Geometric Variational Problems

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Chapter 1

Introduction

These are notes of courses given in 2015 and 2018 at the University of Freiburg. In 2007 Tobias Lamm lectured on this topic at FU Berlin, and I followed his notes. The subject of the course is the regularity theory for two-dimensional geometric variational problems, in particular compensation methods due to Henry Wente, Frédéric Hélein and Tristan Rivière. Along the lines we introduce certain Hardy and Lorentz spaces, and present the construction of a Coulomb gauge following Karen Uhlenbeck.

Chapter 2

Geometric variational problems

The purpose of this Chapter is to introduce the key examples of two-dimensional geometric variational problems. More background information on these examples can be found in the book of Jost [33].

2.1 The two-dimensional Dirichlet energy

The Dirichlet energy of a map $u \in C^1(U, \mathbb{R}^n)$ on an open subset $U \subset \mathbb{R}^2$ is defined by

$$\mathcal{E}(u) = \frac{1}{2}\int_U |Du|^2 \quad \text{where } |Du|^2 = \operatorname{tr}(Du^{\mathrm{T}} Du). \tag{2.1}$$

The definition applies in any dimension, but the following interesting feature is specific to the case of dimension two.

Theorem 2.1.1 (Conformal invariance). *Let $f : U \to V$ be a conformal diffeomorphism between open sets $U, V \subset \mathbb{R}^2$. Then*

$$\mathcal{E}(v \circ f) = \mathcal{E}(v) \quad \textit{for all } f \in C^1(V, \mathbb{R}^n).$$

Proof. Denote by $g = (g_{ij}) = Df^{\mathrm{T}} Df$ the Gram matrix (or induced Riemannian metric) associated to f. The condition that f is conformal means that the tracefree part g° vanishes, in other words we can write

$$g_{ij} = e^{2\lambda}\delta_{ij} \quad \text{where } \lambda : U \to \mathbb{R}.$$

Using $Df^{\mathrm{T}} = e^{2\lambda} Df^{-1}$ we compute

$$|D(v \circ f)|^2 = \operatorname{tr}\big(Df^{\mathrm{T}} Dv^{\mathrm{T}} \circ f Dv \circ f Df\big) = \operatorname{tr}\Big(Df Df^{\mathrm{T}} (Dv^{\mathrm{T}} Dv) \circ f\Big) = e^{2\lambda}\, |Dv|^2 \circ f.$$

But $e^{2\lambda} = \sqrt{\det g} = |\det Df|$ in dimension two, thus the transformation formula yields

$$\frac{1}{2}\int_U |D(v \circ f)|^2 = \frac{1}{2}\int_U |Dv|^2 \circ f\, |\det Df| = \frac{1}{2}\int_V |Dv|^2.$$

□

For $u \in C^1(U,\mathbb{R}^n)$ with $\mathcal{E}(u) < \infty$ and $\varphi \in C^1_c(U,\mathbb{R}^n)$ we have the first variation formula

$$\frac{d}{dt}\mathcal{E}(u+t\varphi)|_{t=0} = \int_U \langle Du, D\varphi\rangle. \tag{2.2}$$

Assuming further $u \in C^2(U,\mathbb{R}^n)$ we can integrate by parts to obtain

$$\frac{d}{dt}\mathcal{E}(u+t\varphi)|_{t=0} = -\int_U \langle \Delta u, \varphi\rangle.$$

In other words the Euler-Lagrange operator for the Dirichlet energy is $-\Delta$. We say that u is a critical point of the Dirichlet energy if the first variation in (2.2) vanishes for all $\varphi \in C^1_c(U,\mathbb{R}^n)$. For $u \in C^2(U,\mathbb{R}^n)$ this is equivalent to u being harmonic, i.e.

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{on } U. \tag{2.3}$$

From the conformal invariance of the Dirichlet energy one derives an equivariance property of the Laplacian as follows. Let $f : U \to V$ be a conformal diffeomorphism and $\varphi \in C^1_c(U,\mathbb{R}^n)$. Then by Theorem 2.1.1

$$\begin{aligned} -\int_U \langle \Delta(v\circ f), \varphi\rangle &= \frac{d}{dt}\mathcal{E}(v\circ f + t\varphi)|_{t=0} \\ &= \frac{d}{dt}\mathcal{E}(v + t\varphi\circ f^{-1})|_{t=0} \\ &= -\int_V \langle \Delta v, \varphi\circ f^{-1}\rangle \\ &= -\int_U \langle (\Delta v)\circ f, \varphi\rangle\,|\det Df|. \end{aligned}$$

Putting $g_{ij} = e^{2\lambda}\delta_{ij}$ we get $|\det Df| = e^{2\lambda}$ as above and conclude

$$(\Delta v)\circ f = e^{-2\lambda}\Delta(v\circ f) \text{ for any conformal diffeomorphism } f : U \to V. \tag{2.4}$$

How comes geometry into play? For $u \in C^1(U,\mathbb{R}^n)$ with $n \geq 2$ the area functional is

$$\mathcal{A}(u) = \int_U \sqrt{\det g} \quad \text{where } g = (g_{ij}) = \langle \partial_i u, \partial_j u\rangle. \tag{2.5}$$

In fact $\sqrt{\det g}$ is the Jacobian Ju, we have explicitely

$$\sqrt{\det g} = \sqrt{|\partial_1 u|^2|\partial_2 u|^2 - \langle \partial_1 u, \partial_2 u\rangle^2} = |\partial_1 u \wedge \partial_2 u| = Ju.$$

In the following we asssume that u is immersed and hence g invertible. The first variation of the area in direction $\varphi \in C^1_c(U,\mathbb{R}^n)$ is then computed as follows. First

$$\frac{\partial}{\partial t}\langle \partial_i(f+t\varphi), \partial_j(f+t\varphi)\rangle|_{t=0} = \langle \partial_i f, \partial_j\varphi\rangle + \langle \partial_j f, \partial_i \varphi\rangle.$$

Using the formula $\partial_t \det g = \det g\,\mathrm{tr}(g^{-1}\partial_t g)$ we obtain, writing $g^{-1} = (g^{ij})$,

$$\frac{d}{dt}\mathcal{A}(u+t\varphi)|_{t=0} = \int_U g^{ij}\langle \partial_j f, \partial_i\varphi\rangle\,\sqrt{\det g}.$$

Note that g^{-1} is again symmetric. Partial integration yields

$$\frac{d}{dt}\mathcal{A}(u+t\varphi)|_{t=0} = -\int_U \langle \vec{H}, \varphi\rangle \sqrt{\det g} \quad \text{where } \vec{H} = \frac{1}{\sqrt{\det g}}\partial_i\big(\sqrt{\det g}\, g^{ij}\partial_j u\big). \tag{2.6}$$

We now compare the area functional to the Dirchlet energy. First we have

$$Ju = |\partial_1 u \wedge \partial_2 u| \leq |\partial_1 u|\,|\partial_2 u| \leq \frac{1}{2}(|\partial_1 u|^2 + |\partial_2 u|^2) = \frac{1}{2}|Du|^2.$$

Thus $\mathcal{A}(u) \leq \mathcal{E}(u)$ with equality if and only if u satisfies the conformality relations, that is

$$\langle \partial_1 u, \partial_2 u\rangle = 0 \quad \text{and} \quad |\partial_1 u|^2 = |\partial_2 u|^2. \tag{2.7}$$

If u is immersed and conformally parametrized, with induced metric $g_{ij} = e^{2\lambda}\delta_{ij}$ and Jacobian $Ju = e^{2\lambda}$, then the mean curvature vector becomes

$$\vec{H} = e^{-2\lambda}\Delta u. \tag{2.8}$$

It follows that a conformally parametrized immersion $u : U \to \mathbb{R}^n$ is a minimal surface if and only if u is Euclidean harmonic. These facts provide a close relation between the Dirichlet energy and the area functional in two dimensions.

The question whether any immersed surface admits a reparametrization which is conformal is of fundamental importance; it has local and global aspects. In 1825 Gauß proved that any real-analytic surface admits locally a conformal reparametrization; this was extended to surfaces of class $C^{1,1}$ by Lichtenstein in 1911. For oriented surfaces the parameter changes are then holomorphic, so that the immersion induces a global complex structure on the parameter domain, which becomes naturally a Riemann surface. The construction of a conformal parametrization is much simpler for minimal surfaces, see [12].

The following result goes in a different direction: it shows that certain critical points of geometric variational problems satisfy automatically the conformality relations. The result plays a crucial rôle in proving that the classical approach to the Plateau problem actually produces a minimal surface. Moreover, this generalizes to other problems for surfaces with precribed mean curvature. Historically, the result was also relevant in the regularity theory, because earlier regularity proofs needed to assume conformality [24]. In the literature one often finds the term *stationary* for a map which is critical with respect to variations of the independent variables.

Theorem 2.1.2. *Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Assume that $u \in C^1(D, \mathbb{R}^n)$ has finite energy and satisfies*

$$\frac{d}{dt}\mathcal{E}(u \circ \phi_t)|_{t=0} = 0,$$

for any smooth family $\phi : \overline{D} \times (-\varepsilon, \varepsilon) \to \overline{D}$ of diffeomorphisms $\phi_t = \phi(\cdot, t)$ with $\phi_0 = \mathrm{id}_{\overline{D}}$. Then u satisfies the conformally relations (2.7).

It is only asserted that u is *weakly* conformal, leaving open whether u is immersed or not. However if u is also harmonic, then points with vanishing Jacobian are isolated and have the character of branchpoints, see [12]. We also point out that it is crucial that D is (conformally equivalent to) a disk.

