-theoretic analysis of algebraically closed valued fields [8], [9], [11]. It was also an aspect of the paper [13], although there the main motivation was to resolve some conjectures about definably compact

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groups in o-minimal structures. In fact a bonus in the current paper is a proof of the compact domination conjecture (formulated in [13]), at least for commutative groups, and various elaborations, which is fairly direct but also uses some of the general machinery we develop.

Both authors have been a bit slow to realize the relevance of Shelah's notion of forking to theories with NIP. We rectify this in the current paper where we will be quite explicit about the role of forking.

Note that a theory T is stable if and only if it is simple and has NIP. In stable theories we have on the one hand the "algebraic" behaviour of nonforking independence, namely symmetry, transitivity, existence of nonforking extensions, as well as local character (any type does not fork over some small set). On the other hand we have (again in stable theories) what can be called broadly "multiplicity theory", the structure and behaviour of nonforking extensions of a type. Included in "multiplicity theory" are alternative characterizations of nonforking, for example a global type p does not fork over a model M iff p is definable over M iff p is finitely satisfiable in M iff p is $Aut(\bar{M}/M)$-invariant (does not split over M) Included also is the finite equivalence relation theorem: a type over an algebraically closed set is stationary, namely has a unique global nonforking extension. In the early texts on stability theory (the original papers and book of Shelah, but also the treatment by Lascar and Poizat), the proofs and development of the algebraic properties of forking were tied up with multiplicity theory (in the form of heirs, coheirs for example). In the work on simple theories, the two strands were distinguished, the algebraic theory being valid in all simple theories, but not the latter. Multiplicity theory did make an appearance in simple theories, but in the (weak) form of the "Independence theorem over a model" and more generally over boundedly closed (rather than algebraically closed) sets. As the validity of the algebraic theory of forking is characteristic of simple theories, it will fail for unstable theories with NIP. One of the points of the current paper is to recover aspects of the multiplicity part of stability theory for theories with NIP. Invariance (rather than stationarity) turns out to be important and already invariant types played a role in the analysis of algebraically closed valued fields. We show for example (in section 2), extending work of Shelah [32] and Adler [1] that assuming NIP if p is a global type and A a small set then p does not fork over A iff p is $Aut(\bar{M}/bdd(A))$-invariant iff p is "Borel definable" over $bdd(A)$. Keisler measures figured a lot in the earlier paper [13] and we entertained the possibility that replacing types by Keisler measures would give a smoother theory and

better chance of recovering stationary-like behaviour (uniqueness of nonforking extensions). In section 4 the results for types are carried over to Keisler measures. This includes the "Borel definability" over $bdd(A)$ of a measure which does not fork over A, the proof of which uses the Vapnik-Chervonenkis theorem. We reduce measures to types in two ways. (a) If the Keisler measure is A-invariant then it corresponds to a Borel measure on the space of $bdd(A)$-invariant types (see 4.6), (b) An invariant Keisler measure is also the "frequency average" of some sequence of invariant types (see 4.8). Though these representations of Keisler measures might seem to make considerations of measures unnecessary, in fact some of our proofs of facts about types go through measures. This point appears in section 5, where among other things we show that definable amenability (the existence of left invariant measures) of a definable group G is equivalent to the existence of generic types in the sense of forking.

In section 3 we study what we call *generically stable types* (and what Shelah has just called stable types), mainly in an NIP context. These are types whose behaviour vis-a-vis nonforking extensions is like that in stable theories, namely a global nonforking extension is both definable and finitely satisfiable. A special case of a generically stable type is a stably dominated type, as introduced in [9], and our results place those of [9] in the appropriate general context. Likewise *generically stable groups*, studied in section 6, generalize the stably dominated groups of [11].

In section 5 we recall the groups with fsg (finitely satisfiable generics) which were introduced in [13], and generalize some results from [13] such as definable amenability, to definable groups with generic types in the sense of forking.

Definably compact groups in a variety of settings (o-minimal, P-minimal, metastable) have either been proved to have or are expected to have fsg. In section 7 we prove the *uniqueness* of translation invariant Keisler measures in fsg groups. We see this as a kind of common generalization of the uniqueness of global translation invariant *types* for stable groups, and of the uniqueness of Haar measure for compact groups, and exemplifies our search for stable-like behaviour at the level of Keisler measures.

It is natural to try to both extend the notion of generic stability from types to measures, as well as to find group-free versions of the fsg property. This is discussed briefly in sections 4 and 7, and the problems will be addressed in a future paper with P. Simon.

Section 8 is devoted to a proof of compact domination for commutative

definably compact groups in o-minimal expansions of real closed fields. In fact we prove a strong version, domination of G by a "semi-o-minimal" compact Lie group G/G^{00}. Direct o-minimal style arguments as well as some of the general theory of invariant types, play a role in the proof. Our proof makes use of a theorem on the existence of definable Skolem functions for "o-minimal subsets of finite-dimensional groups" which is proved in the appendix.

Much inspiration for our work on measures comes from Keisler's work [17] and [18]. But our emphasis differs from Keisler's. Among the main points of Keisler is that if $\phi(x, y)$ is a stable formula, then any ϕ-measure is a "convergent weighted sum" of ϕ-types. (Here a ϕ-measure over M is a finitely additive measure on the Boolean algebra generated by formulas of the form $\phi(x, a)$ for $a \in M$.) So all the properties of nonforking in a stable theory (symmetry, stationarity of types over models,...) pass automatically to measures. Keisler does consider measures in (possibly unstable) theories with NIP. The good class of measures he identifies are so-called "smooth" or "minimally unstable" measures. Loosely (and possibly incorrectly) speaking, a smooth measure over M is a measure over M which has a unique extension to the "unstable part " of $\bar{M}$. He proves that such measures exist. So the only freedom in taking extensions of a smooth measure is with respect to stable formulas, and thus in a sense forking theory for smooth measures essentially reduces to forking theory for measures in stable theories. On the other hand our main focus will be global measures which are invariant over some small set. Only in special cases will these also be smooth.

We will use standard notation. Namely T denotes a complete theory in a language L and we work in a saturated model $\bar{M}$ of T. $x, y, z..$ usually denote finite tuples of variables, A, B, C small subsets of $\bar{M}$ and $M, N, ..$ small elementary submodels of $\bar{M}$. There is no harm in working in $\bar{M}^{eq}$, but sometimes we will assume our theory is one-sorted. A reader would benefit from having some familiarity with stability theory and stable group theory, a reference for which is the first chapter of [29]. However the paper is reasonably self-contained and can on the whole be read independently of [13].

When it is convenient we denote the space of complete types over A in free variable x by $S_x(A)$. By a global (complete) type we mean some $p(x) \in S(\bar{M})$. Recall that a partial type $\Sigma(x)$ (over some set of parameters and closed under finite conjunctions) is said to divide over a set A, if there is

$\phi(x,b) \in \Sigma(x)$ and an A-indiscernible sequence $(b_i : i < \omega)$ of realizations of $tp(b/A)$ such that $\{\phi(x,b_i) : i < \omega\}$ is inconsistent. $\Sigma(x)$ is said to fork over A if Σ implies some finite disjunction of formulas, each of which divides over A. Note that if $p(x)$ is a complete type over some $|A|^+$-saturated model M and $A \subseteq M$ then $p(x)$ divides over A iff $p(x)$ forks over A. Also if A is any (small) set of parameters, and $p(x) \in S(A)$, then p does not fork over A if and only if $p(x)$ has a global extension which does not divide (equivalently does not fork) over A.

A rather more basic notion is splitting: Let $p(x) \in S(B)$ and $A \subseteq B$. We say that p does not split over A if for any L-formula $\phi(x,y)$ and $b, c \in B$, if $tp(b/A) = tp(c/A)$ then $\phi(x,b) \in p$ iff $\phi(x,c) \in p$. This tends to be more meaningful when B is some $|A|^+$-saturated model.

We will be discussing various kinds of strong types, so let us fix notation. First by $Aut(\bar{M}/A)$ we mean the group of automorphisms of $\bar{M}$ which fix A pointwise. Two tuples (of the same length or elements of the same sort) a and b are said to have the same strong type over a set A of parameters, if $E(a,b)$ for each finite A-definable equivalence relation E on the relevant sort (where finite means having only finitely many classes). It is well-known that $stp(a/A) = stp(b/A)$ if and only if a and b have the same type over $acl(A)$ where $acl(-)$ is computed in $\bar{M}^{eq}$. To be consistent with later notation it might be better to call strong types, Shelah strong types or profinite strong types.

We say that a and b have the same compact strong type or KP strong type over A if $E(a,b)$ for each bounded equivalence relation E on the relevant sort which is type-definable over A, that is defined by a possible infinite set of formulas over A. Here bounded means having strictly less than $|\bar{M}|$-many classes which is equivalent to having at most $2^{|T|+|A|}$-many classes. An ($\emptyset$-) hyperimaginary e is by definition the class of a $\emptyset$-type-definable equivalence relation. We define $bdd(A)$ to be the set of hyperimaginaries which have small orbit under $Aut(\bar{M}/A)$. It follows from [19] (see 4.18 there) that a and b have the same KP strong type over A iff they have the same type over $bdd(A)$ (where types over hyperimaginaries are made sense of in [10] for example). In particular if a and b have the same KP strong type over A, then for any A-type definable set X to which a, b belong and bounded A -type-definable equivalence relation E on X, a and b are in the same E-class. All this makes sense for type-definable equivalence relations on sets of infinite tuples, although in [19] we pointed out that a bounded infinitary hyperimaginary is "equivalent" to a sequence of bounded finitary hyperimaginaries.

Finally we say that tuples a and b have the same Lascar strong type over A, if $E(a,b)$ for any bounded equivalence relation E which is invariant under $Aut(\bar{M}/A)$. The relation "equality of Lascar strong type over A" is the transitive closure of the relation $R_A(-,-)$, where $R_A(x,y)$ says: x and y are elements of some infinite A-indiscernible sequence.. Following Lascar we let $Autf(\bar{M}/A)$ denote the group of automorphisms of $\bar{M}$ which fix all Lascar strong types over A. Tuples a, b from $\bar{M}$ (which are possibly infinite) will have the same Lascar strong type over A if and only if they are in the same orbit under $Autf(\bar{M}/A)$.

Note that Lascar strong type refines KP strong type refines strong type refines type (all over A). If $A = M$ is a model, Lascar strong type coincides with type. In a stable theory Lascar strong types over A coincide with strong types over A. This is conjectured to be true in simple theories too, and was proved in the supersimple case. In the general simple case we only know that Lascar strong types coincide with KP strong types. In any case in the current paper we will consider such questions for theories with NIP.

We will also be refereeing to various kinds of "connected components" of a definable (or even type-definable group) G. Suppose A includes the parameters over which G is (type)-defined. Let G^0_A be the intersection of all (relatively) A-definable subgroups of G of finite index. Let G^{00}_A denote the smallest type-definable over A subgroup of G of "bounded" index (equivalently index at most $2^{|L|+|A|}$). Let G^{000}_A denote the smallest subgroup of G of bounded index which is $Aut(\bar{M}/A)$-invariant. If for example G^0_A does not depend on A, but only on G we say that G^0 exists. Likewise for G^{00} and G^{000}.

There is an analogy between definable groups G (and their quotients such as G/G^{00} etc.) and automorphism groups (and their corresponding quotients) which permeates this paper. A basic construction from the first author's thesis produces an automorphism group from a definable group (action). We recall this now, calling it

Construction C.

We start with a theory T which includes a transitive $\emptyset$-definable group action of G on X. Assume that X is stably embedded and that there is some finite subset $\{a_1, .., a_m\}$ of X such that the pointwise stabilizer in G of $\{a_1, .., a_m\}$ is trivial Fix a (saturated) model M of T, and add a new sort X' and a bijection $h : X \to X'$. Call the resulting structure M' and its theory T'. Let M'' be the reduct of M' which has all existing relations on M

as well as all relations on X' and $X \times X'$ which are $\emptyset$-definable in M' and G-invariant (under the action of G induced by h and the action of G on X). Let $T_X = Th(M'')$. Then we have:
Conclusion. In M'' the action of G on X' is by automorphisms, namely for every $\sigma \in G$, $\sigma|X'$ extends to an automorphism of M''. In fact the restriction of $Aut(M'')$ to X' equals the semidirect product of G and the restriction of $Aut(M)$ to X.

Note that after naming $a_1, .., a_m, b_1, .., b_m$ we recover h, and so M' is "internal" to M'': Let $h^{m+1} : X^{m+1} \to X'^{m+1}$ be the bijection induced by h, and let H be the union of the G-conjugates of h^{m+1}. So H is $\emptyset$-definable in M'', and note that $h(x) = y$ iff $(x, a_1, .., a_m, y, b_1, .., b_m) \in H$.

In the case where the action of G on X is regular we have the following equivalent construction. Simply add a new sort X' with a regular action of G on X'. In this structure G^{opp} acts on X' as automorphisms and the full automorphism group of the new structure is the semidirect product of $Aut(M)$ and G.

We will repeatedly use the following characteristic property of theories with NIP.

Fact 1.1. *Suppose T has NIP. Then for any formula $\phi(x, y)$, there is $N < \omega$, such that if $(a_i : i < \omega)$ is an indiscernible sequence, then there does not exist b such that $\neg(\phi(a_i, b) \leftrightarrow \phi(a_{i+1}, b))$ for $i = 0, .., N - 1$.*

In particular if $(a_i : i < \omega)$ is totally indiscernible (or an indiscernible set), then for any b, either $|\{i < \omega :\models \phi(a_i, b)\}| \leq N$ or $|\{i < \omega :\models \neg\phi(a_i, b)\}| \leq N$.

At some point we will, assuming NIP, refer to $Av(I/M)$ where I is some infinite indiscernible sequence (with no last element). It is the complete type over M consisting of formulas with parameters from M which are true on a cofinal subset of I. This makes sense by Fact 1.1.

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2 Forking and Lascar strong types

Forking in NIP theories typically has a different character from forking in simple theories (although the definition, as in the introduction, is the same). In simple theories, forking is associated to a "lowering of dimension". In NIP theories forking can come from just a lowering of order of magnitude within a given dimension. Although dimension is no less important in C-minimal and o-minimal theories than in strongly minimal ones, we do not at the moment know the right NIP based notion that specializes to lowering of dimension in these cases. (Thorn forking is of course a very useful notion but does not apply to the C-minimal case.)

This section builds on work of Poizat [31], Shelah [33] and Adler [1]. Many of our key notions make an explicit or implicit appearance in Chapter 12 of the Poizat reference. For completeness we will begin by restating some of the results by the above mentioned people. The first is a striking characterization of forking in NIP theories from [1] but with roots in [33].

Proposition 2.1. *(Assume NIP) Let $p(x) \in S(\bar{M})$ be a global type and A a (small) set. Then*
(i) p does not fork over A iff p is $Autf(\bar{M}/A)$-invariant, in other words if $p(x)$ is fixed by any automorphism of $\bar{M}$ which fixes all Lascar strong types over A.
(ii) In particular if $A = M$ is a model, then p does not fork over M iff p is invariant under $Aut(\bar{M}/M)$, in other words p does not split over M.

Proof. (i) Right implies left: suppose $\phi(x, y) \in L$ and $(b_i : i < \omega)$ is an A-indiscernible sequence of realizations of $tp(b/A)$ where $\phi(x, b) \in p$. We may assume that $b = b_0$. As $Lstp(b_i/A) = Lstp(b_0/A)$ for all i, $\phi(x, b_i) \in p(x)$ for

all i, so trivially $\{\phi(x, b_i) : i < \omega\}$ is consistent.
Left implies right: Suppose first that b_0, b_1 are the first two members of an A-indiscernible sequence $(b_i : i < \omega)$, and $\phi(x, y) \in L$. We claim that $\phi(x, b_0) \in p$ iff $\phi(x, b_1) \in p$. If not then without loss of generality $\phi(x, b_0) \wedge \neg\phi(x, b_1) \in p$. But note that $((b_i, b_{i+1}) : i = 0, 2, 4, ..)$ is also an A-indiscernible sequence. So as p does not divide over A, $\{\phi(x, b_i) \wedge \neg\phi(x, b_{i+1}) : i = 0, 2, 4, ..\}$ is consistent, but this contradicts NIP (see Fact 1.1.) So our claim is proved. Now if $Lstp(b/A) = Lstp(c/A)$ then we can find $b = b_0, b_1, .., b_n = c$, such that (b_i, b_{i+1}) are the first two members of an A-indiscernible sequence, for each $i = 0, .., n - 1$. So by our claim, $\phi(x, b) \in p$ iff $\phi(x, c) \in p$. This completes the proof of (i).

(ii) is immediate because types over models and Lascar strong strong types over models coincide.

Definition 2.2. *Let $p(x) \in S(\bar{M})$ be a global type.*
(i) We say that p is invariant over the small subset A of $\bar{M}$ if p is $Aut(\bar{M}/A)$ -invariant.
(ii) We say that p is invariant if it is invariant over some small set.

Invariant types were studied by Poizat as "special" types. By Proposition 2.1, if T has NIP then the invariant global types coincide with the global types which do not fork over some small set. If the global type p is A-invariant then we have a kind of defining schema for p, namely for each $\phi(x, y) \in L$ we have some family $D_p\phi$ of complete y-types over A such that for any $b \in \bar{M}$, $\phi(x, b) \in p$ iff $tp(b/A) \in D_p\phi$. So we can apply the schema D_p to not only supersets B of A living in $\bar{M}$ but also to sets $B \supseteq A$ living in a proper elementary extension $\bar{M}'$ of $\bar{M}$. In any case for any such set B, by $p|B$ we mean the complete type over B resulting from applying the schema D_p to B. We will see subsequently that under the NIP hypothesis the defining schema D_p will be "Borel". Given invariant global types $p(x) \in S(\bar{M})$, $q(y) \in (\bar{M})$ we can form the product $p(x) \otimes q(y) \in S_{xy}(\bar{M})$ as follows: Let $\phi(x, y)$ be over $\bar{M}$. We may assume $\phi(x, y)$ to be over small A where both p, q are A-invariant. We put $\phi(x, y) \in p(x) \otimes q(y)$ if for some (any) b realizing $q(y)|A$, $\phi(x, b) \in p(x)$.

Alternatively, if we are willing to consider elements of some $|\bar{M}|^+$-saturated model containing $\bar{M}$, define $p(x) \otimes q(y)$ to be $tp(a, b/\bar{M})$ where b realizes $q(y)$ and a realizes $p|(\bar{M}b)$.

Note that if the global types $p(x)$, $q(y)$ are invariant, then so is $p(x) \otimes q(y)$. We see easily that $\otimes$ is associative. However it need not be commutative.

Namely considering both $p(x) \otimes q(y)$ and $q(y) \otimes p(x)$ as elements of $S_{xy}(\bar{M})$, they may not be equal.

For an invariant global type $p(x)$, and disjoint copies $x_1, .., x_n$ of the variable x we define $p^{(n)}(x_1, .., x_n)$ inductively by: $p_1(x_1) = p(x_1)$ and $p_n(x_1, .., x_n) = p(x_n) \otimes p_{n-1}(x_1, .., x_{n-1})$. We let $p^{(\omega)}(x_1, x_2, ...)$ be the union of the $p_n(x_1, .., x_n)$ which will be a complete infinitary type over $\bar{M}$.

Assuming that $p(x) \in S(\bar{M})$ is A-invariant, then by a *Morley sequence* in p over A, we mean a realization $(a_1, a_2, a_3, ...)$ in $\bar{M}$ of $p^{(\omega)}|A$.

Lemma 2.3. *Let $p(x) \in S(\bar{M})$ be invariant. Then*
(i) Any realization $(b_1, b_2, ...)$ of $p^{(\omega)}$ (in an elementary extension of $\bar{M}$) is an indiscernible sequence over $\bar{M}$.
(ii) Suppose $A \subset \bar{M}$ is small and p is A-invariant. If $a_1, a_2, ...$ from $\bar{M}$ are such that a_{n+1} realizes $p|(Aa_1, .., a_n)$, then $(a_1, a_2, ...)$ is a Morley sequence in p over A. In particular $tp(a_1, a_2, .../A)$ depends only on p and A.

Proof. Straightforward and left to the reader.

Remark 2.4. *(Assume NIP) More generally we can define a Morley sequence of $p \in S(\bar{M})$ over A, assuming just that p does not fork over A, to be a realization in $\bar{M}$ of $Lstp(b_1, b_2, .../A)$ where $(b_1, b_2, ..)$ realizes $p^{(\omega)}$ (in a model containing $\bar{M}$). This is consistent with the previous definition.*

Lemma 2.5. *(Assume NIP) (i) Suppose $p(x)$, $q(x)$ are A-invariant global types. Then $p = q$ iff $p^{(\omega)}|A = q^{(\omega)}|A$ iff for all n and realization e of $p^{(n)}|A$, $p|Ae = q|Ae$.*
(ii) Suppose $Q(x_0, x_1, ...)$ is the type over A of some A-indiscernible sequence. Then $Q = p^{(\omega)}|A$ for some A-invariant global type $p(x)$ if and only if whenever I_j for $j \in J$ are realizations of Q then there is an element c such that (I_j, c) is A-indiscernible for all $j \in J$.

Proof. (i) It suffices to prove that if $p|Ae = q|Ae$ for any realization e of any $p^{(n)}|A$, then $p = q$. Supposing for a contradiction that $p \neq q$ there is $\phi(x, b) \in p$, $\neg\phi(x, b) \in q$. Let $a_1, a_2, ...$ in $\bar{M}$ be such that a_i realizes $p|(Aa_1..a_{i-1}b)$ for i odd, and a_i realizes $q|(Aa_1..a_{i-1}b)$ for i even. Our assumption, together with Lemma 2.3(ii), implies that $(a_1, a_2, ..)$ is a Morley sequence in p over A, hence by 2.3(i) indiscernible over A. But $\phi(a_i, b)$ holds iff i is odd, contradicting Fact 1.1.
(ii) Left implies right is clear and does not require NIP. (Let c realize

$p|(A \cup \cup_j I_j)$.) For the other direction, assume Q has the given property. Define the global type p by: $\phi(x,b) \in p$ iff any realization I of Q extends to an indiscernible sequence I' such that $\phi(x,b)$ is eventually true on I'. Then NIP and our assumptions on Q yield that p is consistent, complete and A-invariant and that Q is the type over A of its Morley sequence.

We continue with some newer material. We first give a rather better and more general result on "Borel definability" than that in [13]. Given a (small) subset A of $\bar{M}$, by a closed set over A we mean the set of realizations in $\bar{M}$ of a partial type over A. An open set over A is the complement (in the relevant ambient sort) of a closed set over A. From these we build in the usual way the Borel sets over A. Alternatively these correspond to the Borel subsets of the relevant Stone space of complete types over A. A global type $p(x)$ will be called "Borel definable over A" if for any L-formula $\phi(x,y)$, the set of b in $\bar{M}$ such that $\phi(x,b) \in p(x)$ is a Borel set over A. So if $p(x)$ is definable over A in the usual sense then p will be Borel definable over A and if p is Borel definable over A then p is A-invariant. In fact, we will be proving *strong* Borel definability over A, in the sense that for any $\phi(x,y) \in L$, the set of b such that $\phi(x,b) \in p$ is a finite Boolean combination of closed sets over A.

Proposition 2.6. *(Assume NIP) Suppose that $p(x) \in S(\bar{M})$ is a global type which is A-invariant. Then p is strongly Borel definable over A.*

Proof. Let $\phi(x,y) \in L$. Let $N < \omega$ be as given for $\phi(x,y)$ by Fact 1.1.
Claim. For any b, $\phi(x,b) \in p$ if and only if for some $n \leq N$ [there is $(a_1,..,a_n)$ realising $p^{(n)}|A$ such that
$\models \phi(a_i,b) \leftrightarrow \neg\phi(a_{i+1},b)$ for $i = 1,..,n-1$, $(*)_n$
and $\models \phi(a_n,b)$,
but there is no $(a_1,..,a_{n+1})$ such that $(*)_{n+1}$ holds].
Proof of claim. Suppose $\phi(x,b) \in p$. By Fact 1.1 choose any realization $(c_i : i < \omega)$ of $p^{(\omega)}|A$ with a maximal finite alternation (at most N_ϕ) of truth values of $\phi(c_i,b)$ for $i < \omega$. Hence, eventually $\phi(c_i,b)$ holds: for if not, let c_ω realize $p|(A \cup \{c_i : i < \omega\} \cup \{b\})$, and we contradict maximality.

The converse holds by the above proof applied to $\neg\phi(x,b)$. So the claim is proved and clearly yields a strongly Borel definition of the set of b such that $\phi(x,b) \in p$.

Remark 2.7. *(i) Define an A-invariant global type $p(x)$ to have NIP if its Morley sequence over A, $(b_i : i < \omega)$ (which has a unique type over A), has*

the property that for every $\phi(x,y) \in L$ there is $n_\phi < \omega$ such that for any c, there at most n_ϕ alternations of truth values of $\phi(b_i, c)$. Then Proposition 2.6 goes through for p.
(ii) Proposition 2.6 also holds when A is a set of hyperimaginaries, such as $bdd(B)$ for some set B of imaginaries.

We now consider (assuming NIP still) global types which do not fork over A, and Lascar strong types over A.

Remark 2.8. *(Assume NIP) Let p, q be global types which do not fork over A. Then $p \otimes q$ does not fork over A. In particular $p^{(n)}$ and $p^{(\omega)}$ do not fork over A.*

Proof. This follows from the well-known fact (valid for any theory T) that if $tp(a/B)$ does not fork over A and $tp(b/Ba)$ does not fork over Aa then $tp(a,b/B)$ does not fork over A. We will give a quick proof of this fact for completeness:
First, using the hypotheses, find a saturated model M containing B such that $tp(a/M)$ does not fork over A, and $tp(b/Ma)$ does not fork over Aa. It is enough to prove:
Claim. $tp(ab/M)$ does not divide over A.
Proof of Claim. Let $c_0 \in M$, and $q(x,y,c_0) = tp(ab/Ac_0)$. Let $(c_i : i < \omega)$ be an A-indiscernible sequence. We must show that $\cup\{q(x,y,c_i) : i < \omega\}$ is consistent. Let $p(x,c_0) = tp(a/Ac_0)$. As $tp(a/M)$ does not fork over A, $\cup\{p(x,c_i) : i < \omega\}$ is consistent, and we may assume (after applying an automorphism which fixes pointwise Aac_0) that $(c_i : i < \omega)$ is Aa-indiscernible. But now, as $tp(b/Ma)$ does not fork over Aa, we may find b' realizing $\cup\{q(a,y,c_i) : i < \omega\}$. So then (a,b') realizes $\cup\{q(x,y,c_i) : i < \omega\}$, and we are finished.

Lemma 2.9. *(Assume NIP) Suppose that $p(x)$ is a global type which does not fork over A. Let c, d realize $p|A$. Then $Lstp(c/A) = Lstp(d/A)$ iff there is an (infinite) sequence a such that both (c,a) and (d,a) realize $p^{(\omega)}|A$.*

Proof. Right implies left is immediate (for, as remarked earlier, elements of an infinite A-indiscernible sequence have the same Lascar strong type over A).
Left to right: Note first that for some $\sigma \in Aut(\bar{M}/A)$, $\sigma(p)(x)$ implies $Lstp(c/A)$. As $\sigma(p)^{(\omega)}|A = p^{(\omega)}|A$, we may assume that already $p(x)$ implies $Lstp(c/A)$. Let $(a_0, a_1, a_2, ...)$ realize $p^{(\omega)}$ (in a bigger saturated model). So

$(a_1, a_2, ...)$ also realizes $p^{(\omega)}$ and does not fork over A. Hence by 2.1 whether or not $\phi(\bar{x}, c)$ is in $p^{(\omega)}$ depends on $Lstp(c/A)$. Hence $tp(c, a_1, a_2, .../A) = tp(a_0, a_1, .../A) = tp(d, a_1, a_2, .../A) = p^{(\omega)}|A$, as required.

Lemma 2.9 says that on realizations of $p|A$, having the same Lascar strong type over A is a type-definable (over A) equivalence relation, hence by our discussion of Lascar strong types in the introduction we see:

Corollary 2.10. *(Assume NIP). (i) Suppose $p(x) \in S(A)$ does not fork over A. Then on realisations of p, Lascar strong type over A coincides with compact (KP) strong type over A.*
(ii) Suppose that T is 1-sorted (namely T^{eq} is the eq of a 1-sorted theory). Suppose that any complete 1-type over any set A does not fork over A. Then over any set A, Lascar strong types coincide with compact strong types, hence T is "G-compact" over any set of parameters.

Proof. (ii) The assumption, together with the discussion in the proof of 2.8, implies that any complete type over any set A does not fork over A. So we can apply (i).

We can now strengthen Proposition 2.1.

Proposition 2.11. *(Assume NIP) Suppose that $p(x)$ is a global type. Then p does not fork over A if and only if p is $bdd(A)$-invariant.*

Proof. Right to left is clear and does not use the NIP assumption. For left to right, assume p does not fork over A. Let σ be an automorphism of $\bar{M}$ fixing $bdd(A)$ pointwise, and we have to show that $\sigma(p) = p$. By Remark 2.8 and Corollary 2.10(i) we have
Claim I. For any realization $\bar{a}$ of $p^{(\omega)}|A$, $Lstp(\bar{a}/A) = Lstp(\sigma(\bar{a})/A)$.

By Claim I and Proposition 2.1(i) it follows that
Claim II. For any realization $\bar{a}$ of $p^{(\omega)}|A$, $p|A\bar{a} = \sigma(p)|A\bar{a}$.

Now let M be a small model containing A, and let $\bar{a}$ realize $p^{(\omega)}|M$.
Claim III. For any c realizing either $p|A\bar{a}$ or $\sigma(p)|A\bar{a}$, $\bar{a}c$ is an A-indiscernible sequence.
Proof. By Claim II, it is enough to prove that $\bar{a}c$ is A-indiscernible for c realizing $p|A\bar{a}$. Note that p does not fork over M hence by 2.1(i) is M-invariant. So by Lemma 2.3, if c realizes $p|M\bar{a}$ then $\bar{a}c$ is M-indiscernible,

hence also A-indiscernible. But c realizes $p|A\bar{a}$ and the latter is a complete type over $A\bar{a}$. So for *any* c realizing $p|A\bar{a}$, $\bar{a}c$ is an A-indiscernible sequence.

Claim IV. Let $\bar{a}'$ be an indiscernible sequence (of realizations of $p|A$) extending $\bar{a}$. Then for c realizing either $p|A\bar{a}'$ or $\sigma(p)|A\bar{a}'$, $\bar{a}'c$ is A-indiscernible.
Proof. This can be seen in various ways. For example it can be deduced from Claim III, using the fact that each of p, $\sigma(p)$ is invariant under $Autf(\bar{M}/A)$, and the fact that any two increasing n-tuples from $\bar{a}'$ have the same Lascar strong type over A.

Now suppose for a contradiction that $p \neq \sigma(p)$. So for some $\psi(x,y) \in L$ and $e \in \bar{M}$, $\psi(x,e) \in p$ and $\neg\psi(x,e) \in \sigma(p)$. Let c_i realize $p|A\bar{a}c_0...c_{i-1}e$ for i even and realize $\sigma(p)|A\bar{a}c_0...c_{i-1}e$ for i odd. By Claim IV, $(\bar{a}, c_i)_i$ is A indiscernible. But $\models \psi(c_i, e)$ iff i is even, contradicting NIP. This concludes the proof of Proposition 2.11.

Finally we will give an analogue of 2.10 (ii) for strong types (which is closely related to material in [16] and [15]). First a preparatory lemma.

Lemma 2.12. *(Assume NIP). Suppose A is algebraically closed, $tp(a/A)$ has a global A-invariant extension, and $e \in acl(Aa)$. Then $tp(ae/A)$ has a global A-invariant extension.*

Proof. Let $p(x) = tp(a/A)$ and fix some global A-invariant (so nonforking) extension $p'(x)$ of $p(x)$. Let $q(x,y) = tp(ae/A)$, and let $q'(x,y)$ be any global extension of $q(x,y)$ whose restriction to x is $p'(x)$. We will show that $q'(x,y)$ is A-invariant. Let $\delta(x,y)$ be a formula over A such that $\delta(a,y)$ isolates $tp(e/Aa)$.
Claim I. q' does not fork over A.
Proof. Let $\phi(x,y,b) \in q'(x,y)$ and let $(b_i : i < \omega)$ be A-indiscernible with $b_0 = b$. We may assume that $\models \phi(x,y,b) \rightarrow \delta(x,y)$. As p' does not fork over A, there is a' realizing $\{\exists y \phi(x,y,b_i) : i < \omega\}$. For each i, let e_i realize $\phi(a', y, b_i)$. As there are finitely many possible choices for the e_i, there is an infinite subset I of ω such that $e_i = e_j$ for $i, j \in I$. So $\{\phi(x,y,b_i) : i \in I\}$ is consistent, which is enough.

As p' is A-invariant so is $p'^{(n)}$ for any n, hence:
Claim II. For all n, $p'^{(n)}|A$ implies a "complete" Lascar strong type over A.

Claim III. For any n, $q'^{(n)}|A$ implies a "complete" Lascar strong type over A.

Proof. As $q'^{(n)}$ does not fork over A, by Lemma 2.9 the relation of having the same Lascar strong type over A, on realizations of $q'^{(n)}|A$ is type-definable over A. But by Claim II clearly there are only finitely many Lascar strong types over A extending $q'^{(n)}|A$: If $((a_1, e_1), .., (a_n, e_n))$, and $((a'_1, e'_1), .., (a'_n, e'_n))$ are realizations of $q'^{(n)}|A$ with distinct Lascar strong types, we may by Claim II assume that $a_i = a'_i$ for $i = 1, .., n$, so there are only finitely many possibilities for the sequence $(e_i)_i$. Hence equality of Lascar strong type on realizations of $q'^{(n)}|A$ is the restriction to $q'^{(n)}|A$ of a finite A-definable equivalence relation. As A is algebraically closed, there is just one Lascar strong type over A extending $q'^{(n)}$, proving Claim III.

The A-invariance of q' follows from Claim III, as in the proof of 2.11.

Proposition 2.13. *(Assume NIP) Let T be 1-sorted and work in T^{eq}. The following are equivalent:*
(i) For any algebraically closed set A and complete 1-type p over A in the home sort, p has a global A-invariant extension.
(iii) For any complete type p over any algebraically closed set A, p has a global A-invariant extension.
(iii) For any A (a) any $p(x) \in S(A)$ does not fork over A, and (b) Lascar strong types over A coincide with strong types over A.

Proof. (i) implies (ii). It is enough to prove that for any n any complete n-type (in the home sort) over any algebraically closed set A has a global A-invariant extension. We prove it by induction on n. Suppose true for n. Let $p(x_1, .., x_n, x_{n+1}) = tp(a_1, .., a_n, a_{n+1}/A)$ with the a_i's elements of the home sort, and A algebraically closed. Let (the infinite tuple) e be an enumeration of $acl(Aa_1, .., a_n)$. By the induction hypothesis and 2.12, $tp(e/A)$ has a global A-invariant extension, realized by e' say (in a bigger saturarated model $\bar{M}'$). Let $a'_1, .., a'_n$ denote the copies of the a_i in e'. By the hypothesis over the algebraically closed base e', there is a'_{n+1} (in the bigger model) with $tp(e', a'_{n+1}/A) = tp(e, a_{n+1}/A)$ such that $tp(a'_{n+1}/\bar{M}e')$ does not split over e'. It follows easily that $tp(e', a'_{n+1}/\bar{M})$ is A-invariant, hence also $tp(a'_1, .., a'_n, a'_{n+1}/\bar{M})$ is the A-invariant extension of p we are looking for.
(ii) implies (iii), and (iii) implies (i) are clear.

Corollary 2.14. *If T is o-minimal or C-minimal then (i) to (iii) of Proposition 2.13 hold.*

Proof. Condition (i) holds in C-minimal theories through the existence of

"generic" 1-types (see [9]). In the o-minimal case (i) holds without even the condition that A is algebraically closed.

3 Generically stable types

Here we make a systematic study of what Shelah has called "stable types" in [32]. We discuss our choice of language a bit later. We begin with some preliminary remarks.

Lemma 3.1. *(Assume NIP) Let $p(x)$ be a global type which does not fork over a small set A.*
(i) Suppose p is definable. Then p is definable over $acl(A)$. In particular p is $acl(A)$-invariant.
(ii) Suppose that p is finitely satisfiable in some small model. Then p is finitely satisfiable in any model which contains A.

Proof. (i) By 2.11 p is $bdd(A)$-invariant. So if p is definable, then for any $\phi(x,y) \in L$ the ϕ-definition of p is over $bdd(A)$ hence over $acl(A)$.
(ii) Let M_1 be a small model in which p is finitely satisfiable. Let M be an arbitrary (small) model containing A. Let $\phi(x,c) \in p$. Let M_1' realize a coheir of $tp(M_1/M)$ over Mc. As p is M-invariant, p is finitely satisfiable in M_1' so there is $a' \in M_1'$ such that $\models \phi(a,c)$. So there is $a' \in M$ such that $\models \phi(a,c)$.

Among our main results is:

Proposition 3.2. *(Assume NIP) Let $p(x) \in S(\bar{M})$, and let A such that p is A-invariant.*
Consider the conditions (i), (ii), (iii), (iv) and (v) below.
(i) $p(x)$ is definable (hence A-definable), and also finitely satisfiable in some/any small model containing A.
(ii) $p^{(\omega)}|A$ is totally indiscernible. That is, if $(a_i : i < \omega)$ is a Morley sequence in p over A, then $(a_i : i < \omega)$ is an indiscernible set (not just sequence) over A.
(iii) For any formula $\phi(x,y)$ there is N such that for any Morley sequence $(a_i : i < \omega)$ of p over A, and c, $\phi(x,c) \in p$ if and only if
$\models \vee_{w \subset 2N, |w|=N} \wedge_{i \in w} \phi(a_i, c)$.
(iv) For all small $B \supseteq A$, p is the unique global nonforking extension of $p|B$,
(v) For all n, $p^{(n)}$ is the unique global nonforking extension of $p^{(n)}|A$.

Then (i), (ii), (iii) and (iv) are equivalent, and imply (v). Moreover if A has the additional property that every complete type over A does not fork over A, then (v) implies each of (i),(ii),(iii),(iv).

Proof. (i) implies (ii): Fix a small model $M \supseteq A$ such that p is finitely satisfiable in M (and of course definable over M), and there is no harm in proving (ii) with M in place of A. Let $(a_i : i < \omega)$ be a Morley sequence in p over M. We will show
(*) for any n and $i \leq n$, a_i realizes the restriction of p' to $Ma_0..a_{i-1}a_{i+1}...a_n$.
Note that (*) will be enough to show by induction that for any n and permutation π of $\{0, .., n\}$, $tp(a_0, .., a_n/M) = tp(a_{\pi(0)}, .., .a_{\pi(n)}/M)$ which, using indiscernibility of the *sequence* $(a_i : i < \omega)$ will prove its total indiscernibility.
So let us prove (*). Note that $tp(a_{i+1}, .., a_n/Ma_0, .., a_i)$ is finitely satisfiable in M. As $tp(a_i/a_0, .., a_{i-1}, M)$ is definable over M, it follows that $tp(a_0..a_{i-1}a_{i+1}..a_n/Ma_i)$ is finitely satisfiable in M, whence $tp(a_i/Ma_0..a_{i-1}a_{i+1}..a_n)$ is an heir of $p|M$, so (as p is definable), realizes $p|(Ma_0..a_{i-1}a_{i+1}..a_n)$. This proves (*).

(ii) implies (iii): This is by the "in particular" clause of Fact 1.1.

(iii) implies (i). Clearly p is definable. But it also follows from (iii) that p is finitely satisfiable in any model M containing A. For suppose $\phi(x, c) \in p$. Let I be a Morley sequence in p over A such that $tp(I/Mc)$ is finitely satisfiable in M. By (iii) $\phi(a, c)$ for some $a \in I$ hence $\phi(a, c)$ for some $a \in M$.

(ii) implies (v). As (ii) for p clearly implies (ii) for $p^{(n)}$, it suffices to prove (v) for $n = 1$, namely that p is the unique global nonforking extension of $p|A$.

Let us first note that because p is A-invariant $p|A$ implies $p|bdd(A)$, and thus, by 2.11, *any* global nonforking extension of $p|A$ is A-invariant.

Now let q be an arbitrary global nonforking extension of $p|A$. Let $I = (a_i : i < \omega)$ be a Morley sequence in p over A. We will prove that I realizes $q^{(\omega)}|A$ which will be enough, by Lemma 2.5, to conclude that $p = q$. So let b realize $q|(A \cup I)$. We prove inductively that
(**) $tp(a_0, .., a_n, b/A) = tp(a_0, .., a_n, a_{n+1}/A)$ for all n.
Note that of course $tp(a_0/A) = tp(b/A) = p|A$ which is in a sense the prebase step.
So assume (**) is true for $n-1$. Suppose that $\models \phi(a_0, .., a_n, b)$. As q does not fork over A we have, by 2.1 and indiscernibility of I that $\models \phi(a_0, .., a_{n-1}, a_i, b)$ for all $i \geq n$. By (iii) (for p) we see that $\phi(a_0, .., a_{n-1}, x, b) \in p(x)$. By the

induction hypothesis and the A-invariance of p, we conclude that $\phi(a_0, .., a_{n-1}, x, a_n) \in p$, so $\models \phi(a_0, .., a_{n-1}, a_{n+1}, a_n)$. Finally total indiscernibility of I yields that $\models \phi(a_0, .., a_{n-1}, a_n, a_{n+1})$ as required.

As condition (i) is preserved after replacing A by any $B \supseteq A$, it follows from what we have proved up to now that each of (i),(ii), (iii) implies (iv).

(iv) implies (iii). Let $I = (a_i : i < \omega)$ be any Morley sequence in p over A. Note that $Av(I/\bar{M})$ is an $A \cup I$-invariant extension of $p|AI$ hence equals p. It follows easily that for each $\phi(x, y)$ there is N such that for any c either $|\{i < \omega :\models \phi(a_i, c)\}| < N$ or $|\{i < \omega :\models \neg\phi(a_i, c)\}| < N$. So we obtain (iii).

Now assume the additional hypothesis on A, and we prove
(v) implies (ii). With some abuse of notation, let $p^{(\omega^*)}(x_0, x_1, ..)$ denote $tp(a_0, a_1, a_2...)/\bar{M})$ where for each n, $(a_{n-1}, a_{n-2}, .., a_0)$ realizes $p^{(n)}$. Let $Q(x_i)_{i\in\omega}$ be the restriction of $p^{(\omega^*)}$ to A. So (iv) implies that
(***) $p^{(\omega^*)}$ is the unique global nonforking extension of Q.
Note that if a realizes $p|A$ and $(a_i)_i$ realizes the restriction of $p^{(\omega^*)}$ to Aa then $(a_0, a_1, ...a_n, ..., a)$ is A-indiscernible. We claim that Q satisfies the right hand side condition of 2.5 (ii), namely whenever I_1, I_2 are realizations of Q then there is c such that (I_j, c) is indiscernible, for $j = 1, 2$. For, by our hypothesis on A, let I_1', I_2' realize a global nonforking extension of $tp(I_1, I_2/A)$. By (***), each of I_1', I_2' realizes $p^{(\omega^*)}$. So choosing $c \in \bar{M}$ realizing $p|A$, realizing in $\bar{M}$ the restriction of $tp(I_1', I_2'/\bar{M})$ to Ac, and using an automorphism, gives the claim. By 2.5 (ii), $Q = q^{(\omega)}|A$ for some A-invariant global type q, which note must extend $p|A$. Hence, by (iv), $q = p$. So if $I = (a_0, a_1, ...)$ realizes Q and a realizes $q|AI$ then a also realizes $p|AI$. Hence both (a, I) and (I, a) are indiscernible sequences over A, which easily implies that I is an indiscernible set over A, giving (ii). The proof is complete.