Proof. We assume that $u : \mathbb{H} \to \mathbb{R}^n$ where $\mathbb{H}$ is the upper half plane. There is an explicit conformal equivalence between D and $\mathbb{H}$, so that the result transfers to D via conformal invariance. Consider a vector field $X \in C^\infty_c(\mathbb{R}^2,\mathbb{R}^2)$ with $X \circ \tau = \tau X$, where $\tau(x,y) = (x,-y)$, in particular $X_2(x,0) = 0$ for all $x \in \mathbb{R}$. The associated flow $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is smooth, and $\phi_t = \phi(\cdot,t)$ is diffeomorphic with inverse ϕ_{-t}. Uniqueness for the initial value problem implies

$$\phi(\tau(z),t) = \tau(\phi(z,t)) \quad \text{for all } z \in \mathbb{R}^2,\ t \in \mathbb{R}.$$

In particular $\phi_t(\mathbb{R}) = \mathbb{R}$ and $\phi_t(\mathbb{H}) = \mathbb{H}$. Substituting $\zeta = (\xi,\eta) = \phi_{-t}(z)$ we have

$$\mathcal{E}(u \circ \phi_t) = \frac{1}{2}\int_{\mathbb{H}} |Du(\phi_t(\zeta))D\phi_t(\zeta)|^2\, d\xi d\eta = \frac{1}{2}\int_{\mathbb{H}} |Du(z)D\phi_{-t}(z)^{-1}|^2 \det D\phi_{-t}(z)\, dxdy.$$

We differentiate under the integral at $t = 0$. As $\phi(z,t) = z$ for $z \notin \operatorname{spt} X$, the integrand and its derivative for $t \in (-\varepsilon,\varepsilon)$ are bounded by $C\,|Du|^2$ which is integrable. We compute

$$\begin{aligned}
\frac{\partial}{\partial t} D\phi_{-t}(z)\cdot v|_{t=0} &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\phi_{-t}(z+sv)|_{s=0,t=0}\\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\phi_{-t}(z+sv)|_{t=0,s=0}\\
&= -\frac{\partial}{\partial s}X(z+sv)|_{s=0}\\
&= -DX(z)\cdot v.
\end{aligned}$$

This implies

$$\frac{\partial}{\partial t} D\phi_{-t}(z)^{-1}|_{t=0} = DX(z), \quad \frac{\partial}{\partial t}\det D\phi_{-t}(z) = -\operatorname{div} X(z).$$

Putting $X = (a,b)$ we find

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u\circ\phi_t)|_{t=0} &= \int_{\mathbb{H}} \big(\langle Du(z), Du(z)DX(z)\rangle - \frac{1}{2}|Du(z)|^2 \operatorname{div} X(z)\big)\, dxdy\\
&= \int_{\mathbb{H}} \big(\langle \partial_i u, \partial_j u\rangle \partial_i X^j - \frac{1}{2}|Du|^2 \partial_i X^i\big)\, dxdy\\
&= \int_{\mathbb{H}} \big(\frac{1}{2}(|u_x|^2 - |u_y|^2)a_x + \langle u_x, u_y\rangle a_y\big)\, dxdy\\
&\quad + \int_{\mathbb{H}} \big(\langle u_x, u_y\rangle b_x - \frac{1}{2}(|u_x|^2 - |u_y|^2)b_y\big)\, dxdy.
\end{aligned}$$

Taking $X|_{\mathbb{H}} \in C^\infty_c(\mathbb{H},\mathbb{R}^2)$ arbitrary, we see that the function

$$h(z) = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle \in L^1(\mathbb{H},\mathbb{C})$$

is a weak solution to the Cauchy-Riemann equations, and hence holomorphic. Next we take $X = (\varphi_y, \varphi_x)$ where $\varphi \in C^\infty_c(\mathbb{R}^2)$ is odd, that is $\varphi(x,-y) = -\varphi(x,y)$. Then X is admissible whence

$$\int_{\mathbb{H}} \langle u_x, u_y\rangle \Delta\varphi\, dxdy = 0.$$

Using odd reflection the function $\langle u_x, u_y\rangle$ extends to a weakly harmonic function which is integrable on $\mathbb{R}^2$. By the mean value formula, we conclude that $\langle u_x, u_y\rangle$ is identically zero. The Cauchy-Riemann equations now yield that $h(z)$ is constant and in fact vanishes, again by integrability. □

2.2 Surfaces of prescribed mean curvature in $\mathbb{R}^3$

Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. For $u \in C^1(\overline{D}, \mathbb{R}^3)$ we consider the functional

$$\mathcal{F}(u) = \frac{1}{2}\int_D |Du|^2 + \int_D u^*\omega, \tag{2.9}$$

where $\omega \in C^1(\mathbb{R}^3, \Lambda^2(\mathbb{R}^3))$ is a given two-form. To interpretate the second term geometrically, let us assume for simplicity that $u \in C^2(\overline{D}, \mathbb{R}^3)$. Consider the cone over u defined by

$$F : D \times [0,1] \to \mathbb{R}^3,\ F(z,t) = tu(z).$$

Writing $d\omega = H\, dV_{\mathbb{R}^3}$ where $dV_{\mathbb{R}^3} = dX^1 \wedge dX^2 \wedge dX^3$ we get by Stokes' theorem

$$\begin{aligned}
\int_{D\times[0,1]} F^*(H dV_{\mathbb{R}^3}) &= \int_{D\times[0,1]} F^* d\omega \\
&= \int_{D\times[0,1]} dF^*\omega \\
&= \int_D F(\cdot,1)^*\omega - \int_D F(\cdot,0)^*\omega + \int_{\partial D\times[0,1]} F^*\omega \\
&= \int_D u^*\omega + \int_{\partial D\times[0,1]} F^*\omega.
\end{aligned}$$

The C^2 assumption was used when interchanging F^* and d. Introducing the multiplicity function $\theta_F(X) = \sum_{F(z,t)=X} \operatorname{sign}\det DF(z,t)$, we get by the transformation formula

$$\int_{F(D\times[0,1])} H\theta_F\, d\mathcal{L}^3 = \int_D u^*\omega + \int_{\partial D\times[0,1]} F^*\omega.$$

The second integral on the right depends only on $u|_{\partial D}$, thus it reduces to a constant when restricting to a class of maps with prescribed boundary values. Up to that constant, the integral $\int_D u^*\omega$ then corresponds to the volume of the cone, weighted with the function H and counted with multiplicities. A special case of interest is

$$\omega = \frac{1}{3} X \llcorner dV_{\mathbb{R}^3} = \frac{1}{3}\big(X^1 dX^2 \wedge dX^3 + X^2 dX^3 \wedge dX^1 + X^3 dX^1 \wedge dX^2\big),$$

in other words $\omega(Y,Z) = \frac{1}{3} dV_{\mathbb{R}^3}(X,Y,Z) = \frac{1}{3}\langle X, Y\wedge Z\rangle$. Then $d\omega = dV_{\mathbb{R}^3}$ and $F^*\omega = 0$ on $\partial D \times [0,1]$ no matter what boundary condition, namely we have

$$F^*\omega\Big(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\Big) = \omega\Big(\frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial t}\Big) = \frac{1}{3}\Big\langle tu\,,\, t\,\frac{\partial u}{\partial \theta} \wedge u\Big\rangle = 0.$$

Therefore in this case we get the classical volume, with multiplicities,

$$\frac{1}{3}\int_D \langle u, u_x \wedge u_y\rangle\, dxdy = \int_D u^*\omega = \int_{F(D\times[0,1])} \theta_F\, d\mathcal{L}^3. \tag{2.10}$$

Next we calculate the first variation of the functional.

Lemma 2.2.1. *Let $\mathcal{F}(u)$ be the functional in (2.9), and put $d\omega = H\, dV_{\mathbb{R}^3}$. Then for any $u \in C^2(\overline{D}, \mathbb{R}^3)$ and $\varphi \in C^2_c(D, \mathbb{R}^3)$ we have*

$$\frac{d}{d\varepsilon}\mathcal{F}(u+\varepsilon\varphi)|_{\varepsilon=0} = \int_D \langle Du, D\varphi\rangle + \int_D H \circ u\, \langle u_x \wedge u_y, \varphi\rangle. \tag{2.11}$$

Proof. We consider the affine homotopy

$$F : D \times [0,\varepsilon] \to \mathbb{R}^3,\ F(z,t) = u(z) + t\varphi(z).$$

Applying Stokes' formula on $D \times [0,\varepsilon]$ we get

$$\int_{D\times[0,\varepsilon]} F^*(H\, dV_{\mathbb{R}^3}) = \int_D (u+\varepsilon\varphi)^*\omega - \int_D u^*\omega + \int_{\partial D\times[0,\varepsilon]} F^*\omega.$$

The last integral on the right vanishes since $\partial_t F(z,t) = \varphi(z) = 0$ on $\partial D \times [0,\varepsilon]$. Taking the derivative at $\varepsilon = 0$ yields

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}\int_D (u+\varepsilon\varphi)^*\omega|_{\varepsilon=0} &= \frac{\partial}{\partial\varepsilon}\int_0^\varepsilon\int_D H(F(x,y,t))\det DF(x,y,t)\, dxdydt|_{\varepsilon=0} \\
&= \int_D H(u(x,y))\det DF(x,y,0)\, dxdy \\
&= \int_D H(u(x,y))\det(u_x, u_y, \varphi)\, dxdy \\
&= \int_D H(u(x,y))\langle u_x \wedge u_y, \varphi\rangle\, dxdy.
\end{aligned}$$

The claim follows by combining with the first variation of the Dirichlet energy. □

We see that regular critical points of $\mathcal{F}$ are solutions of the elliptic system

$$\Delta u = (H \circ u)\ u_x \wedge u_y \quad \text{in } D. \tag{2.12}$$

If u is in addition a conformal immersion, with induced metric $g_{ij} = e^{2\lambda}\delta_{ij}$, then we can rewrite the equation in the form

$$e^{-2\lambda}\Delta u = (H \circ u)\ \nu \quad \text{where } \nu = \frac{u_x \wedge u_y}{|u_x \wedge u_y|}.$$

We know from (2.8) that the left hand side is the mean curvature vector, hence the surface u has prescribed mean curvature $H \circ u$. A special case is when H is constant, then u is called a constant mean curvature or CMC surface. One may then take as differential form

$$\omega = \frac{H}{3} X \llcorner dV_{\mathbb{R}^3} \quad (H \text{ constant}).$$

The partial differential equation (2.12) is called the prescribed mean curvature or constant mean curvature equation, respectively. This terminology does not require solutions to be conformally parametrized, but the geometric interpretation is available only then. To obtain geometric solutions, one may potentially use Theorem 2.1.2. Namely for any C^1 diffeomorphism $\phi : D \to D$ which preserves orientation the transformation formula yields

$$\int_D (u\circ\phi)^*\omega = \int_D \phi^* u^*\omega = \int_D u^*\omega.$$

If u is a critical point of $\mathcal{F}$ with respect to variations $u \circ \phi_t$, then one conludes

$$\frac{d}{dt}\mathcal{E}(u \circ \phi_t)|_{t=0} = \frac{d}{dt}\Big(\mathcal{F}(u \circ \phi_t) - \int_D \phi_t^* \omega\Big)|_{t=0} = 0.$$

The conformality relations now follow from Theorem 2.1.2. This applies, for example, to minimizers of $\mathcal{F}$ under Plateau boundary conditions.