Remark 3.3. *(i) Assuming NIP, we will call a global type $p(x)$* generically stable *if it is both definable and finitely satisfiable in some small model, namely p satisfies 3.2(i) for some A. We may also want to talk about generically stable types without a NIP assumption, in which case we will mean a global p such that for some A (i), (ii), (iii) and (iv) of 3.2 are satisfied. We leave the reader to study the implications between (i)-(iv) in the absence of NIP.*
(ii) (Assume NIP.) Suppose the global type p is generically stable, and p does not fork over B. Then by 3.1, p satisfies 3.2(i) with $A = acl(B)$. Hence

by Proposition 3.2, we recover the finite equivalence relation theorem: any two global nonforking extensions of $p|B$ are distinguished by some finite B-definable equivalence relation.
(iii) Proposition 3.2 goes through assuming only that p has NIP (as in Remark 2.7).
(iv) In $ACVF$, for any set A, any complete type over A has a global nonforking extension. (See [9] or Proposition 2.13.) Hence (i)-(v) of 3.2 are equivalent in $ACVF$, for any A.
(v) In 3.2(iv) it is not enough to require just that $p|A$ has a unique global nonforking extension.

Explanation of (v). So we give an example of a NIP theory and a type $p(x) \in S(A)$ with a unique global A-invariant extension which is *not* definable. We consider the basic C-minimal theory consisting of a dense linear ordering $(I, <)$ with greatest element ∞, and another sort on which there are equivalence relations E_i indexed (uniformly) by $i \in I$ and with E_j infinitely refining E_i if $i < j$ (plus some other axioms, see [12]). By a definable ball we mean an E_i-class for some E_i. By a type-definable ball we mean a possibly infinite intersection of definable balls. We can produce a model M and a type-definable over M ball B such that B contains no proper M-definable ball. Let the global type $p_B(x) \in S(\bar{M})$ be the "generic type" of B, namely p says that $x \in B$ and x is not in any proper definable sub-ball of B. Then p is the unique M-invariant extension of $p|M$ but is not definable.

In the NIP context, our generically stable types coincide with what Shelah [32] calls *stable* types. However there is already another meaning for a complete or even partial type $\Sigma(x)$ to be stable. It is that *any* extension of $\Sigma(x)$ to a complete global type is definable (over some set, not necessarily the domain of Σ). This notion is also sometimes called "stable, stably embedded" (although mainly in the case where the partial type is a single formula). One family of examples of generically stable types come through stable domination in the sense of [9]. Recall that $q(x) \in S(A)$ (A algebraically closed) is said to be stably dominated if there is stable partial type $\Sigma(x)$ over A (stable in the strong sense mentioned above), and an A-definable function f from the set of realizations of p to the realizations of Σ such that, if $q \in S(A)$ is $f(p)$, and a realizes p, then whenever $f(a)$ is independent from B over A then $tp(a/A, f(a))$ has a unique extension over $Bf(a)$. A stably dominated type is generically stable, as is easily verifiable. Generically stable

but unstable types occur in algebraically closed valued fields through stable domination, namely via the stable part of the structure which is essentially the residue field. In "pure" unstable NIP theories such as o-minimal and weakly o-minimal theories, or p-adically closed fields there are no (nonalgebraic) generically stable types. On the other hand in simple theories any stationary type (type with a unique nonforking extension) is easily seen to be generically stable.

We finish this section with some remarks on invariant types and symmetry.

Lemma 3.4. *Suppose $p(x), q(y)$ are global types such that p is finitely satisfiable in some small model, and q is definable. (So both p and q are invariant.) Then $p_x \otimes q_y = q_y \otimes p_x$.*

Proof. Let M be a small model such that q is definable over M and p is finitely satisfiable in M. Let (a, b) realize $(p(x) \otimes q(y))|M$, namely b realizes $q|M$ and a realizes $p|Mb$. We want to show that (a, b) realizes $(q(y) \otimes p(x))|M$. Suppose not. Then there is a formula $\phi(a, y) \in q$ such that $\models \neg\phi(a, b)$. Let $\psi(x)$ (a formula over M) be the $\phi(x, y)$-definition for $q(y)$. So $\models \neg\psi(a)$. As $tp(a/Mb)$ is finitely satisfiable in M, there is $a' \in M$ such that $\models \psi(a') \wedge \neg\phi(a', b)$, which is a contradiction as $tp(b/M) = q|M$. (The reader should note that this is just a restatement of uniqueness of heirs for definable types.)

Finally we return to generically stable types:

Proposition 3.5. *Suppose T has NIP, and that $p(x)$, $q(y)$ are global invariant types such that $p(x)$ is generically stable. Then $p_x \otimes q_y = q_y \otimes p_x$*

Proof. Suppose $\phi(x, y) \in L_{\bar{M}}$ and $\phi(x, y) \in p(x) \otimes q(y)$. Let M be a small model such that each of p and q do not fork over M and ϕ is over M. By assumption, p is definable over M. Moreover
(*) if $\psi(y)$ is the $\phi(x, y)$-definition of p, then for any realisation c of ψ, and any Morley sequence $(a_i : i < \omega)$ of p over M, $\models \psi(a_i, c)$ for all but finitely many $i < \omega$.

From our assumption that $\phi(x, y) \in p(x) \otimes q(y)$, it follows that $\psi(y) \in q$. Now suppose for a contradiction that $\neg\phi(x, y) \in q(y) \otimes p(x)$. Let $(a_i : i < \omega)$ be a Morley sequence in p over M. So then $\neg\phi(a_i, y) \in q$ for all i, so there is b realizing $q|(M \cup \{a_i : i < \omega\})$ such that $\models \neg\phi(a_i, b)$ for all i. As b realizes $\psi(y)$ this contradicts (*), and completes the proof.

4 Measures and forking

In this section we generalize some of the results on types in the previous sections to Keisler measures.

Recall that a Keisler measure μ on a sort S over a set of parameters A, is a finitely additive probability measure on A-definable subsets of S (or on formulas over A with free variable in sort S), namely $\mu(X) \in [0,1]$ for all A-definable X, $\mu(S) = 1$ and the measure of the union of two disjoint A-definable sets X and Y is the sum of the measures of X and of Y. A complete type over A (in sort S) is a special case of a measure. We sometimes write $\mu(x)$ or μ_x to mean that the measure is on the sort ranged over by the variable x. A Keisler measure over $\bar{M}$ is called a global (Keisler) measure.

Let us emphasize that a Keisler measure μ_x over A is the same thing as a regular Borel probability measure on the compact space $S_x(A)$. Regularity means that for any Borel subset B of $S(A)$, and $\epsilon > 0$, there are closed C and open U such that $C \subseteq B \subseteq U$ and $\mu(U \setminus C) < \epsilon$. Note that μ_x defines a finitely additive probability measure (still called μ) on the algebra of clopens of $S(A)$. Theorem 1.2 of [17] extends μ to a Borel probability measure β on $S_x(A)$ using the Loeb measure construction in a mild way. And Lemma 1.3(i) of [17] says that this β is regular. On the other hand a Borel probability measure β on $S_x(A)$ gives, by restriction to the clopens, a Keisler measure μ_x over A. Moreover if β is regular then for any closed set C, $\mu(C)$ will be the infimum of the $\mu(C')$ where C' is a clopen containing C. Hence β is determined by μ.

An important, and even characteristic, fact about NIP theories is that for any global Keisler measure μ_x there are only boundedly many definable sets up to μ_x-equivalence. (See Corollary 3.4 of [13].)

As in [13] notions relating to types generalize naturally to measures. There we discussed the notions of a measure being definable, and of being finitely satisfiable: For example if μ is a global Keisler measure then μ is definable over A, if for each closed subset of $[0,1]$ and L-formula $\phi(x,y)$, $\{b \in \bar{M} : \mu(\phi(x,b)) \in C\}$ is type-definable over A. We say μ is finitely satisfiable in M if every formula with positive measure is realized by a tuple from M. But we also have the notion of forking:

Definition 4.1. *Let μ be a Keisler measure over B, and $A \subseteq B$. We say that μ does not divide over A if whenever $\phi(x,b)$ is over B and $\mu(\phi(x,b)) > 0$*

then $\phi(x,b)$ does not divide over A. Similarly we say μ does not fork over A if every formula of positive μ-measure does not fork over A.

Remark 4.2. *Suppose μ is a global Keisler measure.*
(i) μ does not fork over A iff μ does not divide over A.
(ii) If μ is either definable over M or finitely satisfiable in M, then μ does not fork over M.

Proof. (i) If $\mu(\phi(x)) > 0$ and ϕ forks over A then ϕ implies a finite disjunction of formulas each of which divides over A. One of those formulas must have positive measure by finite additivity.
(ii) If μ is either definable over or finitely satisfiable in M, then for any $\phi(x,y) \in L$ and b, $\mu(\phi(x,b))$ depends on $tp(b/M)$. (In the case that μ is definable over M this is immediate. If μ is finitely satisfiable in M, then $tp(b_1/M) = tp(b_2/M)$ implies that the measure of the symmetric difference of $\phi(x,b_1)$ and $\phi(x,b_2)$ is 0, so again $\phi(x,b_1)$ and $\phi(x,b_2)$ have the same measure.) So if $(b_i : i < \omega)$ is an M-indiscernible sequence and $\mu(\phi(x,b_0)) = r > 0$, then for all i, $\mu(\phi(x,b_i)) = r$. By Lemma 2.8 of [13], $\{\phi(x,b_i) : i < \omega\}$ is consistent.

Proposition 2.1 readily generalizes to measures.

Proposition 4.3. *(Assume NIP) Suppose μ is a global Keisler measure, and A a small set. Then the following are equivalent:*
(i) μ does not fork over A,
(ii) μ is invariant under $Autf(\bar{M}/A)$,
(iii) whenever $Lstp(b_1/A) = Lstp(b_2/A)$, then $\mu(\phi(x,b_1)\Delta\phi(x,b_2)) = 0$.

Proof. (i) implies (iii). Suppose that μ does not fork over A. Let b_0, b_1 begin an indiscernible sequence $(b_i : i < \omega)$. We claim that $\mu(\phi(x,b_0)\Delta\phi(x,b_1)) = 0$. If not then without loss $\mu(\phi(x,b_0) \wedge \neg\phi(x,b_1)) > 0$. As $((b_i, b_{i+1}) : i = 0,2,4,..)$ is A-indiscernible and μ does not fork over A, $\{\phi(x,b_i) \wedge \neg\phi(x,b_{i+1}) : i = 0,2,...\}$ is consistent, contradicting NIP. So we clearly obtain (iii).
(iii) implies (ii) is immediate.
(ii) implies (i). Assume (ii) and suppose that $\mu(\phi(x,b)) = \epsilon > 0$ and that $(b = b_0, b_1, ...)$ is A-indiscernible. So the b_i' have the same Lascar strong type over A, hence $\mu(\phi(x,b_i)) = \epsilon$ for all i. By Lemma 2.8 of [13] again, $\{\phi(x,b_i) : i < \omega\}$ is consistent.

So we see that the global Keisler measures which do not fork over some small set coincide with those which are invariant over some small model, and we can just call them invariant global measures (as we did for types). There is an obvious notion of Borel definability for a measure. Namely we say that (global) μ is Borel definable over A if for any $\phi(x,y)$ and closed subset C of $[0,1]$ the set of $b \in \bar{M}$ such that $\mu(\phi(x,b)) \in C$ is Borel over A. We will prove at the end of this section that (assuming NIP) any global invariant Keisler measure is Borel definable.

We can therefore define a product on invariant measures (global Keisler measures invariant over some small set of model) by integration: Suppose μ_x is a Borel definable (over M) global Keisler measure and λ_y is another global Keisler measure. Fix $\phi(x,y) \in L_{\bar{M}}$. For any $b \in \bar{M}$ let $f(b) = \mu(\phi(x,b))$ Then f is a Borel function over M on the sort to which the b's belong, so we can form the integral $\int f(y)d\lambda_y$, and we call this $(\mu \otimes \lambda)(\phi(x,y))$. When μ, λ are global complete types, this agrees with the product as defined in section 2.

Remark 4.4. *It is natural to attempt to generalize the material from section 3 to measures. So (assuming NIP) we call a global Keisler measure μ* generically stable *if μ is finitely satisfiable in some small model, and also definable. The analogues of Proposition 3.2 and Proposition 3.5 will be treated in a future work joint with P. Simon. Likewise for the measure analogue of the symmetry Lemma 3.4.*

For now, we continue the generalization of the results from section 2 to measures. We also begin to relate invariant measures (those measures which do not fork over a small set) to invariant types. We first generalize Proposition 2.11.

Proposition 4.5. *(Assume NIP) Let μ_x be a global Keisler measure which does not fork over A. Then μ is $bdd(A)$-invariant.*

Proof. Let B be the Boolean algebra of formulas $\phi(x,b)$ over $\bar{M}$ quotiented by the equivalence relation $\phi(x,b) \sim \psi(x,c)$ if $\mu(\phi(x,b)\Delta\psi(x,c)) = 0$. Let $[\phi(x,b)]$ denote the class of $\phi(x,b)$, i.e. the image of $\phi(x,b)$ in B. Note that B is small, because by Proposition 4.3,
(*) if $Lstp(b/A) = Lstp(c/A)$ then $[\phi(x,b)] = [\phi(x,c)]$.
For any ultrafilter U on B, we obtain a complete global type $p_U(x)$ by

putting $\phi(x,b) \in p_U$ iff $[\phi(x,b)] \in U$. Note that by (*) p_U is invariant under $Autf(\bar{M}/A)$, so does not fork over A, so is in fact $bdd(A)$-invariant.

We claim that in fact if b and c have the same compact strong type over A (namely the same type over $bdd(A)$) then $\mu(\phi(x,b)\Delta\phi(x,c)) = 0$, which will show that μ is $bdd(A)$-invariant. If not we have, without loss, $\mu(\phi(x,b) \wedge \neg\phi(x,c)) > 0$, so there is an ultrafilter U containing $[\phi(x,b) \wedge \neg\phi(x,c)]$, and we obtain the global type p_U which as pointed out above is $bdd(A)$-invariant. As $\phi(x,b) \in p_U$, also $\phi(x,c) \in p_U$, a contradiction.

Note that given small A the collection of global x-types which do not fork over A is closed in $S_x(\bar{M})$ and coincides with the space of global types invariant over $bdd(A)$. As in section 8 we call this space $S_x^{bdd(A)}(\bar{M})$. In the same vein as Proposition 4.5 we have.

Proposition 4.6. *Fix small A. Then there is a natural bijection between global Keisler measures which do not fork over A and regular Borel probability measures on $S_x^{bdd(A)}(\bar{M})$.*

Proof. We have already observed the bijection between global Keisler measures and regular Borel probability measures on the space of global types. Let μ_x be a global Keisler measure which does not fork over A, and let β be the regular Borel measure on $S_x(\bar{M})$ corresponding to it. It is enough to show that the restriction of β to $S_x^{bdd(A)}(\bar{M})$ determines μ. But this is clear. For if $\phi(x), \psi(x)$ are formulas over $\bar{M}$ and $\mu(\phi\Delta\psi) > 0$, then there is some $p \in S_x^{bdd(A)}(\bar{M})$ containing (without loss) $\phi \wedge \neg\psi$.

Proposition 4.7. *(Assume NIP) Let $p(x)$ be a complete type over A. Then the following are equivalent:*
(i) p does not fork over A (that is p has a global nonforking extension),
(ii) there is a global Keisler measure μ which extends p and is A-invariant.

Proof. (ii) implies (i): Let μ be a global A-invariant Keisler measure which extends p. Suppose, for a contradiction, that $p(x)$ forks over A. So there are $\phi(x) \in p(x)$, and formulas $\psi_1(x), .., \psi_n(x)$ (with parameters), each dividing over A such that $\models \phi(x) \rightarrow \vee_i \psi_i(x)$. As $\mu(\phi(x)) = 1$, $\mu(\psi_i(x)) > 0$ for some $i = 1, .., n$. But as μ is A-invariant, μ does not divide over A, contradiction.
(i) implies (ii). Let $p'(x)$ be a global nonforking extension of p. By 2.11 p' is $bdd(A)$-invariant. By Proposition 2.6, p' is Borel definable over $bdd(A)$.

Let G be the compact Lascar group over A, namely $Aut(bdd(A)/A)$. G is a compact group which thus has a (unique left and right invariant) Haar measure h. Let S be the space of all complete types over $bdd(A)$. So S is a compact space acted on continuously by G.

We now define the A-invariant global measure μ extending $p(x)$. Let $\phi(x, b)$ be a formula over $\bar{M}$. Let $q(y) = tp(b/A)$, and let $Q \subset S$ be the space of complete types over $bdd(A)$ extending q (a closed subset of S). The Borel definability of p' over $bdd(A)$ says that the set X of $tp(b'/bdd(A))$ such that $\phi(x, b') \in p'$ is a Borel subset of S. Hence $X \cap Q$ is also a Borel subset of Q. The compact space Q is acted on continously and transitively by G, hence has the form G/H for some closed subgroup of G. Let π be the canonical projection from G onto Q. As $X \cap Q$ is Borel in Q, $\pi^{-1}(X \cap Q)$ is a Borel subset of G, hence $h(\pi^{-1}(X \cap Q))$ is defined and we define this to be $\mu(\phi(x, b))$. Note that if $tp(b'/A) = tp(b/A)(= q)$ then by construction $\mu(\phi(x, b')) = \mu(\phi(x, b))$, so μ is A-invariant. Also if $\phi(x, b) \in p(x)$ (namely $b \in A$), then Q is a singleton ($\{tp(b/A)\}$) and $X \cap Q = Q$, whereby $\mu(\phi(x, b)) = 1$. So μ extends p. We leave it to the reader to check finite additivity of μ.

In the stable case it is not so hard to see, via the finite equivalence relation theorem for example, that the A-invariant measure μ extending p in (ii) above is unique. Likewise if p has a generically stable global nonforking extension. It would be interesting to find examples of uniqueness of μ, when p does not have a generically stable nonforking extension. We will come back to the issue of when μ is unique later in the paper.

We now return to the question of the Borel definability of invariant measures. We will make use of the Vapnik-Chervonenkis theorem [36] which we now describe.

Let (X, Ω, μ) be a probability space, that is a set X equipped with a σ-algebra Ω of subsets (events) of X and a countably additive probability measure μ with values in $[0, 1]$. Namely $\mu(\emptyset) = 0$, $\mu(X) = 1$, $\mu(E)$ is defined for any $E \in \Omega$ and if $E_i \in \Omega$ for $i < \omega$ are pairwise disjoint and $E = \cup_i E_i$ then $\mu(E) = \sum_i \mu(E_i)$. For $k > 0$, let μ^k be the product measure on X^k. Also, for $k > 0$, $A \in \Omega$ and $\bar{p} = (p_1, .., p_k) \in X^k$, let $fr_k(A, \bar{p})$ be the proportion of p_i's which are in A, namely $|\{i : p_i \in A\}|/n$. For any $A \in \Omega$, let $g_{A,k} : X^k \to [0, 1]$ be defined by $g_{A,k}(\bar{p}) = |fr_k(A, \bar{p}) - \mu(A)|$. Also let $h_{A,k} : X^{2k} \to [0, 1]$ be defined by $h_{A,k}(\bar{p}, \bar{q}) = |fr_k(A, \bar{p}) - fr_k(A, \bar{q})|$. Note that $g_{A,k}$ is μ^k-measurable and $h_{A,k}$ is μ^{2k}-measurable. We will say (somewhat nonstandardly) that a family $\mathcal{C}$ of events has NIP if there is a

natural number d such that for no subset F of X of cardinality d do we have that $\{F \cap A : A \in \mathcal{C}\}$ is the full power set of F. For $\mathcal{C}$ with NIP the smallest such d will be called the VC-dimension of $\mathcal{C}$.

With this notation, Theorem 2 of [36] gives:

VC Theorem. Suppose that the family $\mathcal{C}$ of events has NIP. Suppose also that for every $k > 0$, $sup_{A\in\mathcal{C}} g_{A,k}$ is μ^k-measurable and $sup_{A\in\mathcal{C}} h_{A,k}$ is μ^{2k}-measurable. Then there is a function $f(-,-)$ such that for any $\epsilon > 0$, and $k < \omega$ $\mu^k(\{\bar{p} : sup_{A\in\mathcal{C}} g_{A,k}(\bar{p}) > \epsilon\}) \leq f(k,\epsilon)$, and for any given ϵ, $f(k,\epsilon) \to 0$ as $k \to \infty$.
Moreover the function f depends only on the VC-dimension of $\mathcal{C}$.

It is convenient to state a version or consequence of the VC Theorem in which the two measurability assumptions are dropped.

First let us note that any family $\mathcal{F}$ of measurable functions from X to $[0,1]$ say, has an *essential supremum*, which is by definition measurable, and can be chosen as the outright supremum of some countable subfamily of $\mathcal{F}$. By an essential supremum of $\mathcal{F}$ we mean a measurable function $g : X \to [0,1]$ such that for each $f \in \mathcal{F}$, $g \geq f$ on a set of μ-measure 1, and whenever $g' : X \to [0,1]$ has the same property, then $g \leq g'$ on a set of measure 1. Such a g can be found as follows. We construct measurable g_α for α a countable ordinal in the following way: let g_0 be some member of $\mathcal{F}$. At limit stages, take suprema. Given g_α, if it is already an essential supremum of $\mathcal{F}$, stop. Otherwise there is some $f \in \mathcal{F}$ such that $f > g_\alpha$ on a set of positive measure. Put $g_{\alpha+1} = sup\{g_\alpha, f\}$. Note that $\int g_\alpha$ is strictly increasing with α, hence the construction has to stop at some countable ordinal β. Then g_β is measurable, is an essential supremum of $\mathcal{F}$, and is, by construction, the supremum of some countable subfamily of $\mathcal{F}$.

Note that any two essential suprema g, g' of $\mathcal{F}$ are equivalent in the sense that they are equal almost everwhere. We will just write $esssup\mathcal{F}$ for a representative of the equivalence class which we will assume to be a supremum of a countable subfamily of $\mathcal{F}$.

VC Theorem.* Suppose that the family $\mathcal{C}$ of events has NIP. Then for any $\epsilon > 0$, $\mu^k(\{\bar{p} : esssup_{A\in\mathcal{C}} g_{A,k}(\bar{p}) > \epsilon\}) \to 0$ as $k \to \infty$.