To derive the prescribed mean curvature equation from the vanishing of the first variation we have imposed strong regularity assumptions on the function u. In contrast, the existence theory will only give us functions $u \in W^{1,2} \cap L^\infty(D, \mathbb{R}^3)$, say. It is a key issue to show that these weak solutions are regular. The special case of constant mean curvature was solved by Wente in 1969 [65], whereas the general case of variable mean curvature (for which $H \circ u$ is a priori only bounded and measurable) was proved by Riviére much later in 2007 [48].

2.3 Harmonic maps

As second example we now introduce harmonic maps. Let $M \subset \mathbb{R}^n$ be an m-dimensional smooth compact submanifold where $1 \leq m \leq n-1$. By definition, a harmonic map $u : D \to M$ is a critical point of the Dirichlet energy under the constraint that $u(D) \subset M$; that is only variations staying in M are allowed.

We first consider the special case of a round sphere $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$. We then have the projection $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{S}^{n-1}$, $\pi(x) = \frac{x}{|x|}$. For $x \in \mathbb{S}^{n-1}$ its derivative is given by

$$D\pi(x)v = P^\top(x)v = v - \langle v, x\rangle x.$$

Now assume that $u \in C^1(D, \mathbb{R}^n)$ has finite Dirichlet energy and maps into $\mathbb{S}^{n-1}$. For $\varphi \in C^1_c(D, \mathbb{R}^n)$ we compute using the chain rule

$$\frac{d}{dt}\mathcal{E}\big(\pi \circ (u + t\varphi)\big)|_{t=0} = \int_D \big\langle Du, D(D\pi(u)\varphi)\big\rangle = \int_D \langle Du, D(\varphi - \langle \varphi, u\rangle u)\rangle.$$

Note that $|u + t\varphi| \geq 1 - |t|\, \|\varphi\|_{C^0(D)} > 0$ for $|t|$ small. Now $\langle Du, u\rangle = 0$ so that

$$\frac{d}{dt}\mathcal{E}\big(\pi \circ (u + t\varphi)\big)|_{t=0} = \int_D \langle Du, D\varphi\rangle - \int_D \big\langle |Du|^2 u, \varphi\big\rangle.$$

If $u \in C^2(D, \mathbb{R}^n)$ we can integrate by parts to get the harmonic map equation

$$-\Delta u = |Du|^2\, u. \tag{2.13}$$

To generalize this computation to the case of a general submanifold $M \subset \mathbb{R}^n$, we need the following tubular neighborhood lemma which is stated without proof.

Lemma 2.3.1. *Let $M \subset \mathbb{R}^n$ be a compact submanifold of class C^2. There exists a neighborhood $U_\varrho(M) = \{Y \in \mathbb{R}^n : \operatorname{dist}(Y, M) < \varrho\}$, such that the nearest point projection $\pi^M : U_\varrho(M) \to M$ is well-defined and given by*

$$\pi^M(X + N) = X \quad \text{for all } X \in M,\ N \in T_X M^\perp \text{ with } |N| < \varrho.$$

Moreover π^M is of class C^1 and has derivative $D\pi^M(X) = P^\top(X)$ for all $X \in M$, where $P^\top(X)$ is the orthogonal projection onto $T_X M$.

Now assume $u \in C^1(D, \mathbb{R}^n)$ has finite Dirichlet energy and maps into the submanifold M. For $\varphi \in C^1(D, \mathbb{R}^n)$ one computes

$$\frac{d}{dt}\mathcal{E}(\pi^M \circ (u + t\varphi))|_{t=0} = \int_D \big\langle Du, D(D\pi^M(u)\varphi)\big\rangle = \int_D \big\langle Du, D(P^\top(u)\varphi)\big\rangle.$$

Let $\gamma(t)$ be a C^1 curve in M, and let $e_i(t)$ be a parallel orthonormal frame along γ. Then $e_i' = A^M(\gamma', e_i)$, and hence for any $v \in \mathbb{R}^n$

$$\big((DP^\top)(\gamma)\gamma'\big)v = \frac{d}{dt}P^\top(\gamma)v = \frac{d}{dt}\langle e_i, v\rangle e_i = \langle A^M(\gamma', e_i), v\rangle e_i + \langle e_i, v\rangle A^M(\gamma', e_i).$$

As in the case of the sphere, we note that Du maps into T_uM, therefore

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\pi^M \circ (u + t\varphi))|_{t=0} &= \int_D \big\langle Du, P^\top(u)D\varphi + \big((DP^\top)(u)Du\big)\varphi\big\rangle \\ &= \int_D \langle Du, D\varphi\rangle + \int_D \big\langle Du, \langle A^M(Du, e_i), \varphi\rangle e_i\big\rangle\end{aligned}$$

We rewrite the last term as

$$\langle A^M(\partial_\alpha u, e_i), \varphi\rangle\langle\partial_\alpha u, e_i\rangle = \langle A^M(\partial_\alpha u, \partial_\alpha u), \varphi\rangle =: \langle A^M(Du, Du), \varphi\rangle.$$

Thus we finally obtain the form

$$\frac{d}{dt}\mathcal{E}(\pi^M \circ (u + t\varphi))|_{t=0} = \int_D \langle Du, D\varphi\rangle + \int_D \langle A^M(Du, Du), \varphi\rangle.$$

Again if $u \in C^2(D, \mathbb{R}^n)$ one arrives at the general harmonic map equation

$$\Delta u = (A^M \circ u)(Du, Du). \tag{2.14}$$

We have introduced harmonic maps as critical points under the constraint $u(D) \subset M$. However, the energy of a map $u \in C^1(D, M)$ depends only on the Riemannian metric $h(\cdot,\cdot) = \langle\cdot,\cdot\rangle|_{TM}$, and not on the particular choice of the isometric embedding of (M, h). From that viewpoint, harmonic maps are in the realm of intrinsic Riemannian geometry. On the other hand, the inclusion $M \subset \mathbb{R}^n$ induces the embedding $C^1(D, M) \subset W^{1,2}(D, \mathbb{R}^n)$, which opens an approach to existence by variational methods. In fact the subset

$$W^{1,2}(D, M) := \{u \in W^{1,2}(D, \mathbb{R}^n) : u(z) \in M \text{ almost everywhere}\}$$

is closed under weak convergence by Rellich's theorem, and the Dirichlet integral is lower semicontinuous under weak convergence. This allows to obtain minimizers in $W^{1,2}(D, M)$ e.g. under Dirichlet boundary conditions. A drawback is that these minimizers may have singularities and may not even admit an approximation by C^1 maps in the $W^{1,2}$ topology locally; here a classical example.

Example 2.3.2. *For B the unit ball in $\mathbb{R}^3$ and $\mathbb{S}^2$ the unit 2-sphere, it is easy to see that*

$$u : B \to \mathbb{S}^2,\ u(x) = \frac{x}{|x|},$$

belongs to $W^{1,2}(B,\mathbb{S}^2)$ *and solves the harmonic map system (2.13) in the weak sense. In fact,* u *minimizes energy in* $W^{1,2}(B,\mathbb{S}^2)$ *under Dirichlet boundary conditions* [35]*. Any neighborhood of the origin is mapped to the full 2-sphere. If* ω *denotes the area form on* $\mathbb{S}^2$ *and* $v : B \to \mathbb{S}^2$ *is smooth, then* $d(v^*\omega) = v^* d\omega = 0$ *which implies*

$$\int_B d\eta \wedge v^*\omega = \int_B d(\eta\, v^*\omega) = 0 \quad \text{for any } \eta \in C^1_c(B).$$

By contrast, we compute for the singular map u

$$\int_B d\eta \wedge u^*\omega = \lim_{\varepsilon \searrow 0} \int_{B\setminus B_\varepsilon(0)} d(\eta\, u^*\omega) = -\lim_{\varepsilon \searrow 0} \int_{\partial B_\varepsilon(0)} \eta\, u^*\omega = -4\pi\, \eta(0).$$

In other words, we have $d(u^*\omega) = 4\pi\,\delta_0$ *in the sense of distributions, detecting the topological singularity. Using dominated convergence, one proves that if* $u_k \to u$ *in* $W^{1,2}(B,\mathbb{S}^2)$*, then* $u_k^*\omega \to u^*\omega$ *in* $L^1(B,\Lambda^2(\mathbb{R}^3))$*. It follows that with respect to the local* $W^{1,2}$ *topology,* u *cannot be approximated by smooth maps* $u_k : B \to \mathbb{S}^2$*.*

2.4 Conformal invariance in 2 dimensions

The goal of this section is to classify all two-dimensional variational integrals of first order which are conformally invariant. We will see that the example of the Dirichlet energy plus a pullback of a 2-form already constitutes the general case, if we allow a general Riemannian metric in the target.

Let $f : \mathbb{R}^n \times \mathbb{R}^{2\times n} \to \mathbb{R}$, $f = f(X,A)$, be an integrand. On any bounded domain $G \subset \mathbb{R}^2$, we have the associated functional

$$\mathcal{F}(u,G) = \int_G f(u(z), Du(z))\, dxdy \quad \text{for } u : G \to \mathbb{R}^n.$$

We say that $\mathcal{F}$ is conformally invariant if for any conformal diffeomorphism $\phi : G \to \phi(G))$ we have the property

$$\mathcal{F}(u \circ \phi^{-1}, \phi(G)) = \mathcal{F}(u,G) \quad \text{for all } u : G \to \mathbb{R}^n. \tag{2.15}$$

The following classification is due to Michael Grüter.

Theorem 2.4.1 ([25])**.** *Consider a functional* $\mathcal{F}(u,G) = \int_G f(u,Du)$*, such that* f *and* $D_A^2 f$ *are continuous on* $\mathbb{R}^n \times \mathbb{R}^{n\times 2}$*. Assume also that*

$$f(X,A) > 0 \quad \text{whenever } A \neq 0.$$

If $\mathcal{F}(u,G) = \int_G f(u,Du)$ *is conformally invariant, then it has the representation*

$$\mathcal{F}(u) = \frac{1}{2}\int_G g(u)(Du,Du) + \int_G u^*\omega, \tag{2.16}$$

where g *is a Riemannian metric and* ω *is a 2-form, both continuous on* $\mathbb{R}^n$*.*

Proof. We compute substituting $w = \phi(z)$

$$\begin{aligned}\mathcal{F}(u \circ \phi^{-1}, \phi(G)) &= \int_{\phi(G)} f\big(u(\phi^{-1}(w)), Du(\phi^{-1}(w))D\phi^{-1}(w)\big)\, dw \\ &= \int_G f(u(z), Du(z)D\phi(z)^{-1}\big)|\det D\phi(z)|\, dz.\end{aligned}$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a conformal diffeomorphism. Taking $G = D_\varepsilon(0)$ and $u(z) = X + Az$ for given $X \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times 2}$, we obtain after dividing by $|D_\varepsilon|$ and letting $\varepsilon \searrow 0$

$$f(X, A) = f(X, AD\phi(0)^{-1})|\det D\phi(0)|.$$

First we use dilation invariance to show that $f(X, A)$ is a quadratic polynomial in A. Namely by taking $\phi(z) = z/t$ for $t > 0$ and differentiating twice at $t = 0$, we see that

$$f(X, A) = \frac{1}{2} D_A^2 f(X, 0)(A, A).$$

Next we chose $\phi(z) = Sz$ for $S \in \mathbb{S}O(2)$ to get $f(X, A) = f(X, AS^{\mathrm{T}})$. Combining yields

$$D_A^2 f(X, 0)(A, A) = D_A^2 f(X, 0)(AS^{\mathrm{T}}, AS^{\mathrm{T}}).$$

For $A = V \otimes e \in \mathbb{R}^{n\times 2}$, we note that

$$(V \otimes (Se))\zeta = V\langle Se, \zeta\rangle = V\langle e, S^{\mathrm{T}}\zeta\rangle = (V \otimes e)S^{\mathrm{T}}\zeta.$$

Thus by taking $Se_1 = e_2$, $Se_2 = -e_1$ we see using polarization that

$$\begin{aligned}D_A^2 f(X, 0)(V \otimes e_2, W \otimes e_2) &= D_A^2 f(X, 0)((V \otimes e_1)S^{\mathrm{T}}, (W \otimes e_1)S^{\mathrm{T}}) \\ &= D_A^2 f(X, 0)(V \otimes e_1, W \otimes e_1), \\ D_A^2 f(X, 0)(V \otimes e_2, W \otimes e_1) &= -D_A^2 f(X, 0)((V \otimes e_1)S^{\mathrm{T}}, (W \otimes e_2)S^{\mathrm{T}}\big) \\ &= -D_A^2 f(X, 0)(V \otimes e_1, W \otimes e_2).\end{aligned}$$

We can now expand

$$\begin{aligned}f(u, Du) &= \frac{1}{2}\big(D_A^2 f(u, 0)(\partial_1 u \otimes e_1, \partial_1 u \otimes e_1) + D_A^2 f(u, 0)(\partial_2 u \otimes e_2, \partial_2 u \otimes e_2)\big) \\ &\quad + D_A^2 f(u, 0)(\partial_1 u \otimes e_1, \partial_2 u \otimes e_2) \\ &=: \frac{1}{2} g(u)(Du, Du) + u^*\omega(e_1, e_2),\end{aligned}$$

where the symmetric form g and the antisymmetric form ω are defined as

$$\begin{aligned}g(X)(V, W) &= D_A^2 f(X, 0)(V \otimes e_1, W \otimes e_1), \\ \omega(X)(V, W) &= D_A^2 f(X, 0)(V \otimes e_1, W \otimes e_2).\end{aligned}$$

By assumption both are continuous on $\mathbb{R}^n$. Finally the assumed positivity of f implies that g is a Riemannian metric, namely we have for $V \neq 0$

$$g(X)(V, V) = D_A^2 f(X, 0)(e_1 \otimes V, e_1 \otimes V) = 2f(X, e_1 \otimes V) > 0.$$

This finishes the proof of the theorem. □

The calculation of the Euler-Lagrange equation for the Riemannian Dirichlet energy is straightforward using local coordinates. We compute for $\varphi \in C_c^\infty(G,\mathbb{R}^n)$

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_g(u+t\varphi)|_{t=0} &= \int_G \Big(g_{jk}(u)\partial_\alpha\varphi^j\partial_\alpha u^k + \frac{1}{2}\partial_i g_{jk}(u)\varphi^i\partial_\alpha u^j\partial_\alpha u^k\Big)\\
&= -\int_G \varphi^j\Big(g_{jk}(u)\Delta u^k + \partial_i g_{jk}(u)\partial_\alpha u^i\partial_\alpha u^k - \frac{1}{2}\partial_j g_{ik}(u)\partial_\alpha u^i\partial_\alpha u^k\Big)\\
&= -\int_G g_{jk}(u)\varphi^j\Big(\Delta u^k + g^{kl}(u)\partial_i g_{lm}(u)\partial_\alpha u^i\partial_\alpha u^m - \frac{1}{2}g^{kl}(u)\partial_l g_{im}(u)\partial_\alpha u^i\partial_\alpha u^m\Big)\\
&= -\int_G g_{jk}(u)\varphi^j\Big(\Delta u^k + \Gamma^k_{im}(u)\partial_\alpha u^i\partial_\alpha u^m\Big).
\end{aligned}$$

Here the Γ^k_{ij} are the Christoffel symbols of the metric g; we used that the term $\partial_\alpha u^i\partial_\alpha u^m$ is symmetric in i and m. For the pullback integral, we proceed as in the case of codimension one. Introducing the 3-Form $\Omega = d\omega$, we have putting $F : D\times[0,\varepsilon]\to\mathbb{R}^n$, $F(z,t) = u(z)+t\varphi(z)$,

$$\begin{aligned}
\frac{d}{d\varepsilon}\int_G (u+\varepsilon\varphi)^*\omega\,|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_{G\times[0,\varepsilon]} F^*\Omega\,|_{\varepsilon=0}\\
&= \int_G (\Omega\circ u)(u_x,u_y,\varphi)\\
&= \int_G g_{jk}(u)\varphi^j g^{kl}(u)(\Omega\circ u)(u_x,u_y,e_l).
\end{aligned}$$

In summary, the Euler-Lagrange operator $L_f(u)$ of the functional (2.16) is given by

$$L_f(u)^k = -\Delta u^k - \Gamma^k_{ij}(u)\partial_\alpha u^i\partial_\alpha u^j + g^{kl}(u)(\Omega\circ u)(u_x,u_y,e_l), \tag{2.17}$$

where Γ^k_{ij} are the Christoffelsymbols of g and $\Omega = d\omega$. The nonlinear operator

$$(\Delta^N u)^k = \Delta u^k + \Gamma^k_{ij}(u)\partial_\alpha u^i\partial_\alpha u^j$$

is sometimes called the tension field or intrinsic Laplacian of u. The system (2.17) is semilinear with principal term given by the standard Laplacian, and a right hand side which is a quadratic form of the gradient, possibly depending nonlinearly on u. The key question is now whether a regularity theory is available for such systems. Here are the bad news.

Example 2.4.2. *Consider the scalar equation*

$$-\Delta u = |Du|^2 \quad \text{on } G = D_{1/e}(0).$$

We claim that the function $u(z) = \log\log\frac{1}{r}$, $r = |z|$, belongs to $W^{1,2}(G)$ and solves the equation in the weak sense. For this we compute

$$\begin{aligned}
u'(r) &= \frac{1}{r\log r},\\
u''(r) &= -\frac{1}{r^2\log r} - \frac{1}{r^2\log^2 r},\\
|Du|^2 &= u'(r)^2 = \frac{1}{r^2\log^2 r},\\
\Delta u &= u''(r) + \frac{1}{r}u'(r) = -\frac{1}{r^2\log^2 r}.
\end{aligned}$$

Away from the origin the equation holds in the classical sense. Substituting $r = e^{-t}$ where $t \in [1, \infty)$ we have

$$\int_0^{1/e} \frac{dr}{r \log^s \frac{1}{r}} = \int_1^\infty \frac{dt}{t^s} = \begin{cases} \infty & \textit{for } s = 1 \\ \frac{1}{s-1} & \textit{for } s > 1. \end{cases}$$

Thus $Du \in L^2(G, \mathbb{R}^2)$. Using cutoff arguments one now proves that Du is the weak derivative, and that the equation holds weakly on the full domain G. We also see that solutions to the Dirichlet problem may be nonunique, since $u = 0$ on ∂G. Furthermore, we have a counterexample to an L^1 theory for the Laplacian: Δu is integrable while $D^2 u$ is not.

The fact that $u(x) = \log \log \frac{1}{r}$ is unbounded is important in the previous example. Namely, if the weak solution was bounded then a regularity result by Ladyzhenskaya and Ural'tseva (1961) would imply that it is Hölder continuous; further regularity would then follow easily. In the case of harmonic maps the boundedness of the weak solution is for granted, by assuming the target manifold to be compact. Nevertheless the result of Ladyzhenskaya and Ural'tseva does not apply, because it is limited to scalar equations. This is seen from the following modification of our example, due to Hildebrandt and Widman.

Example 2.4.3. *The map $u(r) = \exp(i \log \log \frac{1}{r})$ is a bounded weak solution to the system*

$$-\Delta u = |Du|^2 \Lambda u \quad \textit{where } \Lambda = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In summary we see that the regularity for the prescribed mean curvature equation and for harmonic maps is subtle, and that the particular structure of the nonlinearity needs to be exploited in some way.

Chapter 3

Harmonic maps into spheres

3.1 A conservation law

For a bounded domain $\Omega \subset \mathbb{R}^m$, consider the set of mappings

$$W^{1,2}(\Omega, \mathbb{S}^{n-1}) = \{u \in W^{1,2}(\Omega, \mathbb{R}^n) : |u(x)| = 1 \text{ almost everywhere}\}. \tag{3.1}$$

As defined in the last section, a map $u \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$ is weakly harmonic if

$$\int_\Omega \langle Du, D\phi \rangle = \int_\Omega |Du|^2 \langle u, \phi \rangle \quad \text{for all } \phi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^n). \tag{3.2}$$

We remark that test functions in $C_c^\infty(\Omega, \mathbb{R}^n)$ are sufficient to deduce the more general form of the equation. In fact, for given $\phi \in W_0^{1,2} \cap L^\infty(\Omega, \mathbb{R}^n)$ choose $\phi_k \in C_c^\infty(\Omega, \mathbb{R}^n)$ such that $\phi_k \to \phi$ in $W^{1,2}$ and pointwise almost everywhere. Then let $\eta \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\eta(z) = z$ for $|z| \leq \|\phi\|_{L^\infty}$. The functions $\psi_k = \eta \circ \phi_k$ are uniformly bounded and converge to ϕ almost everywhere. Moreover

$$\|D\psi_k - D\phi\|_{L^2} \quad \leq \quad \|D\eta \circ \phi_k (D\phi_k - D\phi)\|_{L^2} + \|(D\eta \circ \phi_k - \mathrm{Id}) D\phi\|_{L^2} \to 0,$$

where dominated convergence was again used in the last term. The general version of (3.2) now follows by testing with ψ_k and passing to the limit. Of course this argument is not specific to our present topic, but applies to any semilinear system with right hand side quadratic in Du.