This *-version is immediately deduced from the first version: Let us fix ϵ. We can find a countable subfamily $\mathcal{C}_0$ of $\mathcal{C}$ with the same VC-dimension, and such that for each k, $esssup_{A\in\mathcal{C}} g_{A,k} = sup_{A\in\mathcal{C}_0} g_{A,k}$. Note that $sup_{A\in\mathcal{C}_0} h_{A,k}$ is measurable. Hence we can apply the VC Theorem with $\mathcal{C}_0$ in place of $\mathcal{C}$ and

we obtain the *-version.

We will apply the above theorems to the following situation. Let M be a model of a NIP theory, $\phi(x,y)$ an L-formula, and μ_x a Keisler measure over M. Take X to be the Stone space $S_x(M)$, Ω the σ-algebra of its Borel subsets, and identify μ_x with the measure induced on X as described at the beginning of this section. Let $\mathcal{C}$ be the family of clopen subsets of X given by the formulas $\phi(x,c)$ as c varies over M. It is easy to see that as T has NIP the family $\mathcal{C}$ has NIP.

Lemma 4.8. *(Assume NIP.)*
Let M be a model and μ_x a Keisler measure over M.
(i) Let $\phi(x,y) \in L$, and let $\epsilon > 0$. Then there is k, and a Borel subset B of $S_x(M)^k$ of positive μ^k measure such that for all $(p_1,..,p_k) \in B$ and any $c \in M$, $\mu(\phi(x,c))$ is within ϵ of the proportion of the p_i which contain $\phi(x,c)$.
(ii) If moreover M is saturated (for example equals $\bar{M}$) and $A \subseteq M$ is small such that μ does not fork over A, then in (i) we can choose B such that for any $(p_1,..,p_k) \in B$, each p_i does not fork over A.

Proof. (i) We will identify $\phi(x,c)$ with the clopen subset of X it defines, and as above write $fr_k(\phi(x,c),\bar{p})$ for the proportion of p_i containing $\phi(x,c)$. We fix our $\epsilon > 0$. By *VC Theorem**, choose k and $\delta > 0$ such that $\mu^k(\{\bar{p} : esssup_{A\in\mathcal{C}} g_{A,k}(\bar{p}) < \epsilon\}) > \delta$. It follows that for each finite set $C = \{c_1,..,c_m\}$ of parameters from M, the Borel set $B_C = \{(p_1,..,p_k) \in S(M)^k : fr_k(\phi(x,c_j),\bar{p}) - \mu(\phi(x,c_j)) < \epsilon$, for $j = 1,..,m\}$ has μ^k-measure $> \delta$. Note that $B_{C'} \subseteq B_C$ when $C \subseteq C'$. As Borel sets are approximated in measure by closed sets, we can use compactness to find a Borel (in fact closed) subset B of $S_x(M)^k$ of μ^k-measure $\geq \delta$ such that *all* $(p_1,..,p_k) \in B$ satisfy the conclusion of (i).

(ii) This is proved exactly as (i) but working in the closed set $S_x^{bdd(A)}(M)$ of $S_x(M)$ consisting of, types over M which do not fork over A, and bearing in mind 4.6.

Corollary 4.9. *(Assume NIP.) Suppose that μ is a global Keisler measure which does not fork over A. Then μ is Borel definable over $bdd(A)$.*

Proof. Fix $\phi(x,y) \in L$, and $\epsilon > 0$. Let $p_1,..,p_n$ be given by Lemma 4.8 (ii) (for $M = \bar{M}$). By Proposition 2.6, each p_i is strongly Borel definable

over $bdd(A)$. Namely for each i there is some finite Boolean combination Y_i of type-definable over $bdd(A)$-sets such that for all c, $\phi(x,c) \in p_i$ iff $c \in Y_i$. It follows that if c, c' lie in exactly the same Y_i for $i = 1, .., n$, then $\mu(\phi(x,c))$ and $\mu(\phi(x,c'))$ differ by $< 2\epsilon$. This suffices to prove Borel definability of μ.

A thorough account of the model-theoretic results related to VC-type theorems will appear in a forthcoming paper.

5 Generics and forking in definable groups

In [13] the fsg (finitely satisfiable generics) property for definable groups was introduced. Strengthenings of the fsg will be discussed in section 6. In this section we discuss weakenings of the fsg, relate the fsg to forking, and in general try to obtain equivariant versions of results from section 2.

Let G be a $\emptyset$-definable group in $\bar{M}$. $S_G(A)$ denotes the set of complete types over A extending "$x \in G$". Recall from [13] and [35] that if T has NIP then G^{00} (the smallest type-definable bounded index subgroup of G) exists. Our notion of "generic" will be from [13]. However in [11] "generic" is used differently (there it is a translation invariant definable type). Also we distinguish it (as in [20]) from "f-generic":

Definition 5.1. *(i) A definable subset X of G is said to be left generic if finitely many left translates of X by elements of G cover G.*
(ii) $p(x) \in S_G(A)$ is left generic if every formula in p is left generic.
(iii) A definable subset X of G is said to be left f-generic if for each $g \in G$, gX does not fork over $\emptyset$.
(iv) $p(x) \in S_G(A)$ is left f-generic if every formula in p is left f-generic.
(v) G has the fsg if there is a global type $p(x) \in S_G(\bar{M})$ such that every left translate gp of p is almost finitely satisfiable over $\emptyset$, namely for every model M_0, every left translate of p is finitely satisfiable in M_0.

Note that if G is the additive group in RCF (or just the underlying group in $DOAG$) then there are no complete generic types, but there are two f-generic types, at $+\infty$ and $-\infty$. (Both these types are definable over $\emptyset$ and invariant under the group operation).

In [13] we pointed out:

Fact 5.2. *G has the fsg if and only if*
(a) The class of left generic definable subsets of G coincides with the class of

right generic sets (we call them simply generic), and
(b) The class of non generic definable subsets of G is a (proper) ideal, and
(c) If X is a generic definable subset of G, then X meets every model M_0.

The fsg is a rather strong property for a definable group. It is a kind of abstract "definable compactness". The following was observed in [13]. (Theorem 8.1 and Remark 3 of Section 8.)

Remark 5.3. *In o-minimal expansions of real closed fields, the definable groups with fsg are precisely the definably compact groups.*

Definable groups in algebraically closed valued fields have been studied by the first author ([11]). From the analysis there, one can with additional work, obtain a characterization of the definable groups G with fsg, at least when G is abelian: they are precisely the (abelian) groups G such that there is a definable homomorphism $h : G \to A$ where A is internal to the value group and is definably compact, and $Ker(h)$ is stably dominated. The p-adically closed field case is considered in [21], where among other things it is pointed out that definably compact groups defined over the standard model have fsg.

Corollary 5.4. *If G has the fsg, then there exists a (simultaneously left and right) global generic type $p \in S_G(\bar{M})$ and moreover any such global generic type is also (left and right) f-generic.*

Problem 5.5. *Suppose G has the fsg and assume NIP if one wants. Is it the case that every left f-generic type is generic?*

In the case of definably compact groups in o-minimal expansions of real closed fields, Problem 5.5 has a positive answer. This follows from the results in [5] and [23] which give that: if X is definable, closed and bounded then X does not fork over M_0 if and only if X meets M_0. So assume X to be left f-generic and we want to prove that X is generic. We may assume X is closed. Let M_0 be any small model. So for any $g \in G$, gX meets M_0. By compactness X is right generic in G (so generic in G).

We now work without the fsg assumption, improving a result from [13].

Proposition 5.6. *(Assume NIP) Suppose that G has a global left f-generic type p. Then*
(i) $Stab_l(p) = G^{00} = G^{000}$, and
(ii) G is definably amenable, that is G has a left invariant global Keisler measure.

Proof. Recall that $Stab_l(p)$, the left stabilizer of p, is by definition the set of $g \in G$ such that $gp = p$. It is a subgroup of G but on the face of it has no definability properties.
(i) Let us fix a small model M_0 for convenience.
Claim I. If $tp(g_1/M_0) = tp(g_2/M_0)$ then $g_1^{-1}g_2 \in Stab_l(p)$.
Proof of Claim I. By f-genericity of p and the NIP assumption, both p and g_1p are $Aut_{M_0}(\bar{M})$-invariant. Let $f \in Aut_{M_0}(\bar{M})$ such that $f(g_1) = g_2$. So $g_1p = f(g_1p) = f(g_1)f(p) = g_2p$. Hence $g_1^{-1}g_2 \in Stab_l(p)$.

Claim II. Suppose $h \in Stab_l(p)$. Then $h = g_1^{-1}g_2$ for some g_1, g_2 with the same type over M_0.
Proof of Claim II. By definition $ha = b$ for some realizations of p (in a model containing $\bar{M}$). So we can find realizations a, b of $p|M_0$ in $\bar{M}$ such that $ha = b$. So $h = ba^{-1}$. Put $g_1 = b^{-1}$ and $g_2 = a^{-1}$ and note they have the same type over M_0.

From Claims I and II we deduce that $Stab_l(p)$ is type-definable (over M_0) as the set of $g_1^{-1}g_2$ for $g_1, g_2 \in \bar{G}$ having the same type over M_0. On the other hand by Claim I, the index of $Stab_l(p)$ in G is bounded by the number of types over M_0. It follows that $Stab_l(p)$ has to be G^{00}, because p determines a coset of G^{00}. The same reasoning yields that $Stab_l(p) = G^{000}$.

(ii) This follows by the same proof as that of Proposition 6.2 of [13], using part (i) together with Corollary 2.6: We can reduce to the case where T is countable. Given X a definable subset of G, let M_0 be a countable model over which X is defined. Note that p does not fork over M_0. The proof of Corollary 2.3 yields that not only is p Borel definable over M_0, but (using countability of the language and of M_0) that any "$\phi(x, y)$-definition of p" is a countable union of closed sets over M_0. In particular $Y = \{g \in G : X \in gp\}$ is a countable union of closed sets over M_0. By part (i) whether or not $g \in Y$ depends only on the coset g/G^{00}. Let $\pi : G \to G/G^{00}$ be the canonical surjection. Then $\pi(Y)$ is a Borel subset of G/G^{00} and we define $\mu(X)$ to be $\mathbf{h}(\pi(Y))$ where $\mathbf{h}$ is the unique normalized Haar measure on the compact group G/G^{00}. μ is clearly additive and is also left-invariant: Given $h \in G$, $\mu(hX) = \mathbf{h}(\{g/G^{00} : hX \in gp\})$. But $hX \in gp$ iff $g = hg'$ for some g' such that $X \in g'p$. Hence $\{g/G^{00} : hX \in gp\} = h/G^{00} \cdot \{g'/G^{00} : X \in gp\}$. So by left-invariance of $\mathbf{h}$, $\mu(hX) = \mathbf{h}(\{g'/G^{00} : X \in gp\}) = \mu(X)$.

Remark 5.7. *Proposition 5.6 also holds when G is type-definable (with appropriate definitions). But for G definable (over $\emptyset$) , we could also deduce 5.6*

from 4.7 and 2.10(i), together with Construction C from the introduction, in the easy form where the group action is regular. (Assume NIP). Let T' the theory T together with an additional sort S and a $\emptyset$-definable regular action of G on S. Let x be a variable of sort S. Then the following are equivalent: (i) In T there is a global left f-generic type, (ii) in T', $x = x$ does not fork over $\emptyset$.

We now aim towards a kind of converse of 5.6. First:

Lemma 5.8. *(Assume NIP) Let G be a definable (or even type-definable) group, and let μ be a global left invariant Keisler measure on G. Let M_0 be any small model over which G is defined. Then there is a global left invariant Keisler measure μ' on G such that $\mu'|M_0 = \mu|M_0$ and μ' is definable (hence $Aut(\bar{M}/M)$-invariant for some small M).*

Proof. The proof follows Keisler's construction in [17] of "smooth measures" assuming NIP.
Step I. We construct M extending M_0 and extension μ_1 of μ to a left $G(M)$-invariant measure over M such that μ_1 has a unique extension to a global left-invariant measure μ'.
Suppose that already μ and M_0 do not work. So μ has two distinct extensions λ, β to left invariant global measures. So there is a definable subset X of G and some positive real number r such that $|\lambda(X) - \beta(X)| > r$. Let μ' be the average of λ and β. Note that μ' has the property
(*) for any M_0-definable subset Y of G, $\mu'(X \Delta Y) > r/4$.

Let M_1 be a small model containing M_0 such that X is over M_1, and let μ_1 be the restriction of μ' to M_1. Continue with μ_1 in place of μ. We claim that at some point we arrive at μ_α over a model M_α which is left $G(M_\alpha)$-invariant and has a unique global left invariant extension. Otherwise by (*) we find a formula $\phi(x, y)$, a positive real number ϵ and $(c_i : i < \omega)$, and a global measure μ'' such that $\mu''(\phi(x, c_i) \Delta \phi(x, c_j)) > \epsilon$ for all $i \neq j$. This (together with NIP) is a contradiction. (See 3.14 of [17] or 3.2 of [13].) So Step I is accomplished.
Step II. μ' from Step I is definable (over M).
This is proved (via Beth's theorem) exactly as in the proof of Lemma 2.6 (definability of minimal measures) in [13].

The following consequence of Lemma 5.8 was also noticed by Krzysztof Krupinski.

Proposition 5.9. *(Assume NIP) Suppose that the definable group G is definably amenable (has a global left invariant Keisler measure). Then, after possibly adding constants for some small model over which G is defined, there is a global left f-generic type of G.*

Proof. By Lemma 5.8, there is a left invariant global Keisler measure μ on G which is moreover definable, over some small model M, hence does not fork over M. Let p be a global type of G such that $\mu(X) > 0$ for every $X \in p$. Then for every $X \in p$ and $g \in G$, $\mu(gX) = \mu(X) > 0$, hence gX does not fork over M. So p is a left f-generic of G after adding names for the model M.

So we obtain the following:

Corollary 5.10. *(Assume NIP) G is definably amenable if and only if G has a global left f-generic type after possibly adding constants. Moreover under these circumstances G^{000} exists and coincides with G^{00}.*

The corollary goes through for type-definable G too. Shelah [35] has recently proved that assuming NIP, then for any abelian type-definable group G, G^{000} exists. As abelian groups are amenable, the corollary above improves Shelah's theorem. On the other hand Gismatullin [7] has recently been able to drop the abelian hypothesis from Shelah's theorem.

The following clarifies the question of the existence of left f-generic types and addition of constants. In particular the proof of (ii) gives another example where forking does not equal dividing.

Proposition 5.11. *(Assume NIP) (i) Suppose G is a definable group and has a global left f-generic type after naming elements of some model. Then G has a global left f-generic type after naming elements of any model.*
(ii) There is a $\emptyset$-definable group G in a theory T with NIP, such that G is definably amenable, but there is no global f-generic type of G.

Proof. (i) We assume G to be $\emptyset$-definable in T. We will use Remark 5.7. That is we work in the theory T' which is T together with a new sort S and a regular right action of G on S. $\bar{M}$ is our monster model of T and then $\bar{M}'$ denotes the corresponding (monster) model of T', that is $\bar{M}$ together with the new sort S. Let x be a variable of sort S. By 5.7 it is enough to prove:
Claim. Suppose $x = x$ forks over the small elementary substructure M_0 of

$\bar{M}$. Then $x = x$ forks over any small elementary substructure of $\bar{M}$ containing M_0.
Proof of claim. Assume for simplicity that $x = x$ implies $\phi(x, a) \vee \psi(x, b)$ where each of $\phi(x, a)$, $\psi(x, b)$ divides over M_0 (in $\bar{M}'$). Let $(a_i : i < \omega)$ and $(b_i : i < \omega)$ be infinite M_0-indiscernible sequences in $tp(a/M_0)$ and $tp(b/M_0)$ respectively, such that $\{\phi(x, a_i) : i < \omega\}$ and $\{\psi(x, b_i) : i < \omega\}$ are each inconsistent. Let M_1 be a small elementary substructure of $\bar{M}$ containing M_0. By an automorphism argument we may assume that $tp(M_1/M_0 \cup \{a_i : i < \omega\} \cup \{b_i : i < \omega\})$ is finitely satisfiable in M_0. But then clearly each of the sequences $(a_i : i < \omega)$ and $(b_i : i < \omega)$ is also M_1-indiscernible. So $x = x$ forks over M_1.
(ii) Let K be a (saturated) real closed field. Let G be $K \times K$, equipped with its addition, and with all relations which are $\emptyset$-definable in $(K, +, \cdot)$ and invariant under the action of $SL(2, K)$. Of course the theory of G with this structure has NIP, and G is definably amenable as it is abelian. Suppose for a contradiction that G had a global f-generic type $p(x)$. In particular $p(x)$ does not fork over $\emptyset$, so is invariant under automorphisms fixing $bdd(\emptyset)$ by section 2. As $SL(2, K)$ acts on G by automorphisms and $SL(2, K)^{00} = SL(2, K)$, it follows that p is $SL(2, K)$-invariant. But then p induces a $PSL(2, K)$ -invariant global type of the projective line over K, which contradicts Remark 5.2 (iv) of [13] and its proof.

One might think that any global left generic type of any definable group is automatically left f-generic (NIP or no NIP). But we need some assumptions to prove it.

Proposition 5.12. *(Assume NIP). Let G be a definable group which is definably amenable. Then any global left generic type of G is also left f-generic.*

Proof. It is clearly sufficient to prove that any left generic definable set X does not divide over $\emptyset$. And for this it is enough to find a small model M_0 such that any left generic definable set X does not divide over M_0. (Given left generic X_a defined over a, and an indiscernible sequence $(a_i : i < \omega)$ in $tp(a/\emptyset)$ we can find a sequence $(a'_i : i < \omega)$, indiscernible over M_0 and with the same type over $\emptyset$ as $(a_i : i < \omega)$.)

The assumption that there is a global left generic type implies easily that the class $\mathcal{I}$ of non left generic definable sets is a proper ideal of the Boolean algebra of all definable subsets of G. By Proposition 6.3(i) of [13], there is a

small model M_0 such that for every left generic definable set X there is some M_0-definable subset Y of G such that the symmetric difference $X \Delta Y$ of X and Y is in $\mathcal{I}$, namely is non left generic.

Now suppose that X_a is a left generic definable subset of G (defined over a). We want to show that X_a does not divide over M_0. Let Y be definable over M_0 such that the symmetric difference of X_a and Y is nongeneric. Replacing X_a by $X_a \cap Y$ we may assume that $X_a \subseteq Y$. Let $(a_i : i < \omega)$ be an M_0-indiscernible sequence with $a_0 = a$. Suppose by way of contradiction $\cap_{i=1,..n} X_{a_i}$ is inconsistent for some n. Then (as Y is defined over M_0), the union of the $Y \setminus X_{a_i}$ for $i = 1, .., n$ covers Y. But Y is left-generic, and each $Y \setminus X_{a_i}$ is non left generic, contradicting $\mathcal{I}$ being an ideal. This proves the proposition.

6 Generically stable groups

In this section we introduce various strengthenings of the fsg leading to the notion of a generically stable group (analogous to a generically stable type). Assuming NIP these strengthenings will be equivalent.

We first give a natural definition of the fsg for type-definable groups.

Definition 6.1. *Let G be a type-definable group. We say that G has fsg if there is some small model M_0 and global type $p(x)$ of G such that p is finitely satisfiable in $G(M_0)$, namely for every formula $\phi(x) \in p$, ϕ is realized by some $g \in G(M_0)$.*

The basic results on definable groups with fsg go through easily for type-definable groups G. Namely we define a relatively definable subset X of G to be left generic if finitely many left translates of X by elements of G cover G.

Lemma 6.2. *Suppose G is type-definable with fsg. Then*
(i) the class of relatively definable left generic subsets of G coincides with the class of relatively definable right generic subsets of G and forms a proper ideal of the class of relatively definable subsets of G.
(ii) There is a global generic type of G, namely a global type p extending $x \in G$ such that any relatively definable subset of G which is in p is generic.
(iii) There is a small model M_0 such that any global generic type of G is finitely satisfiable in $G(M_0)$, and such that moreover if X is a relatively

definable generic subset of G then finitely many (left or right) translates of X by elements of $G(M_0)$ cover G.
(iv) G^{00} exists and is both the left and right stabilizer of any global generic type of G.

Definition 6.3. *Let G be a definable group.*
(i) We say that G has fsg^+ if G has fsg and some global generic type is definable.
(ii) We say that G has fsg^{++} if G^{00} exists and has fsg as a type-definable group in its own right.
(iii) We say that G is generically stable if G has fsg and some global generic type is generically stable.

Note that (ii) seems to be a rather minor variation on G having fsg but in fact it is a substantial strengthening. Note in particular that if $G = G^{00}$ and G has the fsg then it has fsg^{++}. Definably compact groups in o-minimal structures satisfy none of the above properties, although they have fsg. We will see soon that (iii) implies (ii) implies (i) and that there are all equivalent assuming NIP.

Lemma 6.4. *Suppose G has fsg^{++} (namely G^{00} exists and has fsg). Then*
(i) G has fsg,
(ii) Any global generic type of G^{00} is a global generic type of G.
(iii) There is a unique global generic type of G which implies "$x \in G^{00}$, say p_0.
(iv) p_0 is definable and is also the unique global generic type of G^{00}.

Proof. (i) Let M_0 be a small model witnessing that G^{00} has fsg. We may assume that M_0 contains representatives of each coset of G^{00} in G (as there are boundedly many). Let p be a global generic type of G^{00}. Let g realize p. Then $g_1^{-1}g \in G^{00}$ for some $g_1 \in G(M_0)$. By assumption $g_1^{-1}gp$ is finitely satisfiable in $G(M_0)$ (in fact in $G^{00}(M_0)$. Hence so is gp. This shows that G has fsg.
(ii) We have shown in (i) that if p is a global generic of G^{00} then also gp is finitely satisfiable in $G(M_0)$ for any $g \in G$ which implies that p is a global generic type of G. (Alternatively if X is a definable subset of G and $X \cap G^{00}$ is generic in G^{00} then finitely many translates of X cover G^{00} whereby finitely many translates of this set of finitely many translates covers G.)
(iii) Suppose for a contradiction that p and q are distinct global generic types

of G each of which implies that "$x \in G^{00}$. Let X be a definable set in p such that $Y = G \setminus X$ is in q. Let $X_0 = X \cap G^{00}$ and $Y_0 = Y \cap G^{00}$. So X_0 and Y_0 are relatively definable subsets of G^{00} which partition G^{00}. By Lemma 6.2 we may assume that X_0 is generic in G^{00}. So finitely many translates of X_0 by elements of G^{00} cover G^{00}. In particular (as $q(x)$ implies $x \in G^{00}$), there is $g \in G^{00}$ such that $gX \in q$. But by Corollary 4.3 of [13] the symmetric difference of X and gX is nongeneric (in G), so as q is a generic type of G, we see that X is also in q, a contradiction.
(iv) Let p_0 be the unique type from (iii). By part (ii), p_0 is also the unique global generic type of G^{00}. Let G^{00} be the intersection of the directed family $(Y_i)_{i \in I}$ of definable subsets of G. For a given L-formula $\phi(x, y)$ and $c \in \bar{M}$, $\phi(x, c) \in p_0$ iff for some $i \in I$ finitely many translates of (the set defined by) $\phi(x, c)$ by elements of $G^{00}(M_0)$ cover Y_i. The same is true for $\neg\phi(x, y)$. By compactness the set of c such that $\phi(x, c) \in p_0$ is definable. So p_0 is definable.

Proposition 6.5. *Let G be a definable group. Then*
(i) If G has fsg^{++} then G has fsg^{+}.
(ii) If G is generically stable then G has fsg^{++}.
(iii) Assume NIP. If G has fsg^{+} then G is generically stable. (Hence assuming NIP the three properties fsg^{+}, fsg^{++} and generic stability are equivalent).

Proof. (i) follows from the previous lemma.
(ii) By translating we obtain a generically stable global generic type $p(x)$ of G such that $p(x) \models$ "$x \in G^{00}$". Fix a small model M_0 such that p is definable over and finitely satisfiable in M_0 (and of course G and G^{00} are defined over M_0). Let M be a small $|M_0|^+$-saturated extension of M_0. Let $(a_i : i < \omega)$ be a Morley sequence in p over M_0. Then $a_i \in G^{00}(M)$ for all i. Suppose $\phi(x, c) \in p$. The generic stability of p implies that $\phi(x, c)$ is satisfied by some a_i, so $\phi(x, c)$ is satisfied in $G^{00}(M)$. If $g \in G^{00}$ then $gp = p$ hence any left translate of p by an element of G^{00} is finitely satisfiable in $G^{00}(M)$, so G^{00} has fsg and G has fsg^{++}.
(iii) is immediate because assuming NIP any global type which is both definable over and finitely satisfiable in some small model, is generically stable.

We now develop some consequences of the "weakest" of the new properties, fsg^{+}.

Proposition 6.6. *Suppose that the definable group G has fsg^{+}. Then*
(i) $G^{00} = G^{0}$.

(ii) There is a unique global generic type p_0 of G extending "$x \in G^{00}$" (and p_0 is definable).
(iii) Every global generic type of G is definable and the set of global generics is in one-one correspondence with G/G^{00}.
(iv) For any L-formula $\phi(x, y)$ there is $N < \omega$ such for any c, $\phi(x, c)$ is generic in G if and only if at most N left (right) translates of $\phi(x, c)$ cover G.

Proof. (i) Fix a definable global generic type p. By 4.3 from [13] we know that $Stab(p) = G^{00}$. For a fixed formula $\phi(x, y)$ let (just for now) $p|\phi$ be the set of formulas of the form $\phi(g^{-1}x, c)$ which are in p for $g \in G$ and $c \in \bar{M}$. Then $Stab_\phi(p)$ denotes $\{g \in G : g(p)|\phi = p|\phi\}$ and is a definable subgroup of G. As $Stab(p) = Stab_\phi(p)$ for $\phi \in L$ it follows that each $Stab_\phi(p)$ has finite index and so G^{00} is the intersection of a family of definable subgroups of finite index, so equals G^0.
(ii) By translating the given definable generic p we can find a definable global generic type which extends "$x \in G^{00}$" and we call it p_0. Note that if $q(x)$ is a global generic extending "$x \in G^{00}$" then so is q^{-1}. Hence it suffices to prove:
Claim. If q is any global generic type extending "$x \in G^{00}$" then $q^{-1} = p_0$.
Proof of claim. Let M_0 be any model. So q is finitely satisfiable in M_0 and p_0 is definable over M_0. Let a realize $p|M_0$ and b realize $q|(M_0, a)$. As $a \in G^{00}$ and $G^{00} = Stab(q)$ it follows that $c = ab$ also realizes $q|(M_0, a)$. By Lemma 3.4, a realizes $p|(M_0, c)$. Again as $c^{-1} \in G^{00}$ we see that $c^{-1}a = b^{-1}$ also realises $p|(M_0, c)$. In particular $tp(b^{-1}/M_0) = p_0|M_0$. As M_0 was arbitrary it follows that $q^{-1} = p_0$. The claim is proved as well as part (ii).
(iii) Now any global generic type of G is a translate of a global generic which implies "$x \in G^{00}$". By (ii) every global generic type of G is a translate of the definable type p_0, hence is also definable.
(iv) If $\phi(x, y) \in L$ and $c \in \bar{M}$, then $\phi(x, c)$ is generic in G if for some $g \in G$, $\phi(g^{-1}, c) \in p_0$. So by definability of p_0 the set of c such that $\phi(x, c)$ is generic is definable, which is enough.

Let us remark that there are groups with $G^{00} = G^0$, with the fsg but without fsg^+ (even in a NIP environment). The easiest example is simply the additive group R^+ of the valuation ring R in a saturated p-adically closed field. Here $(R^+)^{00}$ is the intersection of the finite index definable subgroups $v(x) \geq n$ for $n \in \mathbb{Z}^+$, but there are many generic types extending $(R^+)^{00}$.

Remark 6.7. *Suppose the definable group G is generically stable. Let X be*

a definable subset of G. Then X is generic if and only if every left (right) translate of X does not divide (or even fork) over $\emptyset$.

Proof. We know left to right by just the fsg. Now suppose that X is nongeneric, and defined over some model M_0. Let p be some generically stable global generic type of G, and let $(a_i : i < \omega)$ be a Morley sequence of p over M_0. Then for any $g \in G$, as $gX \notin p$, we see that $a_i \notin gX$ for some $i < \omega$. It follows that $\{a_i^{-1}X : i < \omega\}$ is inconsistent. As $(a_i^{-1} : i < \omega)$ is indiscernible over M_0 we see that X divides over M_0 so over $\emptyset$.

Section 7 of this paper is devoted to establishing the uniqueness of invariant (under the group action) measures for definable groups with fsg (assuming NIP). In the case of generically stable groups this can be seen quickly and we do it now.

Lemma 6.8. *Suppose the definable group G is generically stable, and μ is a left invariant global Keisler measure on G. Then μ is generic, in the sense that for any definable subset X of G, if $\mu(X) > 0$ then X is generic.*

Proof. Suppose that X is a nongeneric definable subset of G, defined over a model M_0. The argument in the proof of Remark 6.7 gives an M_0-indiscernible sequence $(a_i : i < \omega)$ such that $\{a_iX : i < \omega\}$ is inconsistent. But $\mu(a_iX) = \mu(X)$ for all i. By Lemma 2.8 of [13], $\mu(X) = 0$.

Lemma 6.9. *Suppose G is a definable group with fsg^+. Then any definable subset of G is a Boolean combination of translates (cosets) of definable subgroups of G of finite index and nongeneric definable sets.*

Sketch of proof. Fix a formula $\phi(x, y) \in L$. By a ϕ-formula we mean a Boolean combination of formulas $\phi(gx, c)$ $(g \in G, c \in \bar{M})$. By a global ϕ-type of G we mean a maximal consistent set of ϕ-formulas over $\bar{M}$ (containing in addition $x \in G$). A global ϕ-type will be called generic iff it contains only generic formulas (iff it extends to a global generic type of G). As in the proof of 6.6, there is a definable subgroup G^0_ϕ of finite index which is the (say left) stabilizer of every global generic ϕ-type. It follows that each of the finitely many cosets of G^0_ϕ contains a unique global generic ϕ-type. Thus every ϕ-set (set defined by a ϕ -formula) is a Boolean combination of cosets of G^0_ϕ and nongeneric definable sets.

Corollary 6.10. *Suppose the definable group G is generically stable. Then there is a unique left invariant (under ther group action) Keisler measure on G which is also the unique right invariant Keisler measure.*

Proof. If μ is a left invariant Keisler measure on G, then μ is determined on definable subgroups of finite index and their translates, and by Lemma 6.8 is 0 on all nongeneric sets. By Lemma 6.9 there is only one possibility for μ. Lemma 6.9 is clearly still true replacing definable subgroups of finite index by normal definable subgroups of finite index. Hence we see that μ is also the unique right invariant Keisler measure on G.

Note that a special case is: if G is a connected stable group, and p is its unique global generic type, then not only is p the unique left (right) invariant global type of G, but it is also the unique left (right) invariant global Keisler measure on G.

7 Uniqueness of invariant measures for groups with fsg

Given a definable group G with fsg, and assuming NIP we have from [13] that there is a left invariant global Keisler measure μ on G (namely G is definably amenable). In fact we constructed such μ which is generic (the definable sets of positive measure are precisely the generics). Clearly μ^{-1} is a right invariant generic global Keisler measure. Our uniqueness theorem (Theorem 7.7 below) generalizes the uniqueness of invariant types for connected stable groups, as well in a sense generalizing the uniqueness of Haar measure for compact groups. (If one is willing to pass to continuous model theory, a compact group is like something finite, hence stable.) The main point is to prove a Fubini (or symmetry) theorem for suitable measures. A. Berarducci also raised the uniqueness issue (in the o-minimal context) in [3] and pointed out the relevance of Fubini. Our methods go through variants of Grothendieck groups.

We begin with a lemma on homomorphisms from vector spaces to the reals. First recall some notation: if A is a $\mathbb{Q}$-vector space, then a subset P of A is said to be a *cone* if P is closed under addition and under scalar multiplication by positive elements of $\mathbb{Q}$.

Lemma 7.1. *Let A be a $\mathbb{Q}$-vector space, B a $\mathbb{Q}$-subspace of A, P a cone in A, and μ_B a nonzero homomorphism from B to $\mathbb{R}$ such that $\mu_B(B \cap P) \geq 0$. Assume that for any $a \in A$ there is $b \in B$ such that $b - a \in P$. Let $c \in A$*

and let $e = inf\{\mu_B(y) : y \in B, y - c \in P\}$.
Then (i) if $e = -\infty$ *then there is no homomorphism* $\mu : A \to \mathbb{R}$ *extending* μ_B *with* $\mu(P) \geq 0$.
(ii) If $e \in \mathbb{R}$, *then there is a homomorphism* $\mu : A \to \mathbb{R}$ *extending* μ_B *such that* $\mu(P) \geq 0$ *and* $\mu(c) = e$.

Proof. First there is no harm in assuming that $0 \in P$.
(i) It is enough to prove the stronger statement that if $\mu : A \to \mathbb{R}$ is any homomorphism extending μ_B, and $\mu(P) \geq 0$ then $\mu(c) \leq e$. But this is immediate: if $e' > e$ then there is $y \in B$ such that $y - c \in P$ and $\mu_B(y) \leq e'$, but then $\mu_B(y) - \mu(c) \geq 0$ so $\mu(c) \leq \mu_B(y) \leq e'$.
(ii) Let us assume $e \in \mathbb{R}$, and by the stronger statement we have just proved all we need to do is to find a homomorphism $\mu : A \to \mathbb{R}$ extending μ_B such that $\mu(P) \geq 0$ and $\mu(e) \geq c$.

As μ_B is nonzero, let $b_1 \in B$ be such that $\mu(b_1) > 0$, and there is no harm in assuming $\mu_B(b_1) = 1$. Replacing c by $mb_1 + c$ for sufficiently large m we may also assume that $e > 0$.

Let $P_B = \{b \in B : \mu_B(b) \geq 0\}$. Note that if $\mu : A \to \mathbb{R}$ is a homomorphism such that $\mu(b_1) = 1$ and μ is nonnegative on P_B then μ extends μ_B. So it suffices to find some homomorphism μ from A to $\mathbb{R}$ such that $\mu(b_1) = 1$, μ is nonnegative on $P + P_B = \{a + b : a \in P, b \in P_B\}$, and $\mu(c) \geq e$. (Because then μ is nonnegative on P_B, and also on P.)
Let $P_c = \{\alpha c - \beta b_1 : \alpha, \beta \in \mathbb{Q},\ \alpha, \beta \geq 0,\ \beta < e\alpha\} \cup \{0\}$.
Claim. $-b_1 \notin P' = P + P_B + P_c$.
Proof of claim. Suppose not.
Case (i). $-b_1 = a + b + \alpha c - \beta b_1$ for $a \in P$, $b \in P_B$, and α, β as in the definition of P_c.
Multiplying by α^{-1} and letting $a' = \alpha^{-1}a$, $b' = \alpha^{-1}b$, and $\gamma = \alpha^{-1}\beta$ we have $-\alpha^{-1}b_1 = a' + b' + c - \gamma b_1$, whence
$\gamma b_1 - (b' + \alpha^{-1}b_1) - c = a' \in P$.
But then $e \leq \mu_B(\gamma b_1 - (b' + \alpha^{-1}b_1)) \leq \gamma < e$ a contradiction.
Case (ii). $-b_1 = a + b$ for $a \in P$, $b \in P_B$.
Then $-a = b_1 + b$, so $-a \in B$ and $\mu_B(-a) > 0$. But then also $a \in B$, so $a \in B \cap P$, so by assumption $\mu_B(a) \geq 0$ whereby $\mu_B(-a) < 0$, again a contradiction.

By the claim we can extend the cone P' to a maximal cone P'' not containing $-b_1$, and note that $b_1 \in P''$, and also each of $P + P_B$, P_c are contained in P''. Note also that P'' defines a linear preorder on A, namely for each $d \in A$,

$d \in P''$ or $-d \in P''$. (If $d \notin P''$ then $-b_1 \in P'' + \mathbb{Q}^+ d$, so $-d \in P'' + \mathbb{Q}^+ b_1$, so $-d \in P''$.) Our assumptions on A, together with the definition of P_B imply that for any $a \in A$ there is $n > 0$ such that $nb_1 - a \in P + P_B$ so also in P''. Thus $\mathbb{Q}^+ b_1$ is cofinal in A. Now it is clear that the homomorphism from $\mathbb{Q}b_1$ to $\mathbb{R}$ which sends b_1 to 1 extends to an preorder preserving homomorphism μ from A to $\mathbb{R}$ (which sends $A_0 = \{x \in A : b_1 - nx \in P'' \text{ for all } n \in \mathbb{Z}\}$ to 0). Then $\mu(P + P_B) \geq 0$, as $P + P_B \subseteq P''$. But also, as $P_c \subseteq P''$ we have that $c - \gamma b_1 \in P''$ for all positive $\gamma < e$. Hence $\mu(c) \geq e$, and we have found the required μ.

We now consider a certain variant of the Grothendieck (semi)-group introduced in section 5 of [13] We will also work at the more general level of definable group actions rather than just definable groups. So we will fix a definable group action (G, X) and a small model M_0 over which (G, X) is definable. Using the notation analogous to that in [13], we will take the relevant semigroup $K_{semi}(X)$ to be the collection of nonnegative cycles $\sum_i k_i X_i$ in X up to piecewise translation by members of $G(M_0)$. $K_0(X)$ will be the corresponding Grothendieck group. (Recall that we define $x_1 \sim_0 x_2$ for $x_1, x_2 \in K_{semi}(X)$ if there is $y \in K_{semi}(X)$ such that $x_1 + y = x_2 + y$ in K_{semi}. Then K_0 is the collection of $\sim_0$-classes together with formal inverses.) When we apply Lemma 7.1 to this situation, the $\mathbb{Q}$-vector space A will be the tensor product of $\mathbb{Q}$ with $K_0(X)$. Define P_0 to be the image of K_{semi} in K_0 (under the canonical map) and then P will be $\{\alpha x : \alpha \in \mathbb{Q}^+, x \in P_0\}$, a cone in A. Define B_0 to be the image in $K_0(X)$ of the "M_0-definable" members of $K_{semi}(X)$, and then B will be the tensor product of $\mathbb{Q}$ with B_0.

We now give a somewhat more concrete representation of the objects defined in the previous paragraph. It will be convenient both notationally and conceptually to introduce a certain category $\mathcal{D}$ in place of the semigroup of nonnegative cycles on X (namely before identification via piecewise translations). First we think of $\mathbb{Z}$ as living in our monster model $\bar{M}$ as the directed union of the finite sets (or sorts) $\{-m, ..., m\}$. We also have the group structure on $\mathbb{Z}$ definable in $\bar{M}$, piecewise. Let $\tilde{X}$ be $X \times \mathbb{Z}$. A definable subset of $\tilde{X}$ is by definition a definable subset of $X \times \{-m, .., m\}$ for some m. Likewise for $\tilde{G} = G \times \mathbb{Z}$, which is now a "definable" group. The natural action of $\tilde{G}$ on $\tilde{X}$ is also "definable", namely for each m, n we have the map $(G \times \{-m, .., m\}) \times (X \times \{-n, .., -n\}) \to (X \times \{-m-n, .., m+n\})$ is definable. The objects of the category are definable subsets of $\tilde{X}$. If Y, Y' are in $\mathcal{D}$ then a morphism f from Y to Y' is an injective map f from Y

into Y' such that there is a partition $Y = Y_1 \cup ... \cup Y_n$ of Y and elements $g_i \in \tilde{G}(M_0) = G(M_0) \times \mathbb{Z}$ such that for each i and $y \in Y_i$, $f(y) = g_i y$. Equivalently (by compactness) f is a definable embedding of Y in Y' such that for each $y \in Y$ there is $g \in \tilde{G}(M_0)$ such that $f(y) = gy$. So Y and Y' will be isomorphic in $\mathcal{D}$ if there is f is above which is a bijection between Y and Y'. Then as a set $K_{semi}(X)$ is the set of isomorphism classes of members of $\mathcal{D}$. The addition on $K_{semi}(X)$ is induced by the following (non well defined) addition on $\mathcal{D}$: if Y, Y' are definable subsets of $X \times \{-m, .., m\}$ then $Y + Y' = Y \cup (2m+1)Y'$. (Likewise for nY.) Connecting with earlier notation, B_0 is the image in $K_0(X)$ of the set of isomorphism classes of M_0-definable elements of $\mathcal{D}$ (and B the tensor product of B_0 with $\mathbb{Q}$). For Y a definable subset of $\tilde{X}$, $[Y]$ denotes its image in $A = K_0(X) \otimes \mathbb{Q}$.

We will be interested in Keisler measures μ on X which are $G(M_0)$-invariant. Note that any such μ extends uniquely to (and is determined by) a $\tilde{G}_0$-invariant finitely additive measure on the definable subsets of $\tilde{X}$.

Lemma 7.2. *The global $G(M_0)$-invariant Keisler measures on X correspond to the homomorphisms $h : A \to \mathbb{R}$ such that $h(P) \geq 0$ and $h[X] = 1$. Namely if μ is a global $G(M_0)$-invariant Keisler measure on X, then there is a unique homomorphism $h : A \to \mathbb{R}$ such that $h([Y]) = \mu(Y)$ for any definable subset Y of X. Moreover this h satisfies $h([X]) = 1$ and $h(P) \geq 0$. Conversely if $h : A \to \mathbb{R}$ is such that $h([X]) = 1$ and $h(P) \geq 0$ then defining $\mu(Y) = h([Y])$ for any definable subset Y of X gives a global $G(M_0)$-invariant Keisler measure μ on X.*

Proof. Clear.

Corollary 7.3. *Suppose that μ is a global $G(M_0)$-invariant Keisler measure on X which is moreover the unique global $G(M_0)$-invariant Keisler measure on X extending $\mu|M_0$. Then for any definable subset D of X (or $\tilde{X}$) with $\mu(D) = \beta$ and $\epsilon > 0$, there are $n \in \mathbb{N}$, M_0-definable $E_0, E_1 \in \mathcal{D}$ and $D' \in \mathcal{D}$ such that*
(i) $\mu(E_1) - \mu(E_0) < n(\beta + \epsilon)$ and
(ii) there is a morphism f (in $\mathcal{D}$) from $E_0 + nD + D'$ to $E_1 + D'$.

Proof. With earlier notation let $h : A \to \mathbb{R}$ be the homomorphism corresponding to μ as given by Lemma 7.2. So by Lemma 7.1, h is the unique homomorphism such that $h(P) \geq 0$, and which extends $h|B$. Let D be a definable subset of X (or $\tilde{X}$) and put $c = [D]$, and let $\mu(D) = h([D]) =$

$\inf\{h(y) : y \in B, y - c \in P\}$. Fix $\epsilon > 0$. So for some $y \in B$ we have $h(y) < \mu(D) + \epsilon$ and $y - c \in P$. Now for large enough positive n we have $n(y - c) = [D'']$ for some definable subset D'' (of $\tilde{X}$), and $ny = [E_1] - [E_0]$ with E_0, E_1 M_0-definable subsets of $\tilde{X}$. So we have:
(a) $\mu(E_1) - \mu(E_0) = h([E_1] - [E_0]) = nh(y) < n(\mu(D) + \epsilon)$, which gives (i).
We also have:
(b) $[E_1] - [E_0] - n[D] = [D'']$.

We then obtain (after possibly multiplying everything by some $m > 0$) some definable D' such that in $\mathcal{D}$, $E_0 + D'' + nD + D'$ is isomorphic to $E_1 + D'$. Ignoring D'' this gives a morphism f in $\mathcal{D}$ from $E_0 + nD + D'$ to $E_1 + D'$ which is (ii).

Let us give a more explicit statement.