We take up the subject of harmonic maps into spheres with a reformulation of the Euler-Lagrange equation due to Yun Mei Chen and Jalal Shatah, see also [29]. In the following arguments, we use several times the Sobolev product rule implying that the space $W^{1,2} \cap L^\infty(\Omega)$ is an algebra. If additionally one of the factors belongs to $W_0^{1,2}(B, \mathbb{R}^n)$, then this is also true for the product.

Theorem 3.1.1 ([9, 53]). *For $u \in W^{1,2}(B, \mathbb{S}^{n-1})$ the following are equivalent:*

(a) *u is weakly harmonic.*

(b) *For any $\Lambda \in \mathbb{R}^{n \times n}$ with $\Lambda^{\mathrm{T}} = -\Lambda$ we have* $\operatorname{div}(Du^{\mathrm{T}} \Lambda u) = 0$ *weakly, that is*

$$\int_B \langle Du \cdot \operatorname{grad} \varphi, \Lambda u \rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(B). \tag{3.3}$$

Proof. Let u be weakly harmonic. Testing with $\varphi\Lambda u \in W_0^{1,2} \cap L^\infty(B,\mathbb{R}^n)$, we obtain

$$\begin{aligned} 0 &= \int_B |Du|^2 \langle u, \varphi\Lambda u\rangle \\ &= \int_B \langle Du, D(\varphi\Lambda u)\rangle \\ &= \int_B \partial_\alpha\varphi \langle \partial_\alpha u, \Lambda u\rangle + \int_B \varphi\langle \partial_\alpha u, \Lambda\partial_\alpha u\rangle \\ &= \int_B \langle Du\cdot \operatorname{grad}\varphi, \Lambda u\rangle. \end{aligned}$$

For the reverse direction, we need to show that these special variations are sufficient to deduce the full harmonic map system. Denote by $e_1,\ldots,e_n$ the standard basis of $\mathbb{R}^n$, and consider the skew-symmetric matrices

$$\Lambda_{ij} = (e_i\otimes e_j - e_j\otimes e_i) \in \mathbb{R}^{n\times n} \quad \text{for } 1\le i,j\le n.$$

For any $\omega\in\mathbb{S}^{n-1}$ the $\Lambda_{ij}\omega$ span $T_\omega\mathbb{S}^{n-1}$, in fact for any $\xi\in T_\omega\mathbb{S}^{n-1}$ we have the expansion

$$\begin{aligned} \xi &= (\xi\otimes\omega - \omega\otimes\xi)\,\omega \\ &= \sum_{i,j=1}^n (\xi^i\omega^j - \omega^i\xi^j)(e_i\otimes e_j)\omega \\ &= \sum_{i,j=1}^n \xi^i\omega^j(e_i\otimes e_j - e_j\otimes e_i)\omega \\ &= \sum_{i,j=1}^n \xi^i\omega^j\Lambda_{ij}\omega. \end{aligned}$$

For a given variation $\phi\in C_c^\infty(B,\mathbb{R}^n)$ we obtain the representation

$$\phi = \langle\phi,u\rangle u + \sum_{i,j=1}^n \varphi_{ij}\Lambda_{ij}u, \quad \text{where } \varphi_{ij} = (\phi^i - \langle\phi,u\rangle u^i)u^j. \tag{3.4}$$

Using assumption (b) and the identity $\langle Du, \Lambda_{ij}Du\rangle = 0$, we see that

$$\int_B \langle Du, D(\varphi_{ij}\Lambda_{ij}u)\rangle = \int_B \langle Du\cdot\operatorname{grad}\varphi_{ij}, \Lambda_{ij}u\rangle + \int_B \varphi_{ij}\langle Du,\Lambda_{ij}Du\rangle = 0.$$

On the other hand, from $\langle Du, u\rangle = 0$ we infer

$$\int_B \langle Du, D(\langle\phi,u\rangle u)\rangle = \int_B |Du|^2\langle u,\phi\rangle.$$

Claim (a) follows using the representation (3.4). □

As an application we show that the set of weakly harmonic maps into the round sphere is closed under weak convergence. This is not obvious from (3.2), in general the weak convergence $Du_k \to Du$ does not even imply $|Du_k|^2 \to |Du|^2$ as measures. However it is a simple consequence of Theorem 3.1.1.

Corollary 3.1.2. *Let $u_k \in W^{1,2}(B, \mathbb{S}^{n-1})$ be weakly harmonic, and suppose $u_k \to u$ weakly in $W^{1,2}(B, S^{n-1})$. Then $u \in W^{1,2}(B, \mathbb{S}^{n-1})$ is also weakly harmonic.*

Proof. We have $Du_k \to Du$ weakly in $L^2(\mathbb{R}^{n\times d})$, and $u_k \to u$ strongly in $L^2(B, \mathbb{R}^n)$ by Rellich's theorem. Then $(\partial_\alpha \varphi)\Lambda u_k \to (\partial_\alpha \varphi)\Lambda u$ strongly in $L^2(B, \mathbb{R}^n)$ for $\varphi \in C_c^\infty(B)$, and

$$\int_B \langle Du \cdot \operatorname{grad} \varphi, \Lambda u\rangle = \lim_{k\to\infty} \int_B \langle Du_k \cdot \operatorname{grad} \varphi, \Lambda u_k\rangle.$$

The result follows by Theorem 3.1.1. □

Equation (3.3) derives in a more systematic way as conservation law associated to rotational symmetry. This relation was explained in the fundamental work of Emmy Noether [42]. Let us explain this in some generality, for a variational integral

$$\mathcal{F}(u) = \int_\Omega f(u, Du) \quad \text{where } f : \mathbb{R}^n \times \mathbb{R}^{n\times m},\ f = f(z,p).$$

An explicit dependence $f = f(x,z,p)$ could be allowed without any changes, it is omitted for simplicity. A diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetry of the Lagrangian f if

$$f(\phi(z), D\phi(z)p) = f(z,p) \quad \text{for all } (z,p) \in \mathbb{R}^n \times \mathbb{R}^{n\times m}. \tag{3.5}$$

In fact we then have

$$\mathcal{F}(\phi \circ u) = \int_\Omega f(\phi\circ u, D\phi \circ u\, Du) = \int_\Omega f(u, Du) = \mathcal{F}(u).$$

From a one-parameter family of symmetries $\phi_t = \phi(\cdot, t)$, $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$, one derives an infinitesimal version of symmetry by differentiating (3.5) at $t = 0$. This yields, viewing $D_p f(z,p) \in \mathbb{R}^{n\times m}$,

$$\langle D_z f(z,p), \eta(z)\rangle + \langle D_p f(z,p), D\eta(z)p\rangle = 0 \quad \text{where } \eta(z) = \frac{\partial\phi}{\partial t}(z, 0). \tag{3.6}$$

This in turn implies infinitesimal invariance, namely for any $U \subset\subset \Omega$ we have

$$\frac{d}{dt}\int_U f(\phi_t \circ u, D(\phi_t\circ u))|_{t=0} = \int_U \big(\langle D_z f(u,Du), \eta\circ u\rangle + \langle D_p f(u,Du), D\eta \circ u\, Du\rangle\big) = 0.$$

Now let u be a critical point of $\mathcal{F}$. Testing the weak Euler-Lagrange equation with the function $\varphi\eta\circ u$ for $\varphi \in C_c^1(\Omega)$ and then using (3.6), we obtain

$$\begin{aligned} 0 &= \int_\Omega \big(\langle D_z f(u,Du), \varphi\eta\circ u\rangle + \langle D_p f(u,Du), D(\varphi\eta\circ u)\rangle\big) \\ &= \int_\Omega \langle D_p f(u,Du), (\eta\circ u)\otimes D\varphi\rangle \\ &= \int_\Omega \langle D_p f(u,Du)^{\mathrm{T}} \eta\circ u, D\varphi\rangle. \end{aligned}$$

Therefore if $f = f(z,p)$ has the infinitesimal symmetry (3.6) and if u is a weak solution to the Euler-Lagrange equations, then one has the conservation law

$$\operatorname{div}\left(D_p f(u,Du)^{\mathrm{T}}\, \eta\circ u\right) = 0 \quad \text{weakly in } \Omega. \tag{3.7}$$

In our special case we have $\phi_t(z) = R(t)z$ where $R(t) = \exp t\Lambda$ for $\Lambda \in \mathbb{R}^{n\times n}$ skew-symmetric. It follows that $\eta(z) = \Lambda z$ and the condition of infinitesimal symmetry holds, namely

$$\langle D_p f(p), D\eta(z)p\rangle = \langle p, \Lambda p\rangle = 0.$$

One conludes that a weak harmonic map $u : \Omega \to \mathbb{S}^{n-1}$ satisfies

$$\operatorname{div}(Du^{\mathrm{T}}\,\Lambda u) = 0 \quad \text{weakly in } \Omega.$$

In his second paper [27] Frédéric Hélein exploited this relation of symmetries and conservation laws further to prove regularity for harmonic maps into manifolds with transitive isometry group [27]. Of course , the key point of Theorem 3.1.1 is that test functions of the type $\varphi\Lambda u$ are sufficient to deduce the full harmonic map system, so that the conservation law is actually equivalent to the Euler-Lagrange equation.