Remark 7.4. *The conclusion of Corollary 7.3 can be restated as:*
For any definable subset D of X with $\mu(D) = \beta$ and for any $\epsilon > 0$ there are $n, m, m', m'' \in \mathbb{N}$, M_0-definable sets E_0, E_1 in $\mathcal{D}$, and some $D' \in \mathcal{D}$, as well as $D_i \in \mathcal{D}$ and $g_i \in \tilde{G}_0$ for $i = 1, , .m$, $D'_j \in \mathcal{D}$ and $g_j \in \tilde{G}_0$ for $j = 1, .., m'$, and $D''_k \in \mathcal{D}$ and $g_{k''} \in \tilde{G}_0$ for $k = 1, .., m''$ such that
(i) $\mu(E_1) - \mu(E_0) < n(\beta + \epsilon)$,
(ii) $nD = \cup_{i=1,..,m} D_i$, $D' = \cup_{j=1,,.m'} D'_j$, $E_0 = \cup_{k=1,..,m''} D''_k$, the D_i, D'_j, D''_k are pairwise disjoint, and D' is disjoint from both D and E_1,
(iii) the sets $g_i D_i$, $g_{j'} D'_j$ and $g''_k D''_k$ are pairwise disjoint subsets of $E_1 \cup D'$.

We can now prove the sought after symmetry (or Fubini) theorem. Recall that if μ_x, λ_y are global Keisler measures on definable sets X, Y respectively, and μ is definable (or even Borel definable), then we can form the global Keisler measure $\mu \otimes \lambda$ on $X \times Y$: for D a definable subset of $X \times Y$, $(\mu \otimes \lambda)(D) = \int \mu(D(y)) d\lambda$, where $D(y) = \{x : (x, y) \in D\}$. We may also write $(\mu \otimes \lambda)(D)$ as $\int_D d\mu d\lambda$.

Proposition 7.5. *Suppose (G, X) is a definable group action, and Y a definable set, all defined over M_0. Suppose μ is a global Keisler measure on X which is definable and satisfies the hypothesis of Corollary 7.3. Suppose λ is a global definable Keisler measure on Y. Then $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$.*

Before beginning the proof, let us define an action of G on $X \times Y$ by $g(x, y) = (gx, y)$ and we claim that both measures $\mu \otimes \lambda$ and $\lambda \otimes \mu$ are $G(M_0)$-invariant for this action: Let D be a definable subset of $X \times Y$, and

$g \in G(M_0)$. So for any $y \in Y$, $(gD)(y) = g(D(y))$, so $\mu((gD)(y)) = \mu(D(y))$, and thus $(\mu \otimes \lambda((gD) = (\mu \otimes \lambda)(D)$. On the other hand, let $f(x) = \lambda(D(x))$. Now clearly $D(g^{-1}(x)) = g(D(x))$, so $f(g^{-1}x) = \lambda(gD(x))$. As μ is $G(M_0)$-invariant, $\int f(x)d\mu = \int f(g^{-1}x)d\mu$, so $(\lambda \otimes \mu)(D) = (\lambda \otimes \mu((gD)$. All this of course extends to the actions of $\tilde{G}$ on $\tilde{X}$ and $\tilde{X} \times Y$ and the relevant measures on $\mathcal{D}_X$ and $\mathcal{D}_{X \times Y}$.

Proof of Proposition 7.5. Let D be a definable subset of $X \times Y$. We have to show that $(\mu \otimes \lambda)(D) = (\lambda \otimes \mu)(D)$. By considering also the complement of D, it suffices to prove that for any $\epsilon > 0$, $(\lambda \otimes \mu)(D) \leq (\mu \otimes \lambda)(D) + \epsilon$.

Fix $\epsilon > 0$. By Corollary 7.3, for each $y \in Y$ we find $n_y, m_y, m'_y m, m''_y$, $(E_0)_y$, $(E_1)_y$, D'_y etc. such that (i), (ii) and (iii) of Remark 7.4 hold for $D_y = \{x : (x, y) \in D\}$. By compactness we may partition Y into definable sets Y_ν, such that the n_y, m_y, m'_y, m''_y, $(E_0)_y, (E_1)_y$, $(g_i)_y, (g'_j)_y, (g''_k)_y$ are constant on each Y_ν. Focus attention on one Y_ν. Let $D_\nu = D \cap (X \times Y_\nu$. Let $D'_\nu \subseteq X \times Y_\nu$ be such that for $y \in Y_\nu$, $D'_y = \{x : (x, y) \in D'_{nu}\}$. So clearly we have:
Claim I. There is a morphism in $\mathcal{D}_{X \times Y_\nu}$, from $(E_0 \times Y_\nu) + nD_\nu + D'_\nu$ into $(E_1 \times Y_\nu) + D'_\nu$.

As $\mu(E_1) - \mu(E_0) < n(\mu(D_y) + \epsilon)$ for all $y \in Y_\nu$, we obtain, on integrating with respect to λ over Y_ν that
Claim II. $(\mu(E_1) - \mu(E_0)\lambda(Y_\nu) \leq n(\mu \otimes \lambda)(D_{vu}) + \epsilon\lambda(Y_\nu)$.

But the left hand side in Claim II coincides with $(\lambda \otimes \mu)(E_1 \times Y_\nu) - (\lambda \otimes \mu)(E_0 \times Y_{\nu)}$. So denoting $\lambda \otimes \mu$ by r we rewrite Claim II as
Claim IV. $r(E_1 \times Y_\nu) - r(E_0 \times Y_\nu) \leq n(\mu \otimes \lambda)(D_{vu}) + \epsilon\lambda(Y_\nu)$.

We have already noted that r on $X \times Y$ is $G(M_0)$-invariant, so applying r to Claim I and using the disjointness there, we obtain:
Claim IV. $r(E_0 \times Y_\nu) + nr(D_\nu) + r(D'_\nu) \leq r(E_1 \times Y_\nu) + r(D'_\nu)$.
From Claims II and IV we obtain
Claim V. $r(D_\nu) \leq (\mu \otimes \lambda)(D_\nu) + \epsilon\lambda(Y_\nu)$.

Summing over ν in Claim V, we obtain
$\lambda \otimes \mu)(D) \leq (\mu \otimes \lambda)(D) + \epsilon$ which is what we wanted. The proposition is proved.

Lemma 7.6. *(Assume NIP) Suppose G is a definable group, defined over a small model M_0, and μ is a Keisler measure on G over M_0, which is left $G(M_0)$-invariant. Then*
(i) There is a global left $G(M_0)$-invariant Keisler measure μ' extending μ,

and a small model M containing M_0 such that μ' is the unique left $G(M_0)$-invariant global Keisler measure extending $\mu'|M$. Again any such μ' is definable.
(ii) Suppose in addition that G has fsg. Then any μ' as in (i) is left invariant, namely left $G(\bar{M})$-invariant.

Proof. (i) is proved in exactly the same way as 5.8.
(ii) By the definability of μ', $H = Stab(\mu')$ is a type-definable subgroup of G. We want to show that it is all of G. If not, it is clear that the complement of H contains a generic definable subset X of G. By the fsg, $X \cap G(M_0)$ is nonempty. But then X contains an element of H (as μ' is $G(M_0)$-invariant). Contradiction.

Combined with the material of the previous section we can now obtain our main result.

Theorem 7.7. *(Assume NIP). Suppose G is a definable group with fsg. Then G has a unique left invariant global Keisler measure, which is also the unique right invariant global Keisler measure of G. This measure is both definable and generic.*

Proof. We already know from [13] that G has some left invariant global Keisler measure. Let μ, λ be left invariant global Keisler measures. We will first show that $\mu = \lambda^{-1}$. Let D be any definable subset of G. We want to prove that $\mu(D) = \lambda(D^{-1})$. Let M_0 be a small model over which both G and D are defined. Let μ'' be a global Keisler measure satisfying (i) of 7.6 for some small M containing M_0. Namely μ'' extends $\mu|M_0$ and is the unique left $G(M_0)$-invariant Keisler measure extending $\mu''|M$. By (ii) of 7.6, μ'' is left $G(\bar{M})$-invariant, in particular left $G(M)$-invariant, so is also the unique left $G(M)$ -invariant extension of $\mu''|M$. So renaming M as M_0, μ'' satisfies the hypothesis of 7.3. By 7.6 (ii) μ'' is definable, and as already mentioned μ'' is left invariant. So as we are currently just interested in $\mu(D)$ we may assume that $\mu = \mu''$. Likewise we may assume that λ is definable. By Proposition 7.5, $\mu \otimes \lambda = \lambda \otimes \mu$ (on $G \times G$). Let $Z = \{(x, y) \in G \times G : x \in yD\}$ which is equal to $\{(x, y) \in G\times : y \in xD^{-1}\}$. So $(\mu \otimes \lambda)(Z) = \int \mu(yD)d\lambda = \mu(D)$ as μ is left invariant. Likewise $(\lambda \otimes \mu)(Z) = \int \lambda(xD^{-1})d\mu = \lambda(D^{-1})$.

So we have succeeded in proving that $\lambda = \mu^{-1}$. This applies in particular when $\lambda = \mu$ yielding that $\mu = \mu^{-1}$ (so μ is also right invariant). Hence $\mu = \lambda$ and is also the unique right invariant global Keisler measure on G.

Finally we point out the extension of Proposition 7.7 to homogeneous spaces.

Proposition 7.8. *(Assume NIP) Suppose (G, X) is a definable homogeneous space, and G has NIP. Then there is a unique G-invariant global Keisler measure on X.*

Proof. Let μ be the unique invariant global; Keisler measure on G given by the previous proposition. So it is definable and satisfies the hypothesis of 7.3. Note that we obtain from μ a G -invariant global Keisler measure λ on X: given $x \in X$, the map taking $g \in G$ to gx gives a (G)-invariant surjection $\pi_x : G \to X$. For Y a definable subset of X, define $\lambda_x(Y) = \mu(\pi_x^{-1}(Y))$. Then λ_x is G-invariant and does not depend on the choice of x so we call it λ. Note that for any $x \in X$, and definable subset Y of X, $\lambda(Y) = \mu(\{g \in G : gx \in Y\})$. Clearly λ is also definable and satisfies the hypothesis of 7.3.

Let λ' be another G-invariant global Keisler measure on X. We want to show $\lambda' = \lambda$. As in the proof of 7.7 we may assume λ' is definable. By 7.5, $\mu \otimes \lambda' = \lambda' \otimes \mu$ on $G \times X$. Let Y be a definable subset of X and let $Z = \{g, x) : gx \in Y\}$. Then $(\mu \otimes \lambda')(Z) = \int_{x \in X}(\mu\{g \in G : gx \in Y\})d\lambda' = \int_{x \in X} \lambda(Y)d\lambda' = \lambda(Y)$.

And $(\lambda' \otimes \mu)(Z) = \int_{g \in G} \lambda'(\{x \in X : gx \in Y\})d\mu = \int_{g \in G} \lambda'(Y)d\mu = \lambda'(Y)$. So $\lambda = \lambda'$.

There is an obvious formulation of X having fsg where X is a definable homogeneous space (for a definable group G): there is a global type p of X such that for every $g \in G$, gp is finitely satisfiable in M_0 for every model M_0.

If G has fsg then clearly X does too.

Question 7.9. *(Assume NIP) (i) Does Proposition 7.8 holds if only X has fsg?*
(ii) Are there examples of transitive group actions (G, X) such that X has fsg, G acts faithfully, and G does not have fsg?

Remark 7.10. *Let us say that a definable group G has the* weak fsg *if there is some small model M_0 over which G is defined such that G has no proper type-definable subgroup containing $G(M_0)$. The following results follow from the proofs above.*
(i) (Assume NIP.) If G is definably amenable with weak fsg then G has a unique left invariant Keisler measure which is also its unique right invariant Keisler measure, and is definable.

(ii) (Assume NIP.) Suppose the definable group is definably amenable and has no proper nontrivial type definable subgroups. Then G has a global left invariant type, which is definable, and is moreover the unique left invariant Keisler measure and the unique right invariant Keisler measure on G.

Proof. (i) is given by Proposition 7.5, and the proofs of Lemma 7.6 and Theorem 7.7.
The assumptions of (ii) imply that G has weak fsg, and so let μ be the unique (left, right) invariant measure on G given by (i). As μ is definable, μ does not fork over some small M_0. Let p be some random global type for μ, namely every formula in p has positive μ-measure. Then clearly every left translate of p does not fork over M_0. By Proposition 5.6, $Stab_l(p) = G^{00}$. But our assumptions imply that $G^{00} = G$, hence p is left invariant. By part (i) $\mu = p$.

Note that statement (ii) in Remark 7.10 above mentions only types, but the only proof we have of it involves measures. In fact, using 7.5 one can conclude that, assuming NIP, any group with weak fsg has fsg (and so the groups in Remark 7.10 (ii) are generically stable groups in the sense of section 6) but the proof uses the theory of generically stable measures which will be treated in a future paper.

Finally in this section we speculate on group free analogues of the fsg property. Here is a possible definition, in which the definable group G is replaced by a complete type $p(x) \in S(A)$. For simplicity we make a blanket assumption of NIP.

Definition 7.11. *We say that $p(x) \in S(A)$ has fsg if p has a global extension $p'(x)$ such that for every $(|L| + |A|)^+$-saturated model M containing A, p' is finitely satisfiable in $p(M)$, that is for each formula $\phi(x, b) \in p'$ there is $a \in M$ which realizes p and satisfies $\phi(x, b)$).*

Remark 7.12. *(i) In Definition 7.11 the global extension p' of p is necessarily a nonforking extension of p.*
(ii) Suppose $A = bdd(A)$ and $p(x) \in S(A)$ has fsg. Then p has a unique global nonforking extension p', and p' is generically stable.
(iii) Suppose the global type p' is generically stable and p' does not fork over A. Let $p = p'|A$. Then p has fsg.

Proof. (i) If $(b_i : i < \omega)$ is an A-indiscernible sequence (in $\bar{M}$) then we can find a $(|L|+|A|)^+$-saturated model M containing A such that $(b_i : i < \omega)$ remains indiscernible over M. If $\phi(x, b_0) \in p'$ then $\models \phi(a, b_0)$ for some $a \in M$ hence $\phi(a, b_{i)}$ for all $i < \omega$. So p' does not fork over A.
(ii) By part (i) and 2.11 (and the assumption that $A = bdd(A)$), p' is A-invariant. We will show that some Morley sequence in p' over A is totally indiscernible (and apply 3.2). Let M be a small $(|L|+|A|)^+$-saturated model containing A. Let $(a_i : i < \omega)$ be a Morley sequence in p' over M (so also one over A). It is enough to prove that for any n, $tp(a_0, .., a_{n-1}, a_n/A) = tp(a_n, a_0, .., a_{n-1}/A)$. So fix $n \geq 1$. Suppose that $\phi(x_0, .., x_{n-1}, x)$ is over A and $\models \phi(a_0, .., a_{n-1}, a_n)$. Then $\phi(a_0, .., a_{n-1}, x) \in p'$, so by hypothesis is realized by some $c \in M$ which realizes p. But then $(c, a_0, a_1, .., a_{n-1})$ also begins a Morley sequence in p' over A, so $tp(c, a_0, .., a_{n-1}/A) = tp(a_0, .., a_{n-1}, a_n/A)$, and we see that $\models \phi(a_n, a_0, .., a_{n-1})$. This suffices.
(iii) follows from the material in section 3 and is left to the reader.

In the context of Remark 7.12(iii) we say that the fsg type $p(x) \in S(A)$ comes from a generically stable type. So (by 7.12(ii)) group actions, namely the action of the compact Lascar group $Aut(bdd(A)/A)$ on the set of extensions of p over $bdd(A)$, will enter the picture whenever an fsg type $p(x) \in S(A)$ does *not* come from a generically stable type.

Another source of fsg types is through transitive group actions and construction C from the section 1. Namely suppose that (G, X) is a $\emptyset$-definable group action in T (with NIP), and we form T_X with additional sort X'. Then in T_X there is a unique (so isolated) 1-type $p(x)$ over $\emptyset$ in X'. If X has fsg in T then it is not hard to see that p has fsg in T_X.

In fact, types $p \in S(A)$ with fsg can be characterized as types of the form $\mu|A$ where μ is a global A-invariant generically stable measure (in the sense of Remark 4.4). This was proved by P. Simon and will again appear in a joint work with the authors.

8 The Compact Domination Conjecture

We will prove

Theorem 8.1. *Assume $\bar{M}$ to be an o-minimal expansion of a real closed field. Let G be a definably connected definably compact commutative group definable in $\bar{M}$. Then G is compactly dominated. That is, let $\pi : G \to G/G^{00}$*

be the canonical surjective homomorphism. Then for any definable subset X of G, $Y_X = \{b \in G/G^{00} : \pi^{-1}(b) \cap X \neq \emptyset$ and $\pi^{-1}(b) \cap G \setminus X \neq \emptyset\}$ has Haar measure 0.

In fact the proof (and subsequent elaborations) will yield a bit more: in the structure $\bar{M}^*$ obtained by expanding $\bar{M}$ by a predicate for G^{00}, G/G^{00} will be "semi-o-minimal" with dimension that of G, and moreover for any definable subset X of G the set Y_X above (which is now definable in $\bar{M}^*$) has dimension $< dim(G/G^{00})$. So in a sense G is o-minimally dominated (by a definable o-minimal compact Lie group). This of course suggests many problems and issues for future work. Also, in the paper [14] joint with Peterzil, the full compact domination conjecture (i.e. for not necessarily commutative definably compact groups) is deduced from Theorem 8.1 and results in [13], using a structure theorem for definably compact groups in o-minimal expansions of real closed fields. Of course Theorem 8.1 builds on and uses the work and contributions of a number of people, including Berarducci, Dolich, Edmundo, Otero, and Peterzil.

Until we say otherwise $\bar{M}$ denotes a (saturated) o-minimal expansion of a real closed field K, G is a definably compact definably connected definable group in $\bar{M}$ of o-minimal dimension n, and π is the canonical surjective homomorphism from G to G/G^{00}. Without loss of generality G is defined over $\emptyset$. M_0 will denote a fixed small model. We will make use in a few places of the fact that G^{00} can be defined by a countable collection of formulas. (This is by the DCC result in [4].) The overall proof has several steps and "new" ingredients, including a beautiful result of Otero and Peterzil ([22]). Some of the lemmas go through or can be formulated at various greater levels of generality but we tend to concentrate on the case at hand. By Lemma 10.5 of [13] in order to prove Theorem 9.1 it suffices to show that if X is a definable subset of G of dimension $< n$ then the Haar measure of $\pi(X) \subseteq G/G^{00}$ is 0. We will aim towards this.

The first step is to show that G^{00} is definable in some weakly o-minimal expansion of $\bar{M}$. Recall that weak o-minimality means that every definable set of elements (rather than tuples) of the universe is a finite union of convex sets (with respect to the underlying ordering). Let $\bar{M}^*$ be the expansion of $\bar{M}$ obtained by adjoining a predicate for the trace on $\bar{M}$ of any set definable with parameters in some elementary extension of $\bar{M}$. In other words, for each L-formula $\phi(x, y)$ (x, y tuples) and complete type $q(x)$ over $\bar{M}$, adjoin a predicate for $\{b : \phi(x, b) \in q(x)\}$. By the results of Baisalov and Poizat [2],

or alternatively of Shelah [33], $Th(\bar{M}^*)$ is weakly o-minimal, with quantifier elimination. The weak o-minimality of a theory means that every model is weakly o-minimal, namely that for some (any) model there is a bound on the number of convex components of definable subsets of 1-space in definable families.

Lemma 8.2. *G^{00} is definable in $\bar{M}^*$.*

Proof. Let p be a global generic type of G (which exists by 5.2 and 5.3). By Proposition 5.6 (and 5.12) we have
(i) $Stab(p) = G^{00}$,
and moreover as mentioned in the proof of 5.6,
(ii) $G^{00} = \{g_1 g_2^{-1} : tp(g_1/M_0) = tp(g_2/M_0)\}$.

Given a formula $\psi(x)$ (over some parameters) defining a subset X of G, let $Stab_\psi(p) = \{g \in G :$ for all $h \in G$, $hX \in p$ iff $ghX \in p\}$. Then clearly $Stab_\psi(p)$ is a subgroup of G. Moreover $Stab(p) = \cap_\psi Stab_\psi(p)$ and
(iii) $Stab_\psi(p)$ is definable in $\bar{M}^*$.

We will show that G^{00} is the intersection of finitely many $Stab_\psi(p)$, even with the ψ over M_0.

For $\psi(x)$ over M_0, let $S_\psi(p)$ be the smallest type-definable over M_0 (in $\bar{M}$) set containing $Stab_\psi(p)$. (Note that as p is $Aut(\bar{M}/M_0)$-invariant, so is $Stab_\psi(p)$.) We don't necessarily know that $S_\psi(p)$ is a subgroup of G, but:
Claim (iv). (ψ over M_0.) $S_\psi(p) \cdot G^{00}$ is a (type)-definable subgroup of G (clearly containing $Stab_\psi(p)$).
Proof. Clearly $\pi(S_\psi(p)) \cdot G^{00}$ is the closure of $\pi(Stab_\psi(p))$ in G/G^{00}, and hence is a closed subgroup of G/G^{00}. Its preimage in G is thus a subgroup of G and coincides with $S_\psi(p) \cdot G^{00}$.

Claim (v). $\cap_\psi(S_\psi(p) \cdot G^{00}) = G^{00}$ (where the ψ's in the left hand side are taken only over M_0).
Proof. By (i), the left hand side contains the right hand side. Let us show the reverse inclusion. Fix $g \in G \setminus G^{00}$. So for every $h \in G^{00}$, $gh^{-1} \notin G^{00}$. So by (ii), for every $h \in G^{00}$ and a realizing $p|M_0$, $gh^{-1}a$ does not realize $p|M_0$. By compactness there is a formula $\psi(x) \in p|M_0$, such that for any $h \in G^{00}$ and a satisfying $\psi(x)$, $gh^{-1}a$ does not satisfy $\psi(x)$. However, for any $g_1 \in Stab_\psi(p)$ clearly there is a such that both a and $g_1 a$ satisfy ψ. Thus for every $g_1 \in S_\psi(p)$ the same is true. So we have shown that (for our given

choice of $g \notin G^{00}$), for all $h \in G^{00}$, $gh^{-1} \notin S_\psi(p)$. Namely $g \notin S_\psi(p) \cdot G^{00}$. This proves Claim (v).

We can now complete the proof of the Lemma. We know that G has the DCC on type-definable subgroups of bounded index (see [4]). Hence by Claim (v), G^{00} is the intersection of finitely many of the $S_\psi(p) \cdot G^{00}$. Thus (as $G^{00} \subseteq Stab_\psi(p) \subseteq S_\psi(p) \cdot G^{00}$) G^{00} is the intersection of finitely many of the $Stab_\psi(p)$. By (iii), G^{00} is definable in $\bar{M}^*$.

Remark 8.3. *(i) The proof above uses only the existence of an f-generic type (rather than a generic type), and hence, by 5.6 and 5.10, Lemma 8.2 goes through assuming only that T has NIP, G is a definable definably amenable group, and G/G^{00} is a compact Lie group.*
(ii) Otero and Peterzil pointed out to us a more direct proof of 8.2, but which makes more use of the fact that G has fsg: Choose a global generic type p of G, and by 4.3 of [13] and the fact that G/G^{00} has DCC, G^{00} is of the form $Stab_{\mathcal{I}}(X_1) \cap ... \cap Stab_{\mathcal{I}}(X_n)$, where the X_i are definable sets in p, and $\mathcal{I}$ denotes the ideal of nongeneric definable subsets of G. (Here $Stab_{\mathcal{I}}(X)$ is the set of $g \in G$ such that the symmetric difference of X and gX is nongeneric, and so is type-definable in $\bar{M}$.) But genericity (so also nongenericity) is "definable" in $\bar{M}^*$, because a definable subset X of G is generic iff for some $g \in G$, $gX \in p$. Hence each $Stab_{\mathcal{I}}(X_i)$ is definable in $\mathcal{M}^*$ which suffices.

The next step is given to us by Lemma 4.3 of [22].

Lemma 8.4. *Let I be the interval $[0, 1)$ in $\bar{M}$ (or K). Then there are one-one definable continuous functions $\gamma_1, .., \gamma_n$ from I to G such that $G = \gamma_1(I) + ... + \gamma_n(I)$ (where $\gamma_j(I)$ denotes the image of I under γ_j).*

In fact the γ_i are generators of the o-minimal fundamental group of G and the proof in [22] has a (co)homological character.

For $j = 1, .., n$, let $I_j = \gamma_j(I)$. As π is a surjective homomorphism we have:

Corollary 8.5. $G/G^{00} = \pi(I_1) + ... + \pi(I_n)$.

We are now in the following interesting situation. G/G^{00} as well as its subsets $\pi(I_j)$ are compact spaces under the logic topology, namely in their capacity as bounded hyperdefinable sets in the saturated model $\bar{M}$, but they are also, by Lemma 8.2, definable (or rather interpretable) sets in the (non-saturated)

weakly o-minimal structure $\bar{M}^*$. We will show in the next step that each $\pi(I_j)$ is a "definable o-minimal set" in $\bar{M}^*$ which piecewise is an interval in $\mathbb{R}$.

Let us first be precise about what we mean by a definable set in an ambient structure being o-minimal. Let N be a structure, and X a definable (or interpretable) set in N. When we say "definable" we mean definable in N with parameters from N. Suppose that $<$ is a definable linear ordering on X. We call X o-minimal in N with respect to $<$, if any definable subset Y of X is a finite union of intervals (a, b) (where possibly $a = -\infty$, $b = +\infty$) and singletons. We will call X strongly o-minimal if in addition there is a finite bound on the number of intervals and points in definable families.

In the next proposition we will mention the existence of definable Skolem functions on a definable set. So let us again be precise about the meaning. Again let N be a structure in language L and X a set definable (or interpretable) in N, for now definable without parameters. We say that X has definable Skolem functions in N (or the formula defining X has definable Skolem functions in $Th(N)$), if for any L-formula $\phi(x, y)$ where x is a variable ranging over X, and y is an arbitrary tuple of variables, there is a partial $\emptyset$-definable function $f_\phi(y)$ such that
$N \models (\forall y)((\exists x \in X \phi(x, y)) \rightarrow (\phi(f_\phi(y), y) \wedge f_\phi(y) \in X))$.

We can also speak of X having Skolem functions in N over some set A of parameters from N. The reader should note that if X has definable Skolem functions in N^{eq} (over some parameters), then X has elimination of imaginaries (over some parameters), namely whenever $Z \subseteq X^n$ and equivalence relation E on Z are definable in N then there is a definable bijection of Z/E with some definable $W \subseteq X^m$.

We now return to the main narrative, with notation following Lemma 8.4. The following Proposition is fundamental. The appendix is devoted to the proof of part (iii).

Proposition 8.6. *(Work in $\bar{M}^*$.) For each j, $\pi(I_j)$ is a finite disjoint union of definable sets $X_1, .., X_r$ and points $c_1, .., c_s$, such that each X_i is equipped with a definable total ordering $<_i$, such that (for each $i = 1, .., r$),*
(i) $(X_i, <_i)$ is (abstractly) isomorphic to $\mathbb{R}$ with the usual ordering,
(ii) X_i is strongly o-minimal (with respect to $<_i$),
(iii) X_i has definable Skolem functions after possibly naming some parameters from $\bar{M}^*$.

Proof. Let us fix j. Note that I_j has a canonical linear ordering coming from the map γ. Let E be the equivalence relation "$x-y \in G^{00}$" on I_j, i.e. coming from π. As E is definable in $\bar{M}^*$ (which has weakly o-minimal theory), each E-class is a union of at most k convex sets for some k. Let Y be the set of elements of I_j which are in the "first" convex subset of their E -class. Then Y is definable, so a finite union of convex definable sets, and $\pi(Y) = \pi(I_j)$. Let E_Y denote the restriction of E to Y. Write Y minimally as a finite disjoint union of (definable) convex sets $Y_1, .., Y_t$. So each E_Y-class is convex and contained in a unique Y_i. Let $X_i = \pi(Y_i)$ and note that $<$ induces a linear ordering $<_i$ on X_i (of course definable in $\bar{M}^*$). Let us restrict our attention to some Y_i such that X_i is infinite.
Claim (I). $<_i$ is dense on X_i (i.e. if $a, b \in X_i$ and $a <_i b$ then there is $c \in X_i$ with $a <_i c <_i b$).
Proof. We may work inside an interval $I' = (a', b')$ of Y_i (i.e. with $\pi(a') \leq_i a$ and $\pi(b') \geq_i b$). So $\pi^{-1}(a) \cap I'$ and $\pi^{-1}(b) \cap I'$ are convex sets which are type-definable in $\bar{M}$ and disjoint. Moreover the first has no greatest element and the second has no smallest element (as every coset of G^{00} is open in G). Hence by compactness (in $\bar{M}$) there is $c' \in I'$ such that $\pi^{-1}(a) \cap I' < c' < \pi^{-1}(b) \cap I'$. Let $c = \pi(c')$.

From Claim (I) we may assume that $(X_i, <_i)$ has no first or last element (by removing them if they exist).

Claim (II). $(X_i, <_i)$ is complete, namely every bounded above (below) subset has a supremum (infimum).
Proof. Again we may work in $\pi(I') \subset X_i$ for some interval $I' = (a', b')$ in Y_i. We consider $\pi(I')$ with the logic topology (equivalently as a closed subset of G/G^{00}). Let (A, B) be a Dedekind cut in $(\pi(I'), <_i)$. For $a \in A$, $A_a = \{x \in \pi(I') : a \leq_i x\}$ is closed in $\pi(I')$, as it is clearly the image of a type-definable (in $\bar{M}$) subset of I. Likewise for $b \in B$, $B_b = \{x \in \pi(I') : x \leq_i b\}$ is closed. Hence by compactness of the space $\pi(I')$, there is a point in the intersection of all the A_a's and B_b's. This suffices.

Claim (III). $(X_i, <_i)$ is separable, namely has a countable dense subset with respect to the ordering $<_i$.
Proof. We know that X with the logic topology is second countable (has a countable basis), because E is given by countably many formulas. (See Remark 1.6 of [4] for example.) We will show that every $<_i$-interval (a, b) in X_i contains an open subset of X. So as X has a countable basis, $(X_i, <_i)$

will have a countable dense subset.

We work one i at a time. The reader should convince himself/herself that there is no harm in assuming that $Y = Y_1$. So $X = X_1$. We relabel $<_1$ as $<_X$. We fix an $<_X$ interval (a, b) in X and we want to find a subinterval which is open in X (with the logic topology). Let Z be the union of all the second convex components of E-classes in I. Let E_Z be the restriction of E to Z. So Z is definable in $\bar{M}^*$ and is a disjoint union of finitely many definable convex subsets $Z_1, .., Z_m$ of I such that each E_Z class is convex and contained in a unique Z_j. Consider $\pi(Z) \cap (a, b)$. If it is finite, then after passing to a smaller interval, we may assume that $\pi(Z) \cap [a, b]$ is empty. This means that for $c \in [a, b]$, $\pi^{-1}(c) \subset Y$. Let $a_0 \in \pi^{-1}(a)$, and $b_0 \in \pi^{-1}(b)$. So (a, b) is the set of E-classes which are contained in the interval (a_0, b_0) in I. Thus (a, b) is open in X and we are finished.

So we may assume (by o-minimality and passing to a smaller interval) that (a, b) is contained in $\pi(Z)$. The ordering $<$ on Z induces a definable ordering $<_2$ on $\pi(Z) \subseteq X$. After ignoring finitely many points, we know (by the description of linear orderings definable on an o-minimal structure), that we can break up (a, b) into finitely many $<_X$intervals, on each of which $<_2$ agrees with $<_X$ or $>_X$. It follows that we can find a subinterval (c, d) of (a, b) and some $j = 1, .., m$ such that $\pi^{-1}([c, d]) \cap Z$ is contained in Z_j, and moreover for some c'_0, d'_0 in Z_j (preimages of c, d), $[c, d]$ is the image under π of the closed $<$-interval between c'_0 and d'_0, and moreover $<_2$ on $[c, d]$ agrees with $<_X$ or $>_X$.

Now if $k = 2$ we are finished: Let $c_0, d_0 \in Y$ be preimages of c, d. Then (c, d) is precisely the set of E-classes contained in $(c_0, d_0) \cup (c'_0, d'_0)$ (or in $(c_0, d_0) \cup (d'_0, c'_0)$ if $<_2$ on $[c, d]$ is $>_X$).

If $k > 2$ we continue, replacing (a, b) by (c, d), and considering now W the union of the third convex components of E-classes. Our choice of (c, d) means that passing to smaller subintervals does not disturb the compatibility with the second convex components of E-classes.

This finishes the proof of Claim (III).

It follows from Claims (I), (II) and (III) that $(X_i, <_i)$ is isomorphic as an ordered set to $\mathbb{R}$ with its usual ordering. So we have proved (i).

(ii) follows quickly. For if Z is a definable (in $\bar{M}^*$) subset of (X_i), then by weak o-minimality of $\bar{M}^*$ $\pi^{-1}(Z) \cap Y_i$ is a finite union of convex sets. By completeness of $(X_i, <_i)$, Z is a finite union of intervals and points. Weak o-minimality of $Th(\bar{M}^*)$ implies that there is a bound on the number of

connected components of Z as it varies in a definable family.

(iii). See the appendix.

Let $X_1^j, ..., X_{r_j}^j$ be the sets obtained for $\pi(I_j)$ in Proposition 8.6. By Proposition 8.6, each X_k^j is strongly o-minimal in $\bar{M}^*$. The reader should convince himself or herself and the general machinery of o-minimality (dimension, cell decomposition, etc.) applies to Cartesian products of the X_k^i and definable (in $\bar{M}^*$) subsets W of such Cartesian products. In fact we will call such a definable set Z, a semi-o-minimal definable set. By (iii) of Proposition 8.6, and the discussion preceding the statement of 8.6, we have "elimination of imaginaries" for such definable sets: namely if Z is a subset of a Cartesian product of the X_k^j and E an equivalence relation on Z, both definable in $\bar{M}^*$ then Z/E is in definable bijection with some definable W which is a subset of a Cartesian product of the X_k^j's. We will apply this to G/G^{00} considered as a group definable (or interpretable) in $\bar{M}^*$. In fact let us write J for this group, so as to distinguish it from G/G^{00} as a bounded hyperdefinable group in $\bar{M}$. By Corollary 8.5 J is in the definable closure (uniformly) of the X_k^j's. Thus there is a definable subset Z of some Cartesian product of the X_k^j, and a definable equivalence relation E on Z such that J is definably isomorphic to Z/E. Hence J is definably isomorphic to some definable W which is a subset of a Cartesian product of the X_k^j's. But J also has a definable group structure, hence is a semi-o-minimal group. Note that J is definably connected (in the sense of having no proper definable subgroup of finite index) as it is divisible. The general theory ([28]) of equipping definable groups in o-minimal structures with a definable group manifold structure applies to J. We conclude using (i) of Proposition 8.6 that J with its definable manifold topology is locally Euclidean, and thus (by the special case of Hilbert's 5th problem due to Pontryagin) is a connected commutative Lie group, whose Lie group dimension coincides with its semi-o-minimal dimension. By Corollary 8.5 the semi-o-minimal dimension of J is at most n. On the other hand a connected commutative Lie group is a finite product of copies of $(\mathbb{R}, +)$ and S_1. As J has the same torsion as n-copies of S_1 it follows that, as a Lie group, J has dimension $\geq n$. It follows that

Corollary 8.7. *J is a semi-o-minimal connected compact Lie group definable in $\bar{M}^*$ with both semi-o-minimal and Lie group dimension n.*

We will point out later that, as expected, the topology on J coincides with the logic topology on G/G^{00}. But we will be able to complete the proof of Theorem 8.1 without using this. The next step is:

Lemma 8.8. *Let Y be a definable (in $\bar{M}$) subset of G of dimension $< n$. Then the semi-o-minimal dimension of $\pi(Y) \subseteq J$ is $< n$.*

Proof. The proof uses the ideas from the proof of Lemma 10.3 of [13]. We give some details anyway. We first note that, given Y as in the hypothesis there is a definable subset Z of $I_1 \times ... \times I_n$ such that the map sending $(x_1, .., x_n) \to x_1 + .. + x_n$ induces a (definable) bijection between Z and Y. So $dim(Z) < n$. Let E be the equivalence relation on G "$x - y \in G^{00}$". By the proof of 8.6 we can and will assume that on each of $I_1, .., I_n$ all E-classes are convex (with respect to canonical total orderings of $I_1, .., I_j$) and that for each $i = 1, .., n$ $\pi(I_i)$ is strongly o-minimal (in $\bar{M}^*$). Let us denote by π_n the natural map from $I_1 \times .. \times I_n$ to $\pi(I_1) \times .. \times \pi(I_n)$.
Claim. $\pi_n(Z)$ has dimension $< n$ in the semi-o-minimal structure $\pi(I_1) \times .. \times \pi(I_n)$ (in $\bar{M}^*$).
Proof. In fact we will prove, by induction on n that if W is any definable (in $\bar{M}$) subset of $I_1 \times .. \times I_n$ of dimension $< n$, then $\pi_n(W) \subset \pi(I_1) \times .. \times \pi(I_n)$ has dimension $< n$. And this is an imitation of the proof of 10.3 in [13]: We may assume W is the graph of a continuous definable function $f : C \to I_n$, where C is an open definable subset of $I_1 \times .. \times I_{n-1}$. Assume for a contradiction that $\pi_n(W)$ has dimension n, in $\pi(I_1) \times .. \times \pi(I_n)$. It follows that $\pi_n(W)$ contains the closure of a subset of the form $U \times (a, b)$, where U is an open rectangular box in $\pi(I_1) \times .. \times \pi(I_{n-1})$ and (a, b) is an interval in $\pi(I_n)$. We may assume that $\pi_{n-1}^{-1}(U) \subseteq C$. Let $c \in I_n$ be such that $a < b < c$, and let $c' \in \pi^{-1}(c) \subseteq I_n$. For any $x \in U$, there are $x_1, x_2 \in \pi_{n-1}(x)$ such that $\pi(f(x_1)) = a$ and $\pi(f(x_2)) = b$. Then $f(x_1) < c' < f(x_2)$, so by "definable connectness" of $\pi_{n-1}^{-1}(x)$ (and continuity of f) there is $x_3 \in \pi_{n-1}^{-1}(x)$ such that $f(x_3) = c'$. We have shown that $\pi_n(\{z \in C : f(z) = c'\})$ contains U, so by induction hypothesis $\{z \in C : f(z) = c'\}$ has dimension $n - 1$. But this is true for infinitely many $c' \in I_n$, a contradiction. The claim is proved.

We now return to our definable subset Y of G such that $Y = \{x_1 + .. + x_n : (x_1, .., x_n) \in Z\}$. As π is a homomorphism, $\pi(Y) \subset G/G^{00} = \{\pi(x_1) + ... + \pi(x_n) : (x_1, .., x_n) \in Z\} = \{y_1 + ... + y_n : (y_1, .., y_n) \in \pi_n(Z)\}$. By the Claim $\pi_n(Z)$ has dimension $< n$, hence $\pi(Y)$ does too. This completes the proof of Lemma 8.8.

Note that the proof of the above lemma (in fact of the Claim) actually yields that $dim(\pi(Y)) \leq dim(Y)$ for Y a definable subset of G.

The final point is (the easy):

Lemma 8.9. *Let $Z \subset J$ be definable in $\bar{M}^*$ of (semi-o-minimal) dimension $< n$, and Haar measurable in G/G^{00}. Then Z has Haar measure 0.*

Proof. The proof will be by induction on $dim(Z) = k$. For $k = 0$, Z is finite so it is clear. Now let Z of dimension $k > 0$.
Claim. For any r there are $a_1, .., a_r \in J$ such that $dim(a_i Z \cap a_j Z) < k$ for all $i \neq j \leq r$.
Proof of claim. Work in a saturated elementary extension of N of $\bar{M}^*$. Let $a_1, .., a_r \in J(N)$ be generic independent (in the o-minimal sense) over the base model, namely $dim(a_1, .., a_r/\bar{M}^*) = nr$. Then one sees easily that $dim(a_i Z(N) \cap a_j Z(N)) < k$ for $i \neq j$. By definability of dimension we can find such $a_1, .., a_r$ in J. This proves the claim.

Note that as Z is measurable in G/G^{00}, so is any intersection of translates of Z. Hence by the induction hypothesis and the claim, each $a_i Z \cap a_j Z$ $(i \neq j)$ has Haar measure 0. But then the measure of the union of the $a_i Z$ is r times the measure of Z. So (by choosing r large) this forces Z to have measure 0. The proof is complete.

Proof of Theorem 8.1. If $Y \subset G$ is definable in $\bar{M}$ with dimension $< n$, then $\pi(Y)$ is closed so measurable in G/G^{00}, but by Lemma 8.8, $\pi(Y)$ has dimension $< n$ as a definable subset of J, hence has Haar measure 0 in G/G^{00} by Lemma 8.9. By 10.5 in [13] we obtain compact domination of G by G/G^{00} (equipped with its Haar measure).

We conclude the paper by proving that the topologies on J and G/G^{00} coincide. Our proof will make use of a little more "theory" some of which is of interest in its own right.

We begin with an arbitrary complete theory T with NIP. For $\phi(x, y) \in L$ by a (complete) global ϕ-type we mean a maximal consistent collection of the formulas of the form $\phi(x, c)$, $\neg\phi(x, c)$ for $c \in \bar{M}$.

Lemma 8.10. *Let M_0 be a small model, $\phi(x, y) \in L$ and $p_0(x)$ a complete global ϕ-type which is $Aut(\bar{M}/M_0)$ invariant. Then p_0 extends to a complete*

global type $p(x)$ which is $Aut(\bar{M}/M_1)$-invariant for some small model M_1 (i.e. in earlier terminology p is invariant).

Proof. As in the proof of Step I of Lemma 5.8, we can find a small model M_1 containing M_0 and a Keisler measure μ_x over M_1 which has a unique extension to a Keisler measure over any larger M_2 which is consistent with p_0. (To say that μ_x is consistent with p_0 means that $\mu_x \cup p_0$ extends to a global Keisler measure. Assuming that M_1 is $|M_0|^+$-saturated, which we can and will do, this is equivalent to requiring that μ_x extends $p_0|M_1$.) So μ_x has a unique extension to a global Keisler measure μ' which extends p_0. Clearly μ' is $Aut(\bar{M}/M_1)$-invariant (as p_0 is). By Proposition 4.6 (see also the proof of 4.5) we obtain some global type $p'(x)$ which is M_1-invariant and extends p_0.

Corollary 8.11. *(Strong Borel definability for invariant ϕ-types.) Let p_0 be a complete global ϕ-type which is M_0-invariant. Then $X = \{b : \phi(x,b) \in p_0(x)\}$ is a finite Boolean combination of type-definable over M_0 sets.*

Proof. By Lemma 8.10 and Proposition 2.6, X is a finite Boolean combination of type-definable over M_1 sets Y_i for some small model M_1 containing M_0. Let Y_i' be $\{b : \exists c \in Y, tp(c/M_0) = tp(b/M_0)\}$. Then Y_i' is type-definable and M_0-invariant hence type-definable over M_0. And X is the same finite Boolean combination of the Y_i'.

Here is the application which will be relevant to our concerns:

Lemma 8.12. *Let $\bar{M}$ be a saturated model, and M_0 a small submodel. Let $\bar{M}^*$ be the Shelah expansion discussed earlier. Suppose that X is definable in $\bar{M}^*$ and is $Aut(\bar{M}/M_0)$-invariant. Then X is a finite Boolean combination of type-definable (over M_0 in the structure $\bar{M}$) sets.*

Proof. . As $Th(\bar{M}^*)$ has quantifier elimination, there is some complete type $p(x)$ over $\bar{M}$ and L-formula $\phi(x,y)$ such that $b \in X$ iff $\phi(x,b) \in p$ (for all $b \in \bar{M}$). Let $p_0 = p|\phi$. So p_0 is a complete global M_0-invariant ϕ -type and we can apply Corollary 8.11

We now return to the setting and notation of Theorem 8.1 and its proof. In particular J is the set G/G^{00} viewed as a definable (compact Lie) group in $\bar{M}^*$ and we just say G/G^{00} for G/G^{00} with the logic topology, another compact Lie group.