We now turn to conservation laws associated to symmetries on the domain. For any diffeomorphism $\phi : U \to \phi(U)$, $U \subset\subset \Omega$, we have by the transformation rule

$$\begin{aligned} \mathcal{F}(u\circ\phi^{-1}, \phi(U)) &= \int_{\phi(U)} f(y, u(\phi^{-1}(y), Du(\phi^{-1}(y))D(\phi^{-1})(y))\,dy \\ &= \int_U f(\phi(x), u(x), Du(x)D\phi(x)^{-1})\,|\det D\phi(x)|\,dx. \end{aligned} \tag{3.8}$$

Chosing a a one-parameter family ϕ_t of diffeomorphisms with $\phi_0 = \mathrm{id}_\Omega$ we compute, putting $\xi = \frac{\partial\phi}{\partial t}(\cdot, 0)$ and noting $\det D\phi_t > 0$,

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}\big(u\circ\phi_t^{-1}, \phi_t(U)\big)|_{t=0} \\ &= \int_U \big(\langle D_x f(\cdot, u, Du), \xi\rangle - \langle D_p f(\cdot, u, Du), Du\cdot D\xi\rangle + f(\cdot, u, Du)\operatorname{div}\xi\big) \\ &= \int_U \big(\langle D_x f(\cdot, u, Du), \xi\rangle - \int_U \big\langle Du^{\mathrm{T}} D_p f(\cdot, u, Du) - f(\cdot, u, Du)\mathrm{Id}_{\mathbb{R}^m}, D\xi\big\rangle. \end{aligned}$$

It is convenient to introduce the abbreviation

$$H(x, z, p) = p^{\mathrm{T}} D_p f(x, z, p) - f(x, z, p)\,\mathrm{Id}_{\mathbb{R}^m} \in \mathbb{R}^{m\times m}, \tag{3.9}$$

or in coordinates

$$H^\alpha_\beta(x, z, p) = p^i_\alpha \frac{\partial f}{\partial p^i_\beta}(x, z, p) - f(x, z, p)\,\delta^\alpha_\beta.$$

We say that u is critical with respect to inner variations, if

$$\frac{d}{dt}\mathcal{F}(u\circ\phi_t^{-1}, \Omega)|_{t=0} = 0 \quad \text{for any flow } \phi_t \text{ of a vector field } \xi \in C^\infty_c(\Omega, \mathbb{R}^m). \tag{3.10}$$

We have $\phi_t = \mathrm{id}$ on $\Omega\backslash\operatorname{spt}\xi$, so $\phi : \Omega\times\mathbb{R} \to \Omega$ is globally defined and smooth, with $\phi_t^{-1} = \phi_{-t}$. By the calculation above, we see that (3.10) implies

$$\operatorname{div} H(\cdot, u, Du) + D_x f(\cdot, u, Du) = 0 \quad \text{weakly in } \Omega. \tag{3.11}$$

In coordinates this takes the form

$$\partial_\beta[H^\alpha_\beta(\cdot,u,Du)] + (\partial_\alpha f)(\cdot,u,Du) = 0 \quad \text{for } \alpha = 1,\dots,m.$$

Equation (3.11) is called Noether's equation in [19], it replaces the Euler-Lagrange equation in the setting of inner variations. Now in view of (3.8), a diffeomorphism $\phi : \Omega \to \phi(\Omega)$ is called a symmetry for the Lagrangian $f = f(x,z,p)$ if

$$f(\phi(x), z, p\, D\phi(x)^{-1})\,|\det D\phi(x)| = f(x,z,p) \quad \text{for all } (x,z,p) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m}.$$

Again we replace this by an infinitesimal version. Assuming a one-parameter family ϕ_t of such symmetries, we get by differentiating and putting $\xi = \frac{\partial \phi}{\partial t}|_{t=0}$

$$\langle D_x f(x,z,p), \xi(x)\rangle - \langle H(\cdot,z,p), D\xi(x)\rangle = 0. \tag{3.12}$$

If $u \in C^1(\Omega,\mathbb{R}^n)$ is critical for inner variations, then any such infinitesimal symmetry ξ yields a conservation law. Namely, testing (3.11) with $\varphi\xi$ where $\varphi \in C^1_c(\Omega)$ we find

$$\begin{aligned} 0 &= \int_\Omega \varphi\big\langle D_x f(\cdot,u,Du),\xi\big\rangle - \int_\Omega \big\langle H(\cdot,u,Du), D(\varphi\xi)\big\rangle \\ &= \int_\Omega \varphi\left(\big\langle D_x f(\cdot,u,Du),\xi\big\rangle - \big\langle H(\cdot,u,Du), D\xi\big\rangle\right) - \int_\Omega \big\langle H(\cdot,u,Du)^{\mathrm{T}}\xi, D\varphi\big\rangle \\ &= -\int_\Omega \big\langle H(\cdot,u,Du)^{\mathrm{T}}\xi, D\varphi\big\rangle. \end{aligned}$$

In other words

$$\operatorname{div}(H(\cdot,u,Du)^{\mathrm{T}}\xi) = 0 \quad \text{weakly in } \Omega. \tag{3.13}$$

As an example, let us reconsider the conformal invariance of the Dirichlet integral. We have for the Dirichlet integral, writing $pe_1 = v$ and $pe_2 = w$,

$$H(p) = \begin{pmatrix} \frac{1}{2}(|v|^2 - |w|^2) & \langle v,w\rangle \\ \langle v,w\rangle & -\frac{1}{2}(|v|^2-|w|^2) \end{pmatrix}.$$

Now u critical with respect to inner variations means

$$\int_\Omega \left\langle H(Du), \begin{pmatrix} \lambda_x & \lambda_y \\ \mu_x & \mu_y \end{pmatrix}\right\rangle = 0 \quad \text{for all } (\lambda,\mu) \in C^1_c(\Omega,\mathbb{R}^2).$$

Clearly, this just says that the function $h = |u_x|^2 - |u_y|^2 - 2i\langle u_x,u_y\rangle$ is holomorphic. The complex quadratic differential $h(z)\,dz^2$ is called the Hopf differential of the map [32]. Next, the condition that a vector field $\xi = (a,b)$ is an infinitesimal symmetry reads

$$\left\langle \begin{pmatrix} \frac{1}{2}(|v|^2 - |w|^2) & \langle v,w\rangle \\ \langle v,w\rangle & -\frac{1}{2}(|v|^2-|w|^2) \end{pmatrix}, \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}\right\rangle = 0 \quad \text{for all } v,w \in \mathbb{R}^n.$$

As expected, this is equivalent to $\xi = a + ib$ being holomorphic. It is easy to see that the (local) flow of ξ is then by biholomorphic transformations.

Clearly the Noether equation will generally not imply the Euler-Lagrange equation, since inner variations only compare to reparametrizations of a given map. For instance, any map is critical for the area integral with respect to inner variations. Naively, one might expect the reverse implication to hold. Namely, writing the Euler-Lagrange equation formally as $D\mathcal{F}(u) = 0$ and applying the chain rule, we should get

$$\frac{d}{dt}\mathcal{F}(u \circ \phi_t)|_{t=0} = D\mathcal{F}(u)(u \circ \phi_t)'(0) = 0.$$

In other words, the Noether equation should follow from the Euler-Lagrange equation by testing with $\varphi = (u \circ \phi_t)'(0)$; this is actually true if u is sufficiently smooth. However, if u is just a weak solution then φ may not be admissible as test function because of lack of regularity, as it involves a derivative of u:

$$\varphi = Du \cdot \xi \quad \text{where } \xi = \frac{\partial \phi}{\partial t}|_{t=0}.$$

This gap is reflected in the regularity theory of harmonic maps, where inner variations played an important role.

The story started with C.B. Morrey who proved regularity of minimizers in two dimensions [37]. Much later R. Schoen improved this by showing that weak solutions are regular, provided they are also critical with respect to inner variations [50]; this is of course fulfilled for minimizers by one-dimensional calculus. Actually, Schoen's proof is essentially based on previous work by M. Grüter who showed regularity of weak H surfaces, that is solutions of the prescribed mean curvature system which are conformally parametrized [24], see Section 2.3 in Jost's book [33]. Eventually F. Hélein found that inner variations are not needed at all to prove regularity in two dimensions [28]. In dimensions $m \geq 3$ R. Schoen and K. Uhlenbeck proved partial regularity of minimizers, saying that the singular set has Hausdorff dimension at most $m-3$ and consists of isolated points for $m = 3$ [51]. F. Lin showed that the map $u(x) = x/|x|$ is in fact minimizing from B^m to $\mathbb{S}^{m-1}$ for any $m \geq 3$ [35]. For weak solutions C. Evans [13] and then F. Béthuel [3] showed that the singular set S has Hausdorff measure $\mathcal{H}^{m-2}(S) = 0$, again provided the map is also critical with respect to inner variations. In fact only dilations are needed, they yield a crucial monotonicity formula. Any hopes for partial regularity of weak solutions in dimension $m \geq 3$ without further assumptions were dashed by T. Rivière [45]. He constructed a harmonic map $u \in W^{1,2}(B^3, \mathbb{S}^2)$ which is everywhere discontinuous. In particular, this solution is not critical with respect to dilations in the domain.

3.2 Wente's inequality

In this section we are back to dimension $m = 2$, in fact we only consider maps defined on the unit disk D. Let us start the discussion with the classical Dirichlet problem for the Poisson equation, assuming that

$$-\Delta u = f \text{ weakly in } D \quad \text{where } u \in W_0^{1,2}(D).$$

A very fundamental estimate, due to Calderon and Zygmund, asserts that if $f \in L^p(D)$ where $1 < p < \infty$, then $u \in W^{2,p}(D)$ and

$$\|u\|_{W^{2,p}(D)} \leq C(p)\, \|f\|_{L^p(D)}.$$

In the case of the harmonic map or prescribed mean curvature systems, this does not apply directly because the right hand side, being quadratic in the gradient, belongs a priori only to $L^1(D)$. The Calderon-Zygmund theory does not extend to the space L^1; the difficulty was already observed in Example 2.4.2. Henry Wente found that this drawback can be compensated if the right hand side has a special algebraic structure, namely when f is a Jacobi determinant. We refer to Hélein's book [29] for an in-depth discussion.

Theorem 3.2.1 ([65])**.** *For $a,b \in W^{1,2}(D)$ given and $\{a,b\} = a_x b_y - a_y b_x$, there exists a unique function $u \in W^{1,2}_0(D)$ solving*

$$\int_D \langle du, d\varphi\rangle = \int_D \{a,b\}\varphi \quad \text{for all } \varphi \in W^{1,2}_0 \cap L^\infty(D), \tag{3.14}$$

The solution belongs belongs to $C^0(\overline{D})$ and satisfies

$$\|u\|_{C^0(\overline{D})} \leq \frac{1}{2\pi}\,\|da\|_{L^2(D)}\,\|db\|_{L^2(D)}, \tag{3.15}$$

$$\|du\|_{L^2(D)} \leq \frac{1}{\sqrt{2\pi}}\,\|da\|_{L^2(D)}\,\|db\|_{L^2(D)}. \tag{3.16}$$

Proof. The uniqueness of the solution is standard. The key to existence are the estimates, assuming that a,b and hence u are smooth on the closed disk; everything else will follow by a simple approximation argument. We observe that the equation is nothing but the third component of the constant mean curvature system for $H=-1$. More precisely, consider the scalar functional, for $\omega = \frac{1}{3}X \llcorner dV_{\mathbb{R}^3}$,

$$\mathcal{F}_{a,b}(u) = \frac{1}{2}\int_D |Df|^2 - \int_D f^*\omega, \qquad \text{where } f = (a,b,u).$$

Then we have from Lemma 2.2.1, noting that $\langle f_x \wedge f_y, e_3\rangle = \{a,b\}$,

$$\frac{d}{dt}\mathcal{F}_{a,b}(u+t\varphi)|_{t=0} = \int_D \langle du, d\varphi\rangle - \int_D \{a,b\}\varphi.$$