Lemma 8.13. *(i) Suppose Z is a definable (in $\bar{M}^*$) subset of J. Then Z is measurable in G/G^{00}.*
(ii) If moreover H is a definable (in $\bar{M}^*$) subgroup of J then H is closed in G/G^{00}.

Proof. (i) Clearly $\pi^{-1}(Z)$ is a definable set in $\bar{M}^*$ and whether or not $x \in \pi^{-1}(Z)$ depends on $tp(x/M_0)$ in $\bar{M}$. Hence it satisfies the assumptions of Lemma 8.12. Hence by Lemma 8.12, $\pi^{-1}(Z)$ is a finite Boolean combination of type-definable sets. We may assume (by multiplying by G^{00}) that each of these type-definable sets is a union of translates of G^{00}. It follows that $Z = \pi(\pi^{-1}(Z))$ is a finite Boolean combination of closed subsets of G/G^{00}, so Borel and measurable.
(ii) As G/G^{00} is separable, any finite Boolean combination of closed sets is a countable intersection of opens, namely a G_δ-set. So applying (i) to H we see that H is a G_δ set in G/G^{00}. Now the closure $\bar{H}$ of H is a subgroup of G/G^{00}. Moreover H and thus each of its translates in $\bar{H}$ is dense in $\bar{H}$. But any two dense G_δ's must intersect. Hence $H = \bar{H}$.

Remark 8.14. *Again the above lemma holds at various levels of generality: (i) holds assuming just T has NIP and (ii) holds if in addition G/G^{00} is separable. Note that (ii) is saying that any subgroup of G which contains G^{00} and is definable in $\bar{M}^*$ is type-definable in $\bar{M}$. In particular the groups $Stab_\psi(p)$ from the proof of Lemma 8.2 are type-definable, so using the DCC we obtain another proof that G^{00} is a finite intersection of the $Stab_\psi(p)$'s.*

Proposition 8.15. *The topologies on J and G/G^{00} coincide.*

Proof. As the group structures on J and G/G^{00} coincide, and both J and G/G^{00} are compact (Hausdorff) groups, it suffices to show that any open neighbourhood U of the identity e, in the sense of J, contains a neighbourhood of e in the sense of G/G^{00}. Let h denote the Haar measure on G/G^{00}. Let W be a *definable* neighbourhood of e in J, such that $W^{-1} \cdot W \cdot W^{-1} \cdot W$ is contained in U. W is clearly generic in J (finitely many translates of W cover J) by compactness of J for example. By Lemma 8.13, W is measurable in G/G^{00}, so $h(W) > 0$. As pointed out in the proof of the CLAIM in section 6 of [13] it follows that $W^{-1} \cdot W$ has interior in G/G^{00}. (A direct proof: As a measurable set is the union of a closed set and a measure 0 set, we may assume that W is closed, hence so is $Z = W^{-1} \cdot W$. If by way of contradiction Z has no interior then the same holds of any finite union of translates of Z.

So we find $a_1, a_2, ...$ in G/G^{00} such that the a_iW are disjoint, contradicting $h(W) > 0$.) So $W^{-1} \cdot W \cdot W^{-1} \cdot W$ contains a neighbourhood of e (in G/G^{00}) as required.

It would be interesting to give a more explicit proof of Proposition 8.15. For example if $dim(G) = 1$ then the analysis in the proof of Proposition 3.5 of [30], yields directly (in hindsight) that the logic topology equals the o-minimal topology on G/G^{00} and coincides with S^1.

Note that once we know Proposition 8.15, then for any definable in $\bar{M}^*$ subset Z of G/G^{00}, Z has Haar measure 0 iff it has dimension $< dim(G/G^{00})$. In any case the proof of Theorem 8.1 together with Proposition 8.15 says that G is o-minimally compactly dominated, i.e.
(*) Let $\bar{M}'$ be the expansion of $\bar{M}$ obtained by adding a predicate for G^{00}. Then G/G^{00} is a semi-o-minimal compact Lie group in $\bar{M}'$ with topology coinciding with its topology as a bounded hyperdefinable group in $\bar{M}$, AND for any definable, in $\bar{M}$ subset X of G, the set of $b \in G/G^{00}$ such that $\pi^{-1}(b)$ meets both X and its complement (which is of course definable in $\bar{M}'$) has dimension $< dim(G/G^{00})$, EQUIVALENTLY, has Haar measure 0.

Let us first remark that (*) also holds for X definable in $\bar{M}^*$. This is because $X = Y \cap G(\bar{M})$ for some subset Y of $G(N)$ definable (with parameters) in a saturated elementary extension N of $\bar{M}$. As $(G/G^{00})(N) = (G/G^{00})(\bar{M})$ and (*) is valid working in N, we deduce that the set of $b \in G/G^{00}$ such that $\pi^{-1}(b)$ meets both X and its complement (in $G(M)$), which again note is a subset of G/G^{00} definable in $\bar{M}^*$, has Haar measure 0. We conclude:

Corollary 8.16. *Let G be definably compact and definable in the saturated o-minimal expansion $\bar{M}$ of a real closed field. Let $\bar{M}^*$ be the Shelah expansion of $\bar{M}$. Let $J = G/G^{00}$ as a definable group in $\bar{M}^*$. Then G is dominated by J in the weakly o-minimal theory $Th(\bar{M}^*)$: namely, working even in a saturated elementary extension of $\bar{M}^*$, if X is a definable subset of G then the set of $b \in J$ such that $\pi^{-1}(b)$ meets both X and its complement, has (o-minimal) dimension $< dim(J)$.*

One can ask what the formal content and implications of the compact domination statement (*) are (either in general, or restricted to the o-minimal context). For example, from [13] we know that if a definable group is compactly dominated then it has the fsg property and a unique invariant Keisler

measure. Of course the proof of Theorem 8.1 (or statement (*)) depends on G having the fsg property as well as the knowledge of torsion points from [6](for definably compact G). It would be interesting to try to recover the torsion points statement directly from compact domination. Namely

Question. (o-minimal context.) Suppose the commutative definably connected definable group G is o-minimally compactly dominated, i.e. statement (*) holds. Can one prove directly that $dim(G) = dim(G/G^{00})$ (and so conclude using the torsion-freeness and divisibility of G^{00} that for each p the p-torsion of G is $(\mathbb{Z}/p\mathbb{Z})^{dim(G)}$)?

A final question is whether given a definable group G in a NIP theory T, there are some reasonable assumptions which imply that G/G^{00} is semi-o-minimal (in the Shelah expansion) or at least o-minimally analysable. A possible assumption would be that G has the DCC on type-definable subgroups of bounded index (equivalently, G/G^{00} with its logic topology is a compact Lie group).

A Appendix: On Skolem functions for o-minimal definable sets

We prove a general result, Proposition A.2 below, and show that it applies to the situation in section 8 to yield Proposition 8.6(iii).

Let us fix a saturated structure N, and a $\emptyset$-definable set X in N such that for some $\emptyset$-definable dense linear ordering without endpoints $<$ on X, X is o-minimal in N with respect to $<$. We will freely adapt o-minimality results for the absolute case (where X is the universe of N) to this relative case.

By X^{eq} we mean $N^{eq} \cap dcl(X)$. It is known that the o-minimal dimension theory on X extends smoothly to X^{eq} (see for example section 3.1 of [25]). Namely for any set A of parameters from N, and $c \in X^{eq}$ we have a natural number $dim(c/A)$, such that dim is subadditive ($dim((c,d)/A) = dim(c/dA)+dim(d/A)$), $dim(c/A) = 0$ iff $c \in acl(A)$, for c an element (rather than tuple) of X, $dim(c/A) = 1$ if $c \notin acl(A)$.

Definition A.1. *We say that X is* untrivial *if whenever $(a_i : i < \omega)$ is a sequence of elements of X such that $a_i \notin acl(a_0, .., a_{i-1})$ for all i then there*

is some n and $b \in X^{eq}$ such that $b \in acl(a_0, .., a_n) \setminus acl(a_0, .., a_{n-1})$ and $a_n \notin acl(b)$.

The main result here is:

Proposition A.2. *Suppose X is untrivial. Then for any elementary substructure N_0 of N, X has definable Skolem functions in N over N_0.*

Before entering the proof of Proposition A.2 we give a lemma which makes use of the fundamental work of Peterzil and Starchenko [26]

Lemma A.3. *Suppose X is untrivial. Then for any $a \in X$ with $a \notin acl(\emptyset)$ there is some interval (c, d) containg a such that the definable set (c, d) has definable Skolem functions in N after naming some parameters from X.*

Proof of Lemma. Let a_i for $i < \omega$ be realizations of $tp(a)$ such that $a_{i+1} \notin acl(a_0, .., a_i)$ for all i. Let n and b be given by untriviality of X. Then a dimension (or independence) calculation shows that $a_n \in acl(a_0, .., a_{n-1}, b)$. We also have $a_n \notin acl(a_0, ..., a_{n-1})$ and $a_n \notin acl(b)$. Let $\bar{b}$ be a finite tuple of elements of X such that $b \in dcl(\bar{b})$ and $\bar{b}$ is independent from a_n over b (i.e. $dim(\bar{b}/ba_n) = dim(\bar{b}/b)$). So we have that $a_n \in acl(a_0, .., a_{n-1}, \bar{b})$ but $a_n \notin acl(a_0, .., a_{n-1})$ and $a_n \notin acl(\bar{b})$. We may assume $a = a_n$. By Claim 1.25 of [24], a is "PS-nontrivial", namely for some infinite open interval I containing a_n and a definable continuous function $F : I \times I \to X$, strictly monotone in each argument. By Theorem 1.1 of [26] there is a convex type definable ordered divisible abelian group $H \subset X$, containing a (where the ordering is the restriction of $<$ to H. Let $c, d \in H$ such that $c < a < d$, and let A be some set of parameters from X containing a, c, d and over which the group operation on H is type-definable. Then it is clear that (c, d) has definable Skolem functions in N over A.

Proof of Proposition A.2. We begin with some reductions. Let N_0 be a (small) definably closed substructure of N (or even N^{eq}). We will say that N_0 satisfies Tarski-Vaught with respect to X if any formula over N_0 which is satisfied in N by some tuple from X is satisfied by a tuple from $N_0 \cap X$. It is clearly enough to restrict ourselves to formulas $\phi(x)$ over N_0 where x is a single variable ranging over X. In any case clearly any elementary substructure of N satisfies Tarski-Vaught with respect to X. So by using compactness, in order to prove A.2 it is enough to prove:

(*) whenever N_0 satisfies Tarski-Vaught with respect to X, then for any tuple $\bar{c}$ from N, $dcl(N_0, \bar{c})$ satisfies Tarski-Vaught with respect to X.

If in the context of (*) $\phi(x)$ is a formula over $N_0\bar{c}$ with x ranging over elements of X, then $\phi(x)$ defines a finite union X_0 of intervals and points from X. The boundary points of X_0 are in $dcl(M_0, \bar{c})$, and X_0 is defined over the set of these boundary points. So in order to prove (*) it suffices to prove:

(**) whenever N_0 satisfies Tarski-Vaught with respect to X, and $\bar{c}$ is a finite tuple of elements of X, then $dcl(N_0, \bar{c})$ satisfies Tarski-Vaught with respect to X.

We can of course prove (**) by adding one element from $\bar{c}$ at a time. Hence it suffices to prove:
(***) whenever N_0 satisfies Tarski-Vaught in N with respect to X, and a is an element of X, then $dcl(N_0, a)$ satisfies Tarski-Vaught in N with respect to X.

The rest of the proof is devoted towards proving (***).

We will suppose that (***) fails and aim for a contradiction. Using o-minimality of X, the failure of (*) is equivalent to the existence of an element $a \in X$, and $b \in dcl(N_0, a)$ (where b might be $+\infty$ or $-\infty$) such that the formula $a < x < b$ (if $b > a$) or $b < x < a$ (if $b < a$) isolates a complete type over N_0a. There is no harm in assuming that $b > a$. We can write $b = g(a)$ for some N_0-definable (possibly constant) function g on P. There are two cases depending whether or not $g(a) \in dcl(N_0)$.
Case (i). $g(a) \notin dcl(N_0)$.
Let $p = tp(a/N_0)$ and P the set of realizations of p in N. Note that $g(a)$ realizes p too, and that g is an N_0-definable strictly monotone increasing function from P onto itself.
Claim I. There is no $\bar{e}$ from X and $N_0\bar{e}$-definable function $f_{\bar{e}}$ such that for all $a' \in P$, $a' < f_{\bar{e}}(a') < g(a')$.
Proof. Otherwise by compactness there is $\theta(x) \in p$, such that
$N \models \forall x(\theta(x) \rightarrow (x < f_{\bar{e}}(x) < g(x)))$. As N_0 satisfies Tarski-Vaught in N with respect to X, there is $\bar{e}'$ from N_0, such that
$N \models \forall x(\theta(x) \rightarrow x < f_{\bar{e}'}(x) < g(x))$.
But then $a < f_{\bar{e}'}(a) < g(a)$, contradicting the fact that $a < x < g(a)$ isolated a complete type over N_0. Claim I is proved.

Claim II. The interval $(a, g(a))$ has definable Skolem functions in N over some set of parameters from X.

Proof. As p is a complete nonalgebraic 1-type of X over N_0, it follows from Lemma A.3 that for some c with $a < c < g(a)$, (a, c) has definable Skolem functions in N after naming some parameters from X. It follows that for any a' realizing p and $c' \in (a', g(a'))$, (a', c') has definable Skolem functions in N over parameters from X. Now (with $c \in (a, g(a))$), $a < c < g(a) < g(c)$. So both (a, c) and $(c, g(a))$ have definable Skolem functions (after naming parameters). Hence so does $(a, g(a))$.

Claim III. There are realizations a_i of p for $i < \omega$ such that, writing I_k for the interval $(a_k, g(a_k))$, we have that for all k, $dcl_{N_0}(I_0 \cup ... \cup I_{k-1} \cup \{a_0, a_1, .., a_{k-1}, a_k\}) \cap I_k = \emptyset$. (Where $dcl_{N_0}(A)$ denotes definable closure in N of $A \cup N_0$.)
Proof. Suppose we have already constructed $a_0, .., a_{k-1}$, and suppose for a contradiction that
($\sharp$): for all $a \in P$, $dcl_{N_0}(I_0 \cup .. \cup I_{k-1} \cup \{a_0, .., a_{k-1}, a\}) \cap (a, g(a)) \neq \emptyset$.
By Claim II, let $\bar{e}$ be a tuple from X such that each of $I_0, .., I_{k-1}$ (and so their union) has definable Skolem functions in N over $\bar{e}$, and we may assume that $a_0, .., a_{k-1}$ are in $\bar{e}$. Note that $\cup_{j=0,..,k-1} I_j$ has definable Skolem functions over $N_0\bar{e}$. Now fix $a \in P$. By ($\sharp$) there is an N_0-definable function $f(w, z_0, .., z_{k-1}, x)$ such that there are tuples $c_0, .., c_{k-1}$ from I_0,..,I_{k-1} respectively such that $f(\bar{e}, c_0, .., c_{k-1}, a) \in (a, g(a))$. Hence there are such $c_0, .., c_{k-1}$ which are in addition contained in $dcl_{N_0}(\bar{e}, a)$. So we have shown that for every $a \in P$, $dcl_{N_0}(\bar{e}, a) \cap (a, g(a)) \neq \emptyset$. By compactness there is a $N_0\bar{e}$-definable function $f_{\bar{e}}(-)$ such that for all $a \in P$, $f_{\bar{e}}(a) \in (a, g(a))$. This contradicts Claim I. Claim III is proved.

It is rather easy to see that in Claim III, $a_k \notin acl_{N_0}(a_0, .., a_{k-1})$. (Alternatively the construction in Claim III goes through with this additional constraint). So we can apply the untriviality of X to find n and $b \in X^{eq}$ such that $b \in acl_{N_0}(a_0, ..., a_n) \backslash acl_{N_0}(a_0, .., a_{n-1})$ (whence $a_n \in acl_{N_0}(a_0, .., a_{n-1}, b)$), and $a_n \notin acl_{N_0}(b)$. This leads quickly to a contradiction as we now show. At this point we will for notational simplicity work over N_0.

First choose c_i, d_i for $i < n$ such that $g^{-1}(a_i) < c_i < a_i < d_i < g(a_i)$, and $(c_0, d_0, ..., c_{n-1}, d_{n-1})$ is independent from (a_n, b) (in the o-minimal sense). Then as $a_n \notin acl(b)$, also $a_n \notin acl(c_0, d_0, ..., c_{n-1}, d_{n-1}, b)$. On the other hand, as $a_n \in dcl(a_0, .., a_{n-1}, b)$ there is a $\emptyset$-definable function f such that $\models \exists x_0, .., x_{n-1}((\wedge_{i<n} c_i < x_i < d_i) \wedge f(b, x_0, .., x_{n-1}) = a_n)$. As $tp(a_n/c_0, d_0, .., c_{n-1}, d_{n-1}, b)$ is not algebraic its set of realizations contains an

open interval around a_n. Hence we can find some $b_n \in I_n$, and $b_0, .., b_{n-1} \in X$ such that for each $i < n$, $c_i < b_i < d_i$ and $f(b_0, .., b_{n-1}, b) = b_n$. Now put $b'_i = b_i$ if $b_i \geq a_i$, and $b'_i = g(b_i)$ if $b_i < a_i$. So $b'_i \in I_i \cup \{a_i\}$ for each i. As $b \in dcl(a_0, .., a_n)$ we conclude that $b_n \in dcl(b'_0, .., b'_{n-1}, a_0, .., a_{n-1}, a_n)$. As $b_n \in I_n$ this contradicts the construction of the a_i. This contradiction completes the proof under Case (i).

Case (ii). $g(a) \in dcl(N_0)$.
So $g(a)$ is either a point of X in N_0 or $+\infty$. Let $d = g(a) \in N_0 \cup \{\infty\})$. So clearly $p(x) = tp(a/N_0)$ is the complete type over N_0 saying that $x \in X$, $x < d$ and $x > c$ for all $c \in X(N_0)$ such that $c < d$.
Claim IV. P (the set of realizations of p) is indiscernible over N_0. Namely for each n, $p(x_1) \cup p(x_2) \cup .. \cup p(x_n) \cup \{x_1 < x_2 < .... < x_n\}$ extends to a unique complete type over N_0.
Proof. By induction. The case $n = 2$ is given to us, as $a < x < d$ isolates a complete 1-type over $N_0 a$. Now assume true for $n \geq 2$, and prove for $n + 1$. Let $a_1 < a_2 < .. < a_n$ realize p. It suffices to show that $a_n < x < d$ isolates a complete 1-type over $N_0 \cup \{a_1, .., a_n\}$, and for that it is enough to prove that $dcl(N_0, a_1, .., a_n) \cap (a_n, d) = \emptyset$. If not $a_n < f(a_1, .., a_n) < d$ for some N_0-definable function f. But by induction hypothesis, $tp(a_n/N_0 a_1..a_{n-1})$ is isolated by $a_{n-1} < x < d$. Hence $N \models (\forall x)((a_{n-1} < x < d) \rightarrow (x < f(a_1, .., a_{n-1}, x) < d))$. Now we use the hypothesis that N_0 satisfies Tarski-Vaught in N with respect to X to find $a'_1 < ... < a'_{n-1}$ in $X(N_0)$, all less than d such that $x < f(a'_1, .., a'_{n-1}, x) < d$ for any x with $a'_{n-1} < x < d$. But our realization a of p is such an x, and we get a contradiction to our Case(ii) hypothesis. This finishes the proof of Claim IV.

However Claim IV is clearly incompatible with the untriviality of X. So the proof in Case (ii) is also complete, as is the proof of Theorem A.2.

We can now give:
PROOF OF PROPOSITION 8.6(iii). We return to the context and notation of 8.6. We already know that X_i with its ordering $<_i$ is strongly o-minimal in $\bar{M}^*$, and is also a definable subset of G/G^{00}. As in section 8, we use J to denote G/G^{00} as a definable (or intepretable) group in $\bar{M}^*$. We will show that X_i is untrivial (as a definable strongly o-minimal set).

Let N_0 denote the structure $\bar{M}^*$, and let N be a saturated elementary

extension. Then $X_i(N)$ is o-minimal in N and a definable subset of the definable group $J(N)$. In fact we will work over the parameter set N_0 over which all the data are anyway defined.

Let $(a_i : i < \omega)$ be elements of $X_i(N)$ which are algebraically independent over N_0. For each i, let $b_i = a_0 \cdot .. \cdot a_i$ where the product is in the sense of the group $J(N)$. So $b_i \in J(N)$. On the other hand $b_i \in dcl(a_0, .., a_i)$ so can be viewed as (is interdefinable with) an element of $X_i(N)^{eq}$. So we can talk about $dim(b_i)$. Let n be as in 8.6, namely the dimension of the original o-minimal group G. By Corollary 8.5 and Proposition 8.6, each element of $J(N)$ is in the definable closure of n elements of G, each of which is a member of some o-minimal definable set (defined over N_0). It follows easily that $dim(b_i)$ is bounded by n. It is easy to see that $dim(b_i)$ is nondecreasing (as the a_i are independent) and $\leq i + 1$. Hence for some $m \leq n$, we have $dim(b_{m-1}) = dim(b_m) = m$. Then $b_m \in acl(a_0, .., a_m)$, and $b_m \notin acl(a_0, .., a_{m-1})$ (for otherwise we conclude that $a_m \in acl(a_0, .., a_{m-1})$). Finally $a_m \notin acl(b_m)$. For otherwise, also b_{m-1} is in $acl(b_m)$ whereby b_m is interalgebraic with (b_{m-1}, a_m) hence has dimension $m + 1$, which it doesn't. We have proved that $X_i(N)$ is untrivial. So we can apply Proposition 8.2 to obtain 8.6(iii).

Remark A.4. *(i) The argument above yields: Suppose X is a definable strongly o-minimal set in a structure N and X definably embeds in a definable group G in N where G has finite thorn rank. Then X is untrivial, so has definable Skolem functions in N after naming parameters.,*
(ii) A recent preprint by Hasson and Onshuus proves that a strongly o-minimal definable set X in a structure N is "stably embedded" in N. So in Proposition A.2 one may assume X to be the universe of N, and the set-up of the proof, although not its content, can be a little simplified.

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