In particular, the equation (3.14) is invariant under orientation-preserving, conformal diffeomorphisms, acting on all variables a, b and u. Moreover, also the quantities in (3.15) are conformally invariant. For the C^0 estimate, this discussion shows that it is sufficient to bound $u(0)$. Namely, for $a \in D$ we consider the disk automorphism

$$\phi(z) = \frac{z+a}{1+\overline{a}z}.$$

By conformal invariance, we can bound $(u\circ\phi)(0) = u(a)$, which gives the result. Now by Green's formula we have, writing $r = |z|$,

$$\begin{aligned}
u(0) &= \frac{1}{2\pi}\int_D (\log r)\, da\wedge db \\
&= \frac{1}{2\pi}\lim_{\varepsilon\searrow 0}\int_{D\setminus D_\varepsilon(0)} d\big((\log r) a\, db\big) - \frac{1}{2\pi}\lim_{\varepsilon\searrow 0}\int_{D\setminus D_\varepsilon(0)} a\,\frac{dr}{r}\wedge db \\
&= -\frac{1}{2\pi}\int_0^1 \frac{dr}{r}\int_0^{2\pi} a(r,\varphi)\frac{\partial b}{\partial\varphi}(r,\varphi)\, d\varphi.
\end{aligned}$$

Let $\widehat{a}(r)$ be the mean value of $a(r,\cdot)$ on $[0,2\pi]$, and estimate

$$\begin{aligned}\Big|\int_0^{2\pi} a(r,\varphi)\frac{\partial b}{\partial\varphi}(r,\varphi)\,d\varphi\Big| &= \Big|\int_0^{2\pi}(a(r,\varphi)-\widehat{a}(r))\frac{\partial b}{\partial\varphi}(r,\varphi)\,d\varphi\Big|\\ &\le \big\|a-\widehat{a}(r)\big\|_{L^2(0,2\pi)}\cdot\big\|\frac{\partial b}{\partial\varphi}\big\|_{L^2(0,2\pi)}\\ &\le \big\|\frac{\partial a}{\partial\varphi}\big\|_{L^2(0,2\pi)}\cdot\big\|\frac{\partial b}{\partial\varphi}\big\|_{L^2(0,2\pi)}.\end{aligned}$$

In the last step we used the Poincaré inequality on $(0,2\pi)$ for functions having zero mean value; this follows easily by Fourier expansion. We conclude

$$\begin{aligned}|u(0)| &\le \frac{1}{2\pi}\int_0^1\big\|\frac{\partial a}{\partial\varphi}\big\|_{L^2(0,2\pi)}\cdot\big\|\frac{\partial b}{\partial\varphi}\big\|_{L^2(0,2\pi)}\frac{dr}{r}\\ &\le \frac{1}{2\pi}\Big(\int_0^1\int_0^{2\pi}\frac{1}{r^2}\big|\frac{\partial a}{\partial\varphi}\big|^2\,rdrd\varphi\Big)^{1/2}\cdot\Big(\int_0^1\int_0^{2\pi}\frac{1}{r^2}\big|\frac{\partial b}{\partial\varphi}\big|^2\,rdrd\varphi\Big)^{1/2}\\ &\le \frac{1}{2\pi}\|da\|_{L^2(D)}\cdot\|db\|_{L^2(D)}.\end{aligned}$$

The L^2 bound for du now follows simply by testing with u and using Cauchy-Schwarz:

$$\int_D|du|^2=\int_D u\{a,b\}\le\|u\|_{C^0(D)}\|da\|_{L^2(D)}\|db\|_{L^2(D)}\le\frac{1}{2\pi}\|da\|^2_{L^2(D)}\|db\|^2_{L^2(D)}.$$

In the smooth case the estimates are settled. Given $a,b\in W^{1,2}(D)$ we approximate $a_k\to a$, $b_k\to b$ in $W^{1,2}(D)$, where a_k,b_k are in $C^\infty(\overline{D})$. Let u_k be the solution of the Dirichlet problem

$$-\Delta u_k=\{a_k,b_k\}\text{ in }D,\quad u_k|_{\partial D}=0.$$

Then u_k-u_l is zero on ∂D and satisfies

$$-\Delta(u_k-u_l)=\{a_k,b_k\}-\{a_l,b_l\}=\{a_k-a_l,b_k\}+\{a_l,b_k-b_l\}.$$

We have $\|da_k\|_{L^2(D)}+\|db_k\|_{L^2(D)}\le C$ for all k. By uniqueness and the estimates, we obtain

$$\begin{aligned}\|u_k-u_l\|_{C^0(\overline{D})}+\|d(u_k-u_l)\|_{L^2(D)} &\le C\left(\|d(a_k-a_l)\|_{L^2(D)}+\|d(b_k-b_l)\|_{L^2(D)}\right)\\ &\to 0\quad\text{as }k,l\to\infty.\end{aligned}$$

Thus $u_k\to u$ in both $W_0^{1,2}(D)$ and $C^0(\overline{D})$, and u satisfies the desired estimates. To get the weak equation for test functions $\varphi\in W_0^{1,2}\cap L^\infty(D)$, we compute

$$\int_D\langle du,d\varphi\rangle=\lim_{k\to\infty}\int_D\langle du_k,d\varphi\rangle=\lim_{k\to\infty}\int_D\varphi\{a_k,b_k\}=\int_D\varphi\{a,b\}.$$

□

A slight extension of Wente's theorem is the following statement for superweak solutions.

Corollary 3.2.2. *Let $a,b\in W^{1,2}(D)$ and assume that $v\in L^1(D)$ solves*

$$-\int_D v\Delta\varphi=\int_D\varphi\{a,b\}\quad\textit{for all }\varphi\in C^\infty(\overline{D}),\ \varphi|_{\partial D}=0. \tag{3.17}$$

Then v is the solution from Theorem 3.2.1, in particular $v\in W_0^{1,2}(D)\cap C^0(\overline{D})$.

Proof. We first note that weak solutions also satisfy the superweak formulation (3.17). In fact, assume that $u \in W_0^{1,p}(D)$, $p \in [1,2]$, solves the boundary value problem

$$\int_D \langle Du, D\varphi\rangle = \int_D \varphi\{a,b\} \quad \text{for all } \varphi \in C^\infty(\overline{D}),\ \varphi|_{\partial D} = 0.$$

Then by Sobolev trace theory, using $u = 0$ on ∂D,

$$\int_D \langle Du, D\varphi\rangle = \int_{\partial D} u\,\frac{\partial \varphi}{\partial r} - \int_D u\,\Delta\varphi = -\int_D u\,\Delta\varphi.$$

We apply this to the solution $u \in W_0^{1,2}(D)$ given by Theorem 3.2.1, and get by subtracting

$$\int_D (u-v)\Delta\varphi = 0 \quad \text{for all } \varphi \in C^\infty(\overline{D}),\ \varphi|_{\partial D} = 0.$$

Now for any $\eta \in C_c^\infty(D)$ we can choose $\varphi \in C^\infty(\overline{D})$ with zero boundary values, such that $\Delta\varphi = \eta$. It follows that $v = u$. □

By the Wente lemma, weak solutions $u \in W^{1,2}(D,\mathbb{R}^3)$ of the constant mean curvature system are continuous; in fact we can always substract a harmonic function to achieve boundary values zero. For the prescribed mean curvature system with variable H, we already mentioned the regularity result of Grüter [24], assuming additionally the conformality relations. His argument is much different, the idea is that the image varifold under the map $u : D \to \mathbb{R}^3$ has weak mean curvature H and hence satisfies a monotonicity formula. Finally, Riviére proved that all weak solutions are regular [46]; his proof will be presented in Chapter 5.

3.3 Regularity of harmonic maps from the disk to $\mathbb{S}^{n-1}$

The regularity of two-dimensional harmonic maps was proved by F. Hélein around 1990, in three papers of increasing generality as regards the target [26, 27, 28]. The first one studied the case of a round sphere, the second addressed homogeneous spaces and finally the third covered all compact submanifolds. Here we take up the case of a round sphere, starting with a simple two-dimensional Hodge lemma. The Hodge decomposition in arbitrary dimension will be discussed in connection with Uhlenbeck's Coulomb gauge theorem, see Theorem 6.1.2.

We denote by d^* the formal adjoint to the exterior derivative d with respect to the L^2 inner product. In $\mathbb{R}^2$ one easily calculates the following table:

$a \in C^1(D)$	$da = a_x\,dx + a_y\,dy$	$d^*a = 0$
$\omega = u\,dx + v\,dy \in C^1(D,\Lambda^1(\mathbb{R}^2))$	$d\omega = (v_x - u_y)\,dx\wedge dy$	$d^*\omega = -(u_x + v_y)$
$\beta = b\,dx\wedge dy \in C^1(D,\Lambda^2(\mathbb{R}^2))$	$d\beta = 0$	$d^*\beta = b_y\,dx - b_x\,dy.$

In particular $d^*da = -\Delta a$ and $dd^*\beta = -\Delta b\,dx\wedge dy$. A rather important observation, in view of the Wente lemma, is that[1]

$$\langle da, d^*\beta\rangle = a_x b_y + a_y(-b_x) = \{a,b\}.$$

We are now ready to state the Hodge lemma.

[1] In the notation of Riviére, $\langle da, d^*\beta\rangle = \langle \nabla a, \nabla^\perp b\rangle$.

Lemma 3.3.1 ([31]). *On the unit disk $D \subset \mathbb{R}^2$, any differential form $\omega \in L^2(D, \Lambda^1(\mathbb{R}^2))$ has a unique L^2-orthogonal decomposition*

$$\omega = da + d^*\beta \quad \text{where } a \in W_0^{1,2}(D),\ \beta \in W^{1,2}(D, \Lambda^2(\mathbb{R}^2)) \text{ with } \int_D \beta = 0.$$

If $d^*\omega = 0$ as distribution, then $a = 0$.

Proof. For any $a \in C_c^\infty(D)$, $\beta \in W^{1,2}(D, \Lambda^2(\mathbb{R}^2))$, we have

$$\int_D \langle da, d^*\beta\rangle = \int_D \langle d(da), \beta\rangle = 0, \text{ hence } \quad dW_0^{1,2}(D) \perp d^*W^{1,2}(D, \Lambda^2(\mathbb{R}^2)).$$

Now if $da + d^*\beta = \omega$, then testing with da gives

$$\|da\|^2_{L^2(D)} = \int_D \langle da, da\rangle + \int_D \langle d^*\beta, da\rangle = \int_D \langle \omega, da\rangle.$$

This shows that $d^*\omega = 0$ implies $a = 0$. Moreover from the Poincarè inequality we get

$$\|a\|_{W^{1,2}(D)} \leq C\,\|da\|_{L^2(D)} \leq C\,\|\omega\|_{L^2(D)}.$$

$\omega = 0$ implies further $d^*\beta = 0$, which proves the uniqueness. For existence we solve

$$d^*da = -\Delta a = d^*\omega \text{ in } D \quad \text{where } a \in W_0^{1,2}(D).$$

Then $d^*(\omega - d\alpha) = 0$, which is the integrability condition to get the desired solution of

$$d^*\beta = \omega - d\alpha, \quad \beta \in W^{1,2}(D, \Lambda^2(\mathbb{R}^2)) \text{ with } \int_D \beta = 0.$$

In fact, the Poincaré lemma does not apply directly due to lack of regularity. But we can solve classically $d^*\beta^\varepsilon = (\omega - d\alpha)_\varepsilon$ on $D_{1-\varepsilon}(0)$, where the right hand side is mollified. The constant coefficient operator d^* commutes with smoothing, so that the integrability condition is preserved. Then we normalize and let $\varepsilon \searrow 0$. □

The proof of regularity for two-dimensional harmonic maps $u : D \to \mathbb{S}^{n-1}$ divides into two steps: first one shows that u is continuous, this is due to Hélein. The second step proving smoothness was known before Hélein's work, it is not specific to harmonic maps but applies to general elliptic systems with right hand side quadratic in Du.

Theorem 3.3.2 ([26]). *2-dimensional harmonic maps $u \in W^{1,2}(D, \mathbb{S}^{n-1})$ are continuous.*

Proof. Our aim is to realize the right hand side of the Euler-Lagrange equation as a sum of terms of the form $\langle da, d^*\beta\rangle$, i.e. Jacobi determinants. Denote the rows of the Jacobi matrix Du by du^i. The Euler-Lagrange equation is

$$-\Delta u^j = \sum_{i=1}^n \langle du^i, du^i\rangle\, u^j.$$

Now du^i is a differential, thus $\omega^{ij} = u^j du^i$ should be a co-differential. This is only possible if $d^*\omega^{ij} = 0$. Now the conservation law, Theorem 3.1.1, comes into play. We know that

$$\operatorname{div}(Du^{\mathrm{T}}\Lambda_{ij}u) = 0 \quad \text{where } \Lambda_{ij} = e_i \otimes e_j - e_j \otimes e_i.$$

We compute for $v \in \mathbb{R}^2$

$$\langle Du^{\mathrm{T}}\Lambda_{ij}u, v\rangle_{\mathbb{R}^2} = \langle \Lambda_{ij}u, Du\cdot v\rangle_{\mathbb{R}^n} = \langle u^j du^i - u^i du^j, v\rangle_{\mathbb{R}^2}.$$

Thus $d^*(u^j du^i - u^i du^j) = 0$ in the sense of distributions, so by the Hodge lemma

$$u^j du^i - u^i du^j = d^*\beta^{ij} \quad \text{where } \beta^{ij} = b^{ij}\, dx\wedge dy \in W^{1,2}(D, \Lambda^2(\mathbb{R}^2)).$$

Luckily, the equation $|u|^2 = 1$ yields the identity

$$0 = d\,\Big(\frac{1}{2}|u|^2\Big) = \sum_{i=1}^n u^i\, du^i.$$

Thus we can write the harmonic map system as

$$-\Delta u^j = \sum_{i=1}^n \langle du^i, u^j du^i - u^i du^j\rangle_{\mathbb{R}^2} = \sum_{i=1}^n \langle du^i, d^*\beta^{ij}\rangle = \sum_{i=1}^n \{u^i, b^{ij}\}.$$

To arrange for zero boundary values, we let $h \in W^{1,2}(D,\mathbb{R}^n)$ be the harmonic extension of $u|_{\partial D}$. Then $v = u - h \in W^{1,2}(D,\mathbb{R}^n)$ solves the problem

$$-\Delta v^j = \sum_{i=1}^n \{u^i, b^{ij}\} \text{ in } D, \quad v = 0 \text{ on } \partial D.$$

We have $v \in C^0(\overline{D},\mathbb{R}^n)$ by Theorem 3.2.1, the Wente lemma. As the harmonic function h is smooth in D, we conclude that $u = v + h$ is also continuous in D. □

In the remaining part of this section we take up the problem of higher regularity for systems of harmonic map type in arbitrary dimensions. This goes back to S. Hildebrandt, K.-O. Widman and M. Wiegner.

Theorem 3.3.3 ([30, 66]). *Let $u \in W^{1,2}\cap L^\infty(B_2(0),\mathbb{R}^n)$ be a weak solution of the equation $-\Delta u = A(u)(Du,Du)$ on $B_2(0) \subset \mathbb{R}^m$, $m \geq 2$. Assume that for constants $a, M < \infty$*

$$|A(z)(p,p)| \leq a\,|p|^2 \quad \textit{for all } |z| \leq M,\ p \in \mathbb{R}^{n\times m}, \tag{3.18}$$

$$\|u\|_{L^\infty(B_2(0))} \leq M. \tag{3.19}$$

Then the following holds:

(1) *Let $\alpha \in (0,1)$. If $aM \leq \varepsilon_0 = \varepsilon_0(\alpha)$ then $u \in C^{0,\alpha}(B_1(0),\mathbb{R}^n)$.*

(2) *For $\alpha \in (\frac{2}{3},1)$ we get further $u \in C^{1,\mu}(B_1(0),\mathbb{R}^n)$, where $\mu = \frac{3}{2}\alpha - 1 \in (0,\frac{1}{2})$.*

Proof. For $x \in B_1(0)$ and $\varrho \in (0,1]$, let $v \in W^{1,2}(B_\varrho(x),\mathbb{R}^n)$ be harmonic with $v - u \in W_0^{1,2}(B_\varrho(x))$. We have the standard estimates

$$\sup_{B_\varrho(x)} |v| \leq \|u\|_{L^\infty(B_\varrho(x))} \leq M,$$

$$\sup_{B_{\varrho/2}(x)} |Dv| + \varrho \sup_{B_{\varrho/2}(x)} |D^2 v| \leq \frac{C}{\varrho^{m/2}}\|Dv\|_{L^2(B_\varrho(x))}.$$

For $w = u - v$ and $\phi \in W_0^{1,2} \cap L^\infty(B_\varrho(x), \mathbb{R}^n)$ we infer

$$\int_{B_\varrho(x)} \langle Dw, D\phi \rangle = \int_{B_\varrho(x)} \langle A(u)(Du, Du), \phi \rangle.$$

Taking $\phi = w$ yields the inequality

$$\int_{B_\varrho(x)} |Dw|^2 = \int_{B_\varrho(x)} \langle A(u)(Du, Du), w \rangle \leq a\, \|w\|_{L^\infty(B_\varrho(x))} \int_{B_\varrho(x)} |Du|^2. \tag{3.20}$$

Now let $\theta \in (0, \frac{1}{2}]$. By the above bounds for v, we have the decay

$$\int_{B_{\theta\varrho}(x)} |Dv|^2 \leq C(\theta\varrho)^m \sup_{B_{\varrho/2}(x)} |Dv|^2 \leq C\theta^m \int_{B_\varrho(x)} |Dv|^2.$$

Using $\|w\|_{L^\infty(B_\varrho(x))} \leq \|u\|_{L^\infty(B_\varrho(x))} + \|v\|_{L^\infty(B_\varrho(x))} \leq 2M$, we estimate

$$\begin{aligned}
(\theta\varrho)^{2-m} \int_{B_{\theta\varrho}(x)} |Du|^2 &\leq 2(\theta\varrho)^{2-m} \int_{B_{\theta\varrho}(x)} |Dv|^2 + 2(\theta\varrho)^{2-m} \int_{B_{\theta\varrho}(x)} |Dw|^2 \\
&\leq C\theta^m (\theta\varrho)^{2-m} \int_{B_\varrho(x)} |Dv|^2 + 4aM(\theta\varrho)^{2-m} \int_{B_\varrho(x)} |Du|^2 \\
&\leq C\theta^2 \big(1 + aM\theta^{-m}\big)\, \varrho^{2-m} \int_{B_\varrho(x)} |Du|^2.
\end{aligned}$$

In the last step we used that $v(x)$ minimizes the Dirichlet energy with given boundary values. Assume for the moment that

$$aM \leq \theta^m, \tag{3.21}$$

so that for any $x \in B_1(0)$, $\varrho \in (0,1]$ we have the inequality

$$\phi(x, \theta\varrho) \leq C\theta^2 \phi(x, \varrho) \quad \text{ where } \phi(x,\varrho) = \varrho^{2-m} \int_{B_\varrho(x)} |Du|^2.$$

Given $\varrho \in (0,1]$ we choose $k \in \mathbb{N}_0$ with $\theta^{k+1} < \varrho \leq \theta^k$, and iterate

$$\begin{aligned}
\phi(x,\varrho) &\leq \theta^{2-m} \phi(x, \theta^k) \\
&\leq \theta^{2-m} (C\theta^2)^k \phi(x,1) \\
&\leq \theta^{2-m-2\alpha} (C\theta^{2-2\alpha})^k\, \varrho^{2\alpha} \int_{B_2(0)} |Du|^2.
\end{aligned}$$

Given $\alpha \in [0,1)$, we chose $\theta = \theta(\alpha) \in (0, \frac{1}{2}]$ with $C\theta^{2-2\alpha} \leq 1$, and take $\varepsilon_0 = \theta^m$ in assumption (1). Then (3.21) holds, and we conclude

$$\varrho^{2-m} \int_{B_\varrho(x)} |Du|^2 \leq C(\alpha) \varrho^{2\alpha} \int_{B_2(0)} |Du|^2.$$

By Morrey's Dirichlet growth theorem $u(x)$ is α-Hölder continuous on $B_1(0)$. To prove that Du is also Hölder continous, we use that the L^∞ bound for w has improved. Namely writing

$u_{x,\varrho}$ for the mean value on $B_\varrho(x)$ we can now estimate, again using the maximum principle for harmonic functions,

$$\|w\|_{L^\infty(B_\varrho(x)} \le \|u-u_{x,\varrho}\|_{L^\infty(B_\varrho(x))} + \|v-u_{x,\varrho}\|_{L^\infty(B_\varrho(x))} \le 2\|u-u_{x,\varrho}\|_{L^\infty(B_\varrho(x))} \le C\varrho^\alpha.$$

Inserting this into (3.20) yields, passing to mean value integrals,

$$\fint_{B_\varrho(x)} |Dw|^2 \le C\varrho^\alpha \fint_{B_\varrho(x)} |Du|^2 \le C\,\varrho^\alpha \varrho^{2\alpha-2} = C\varrho^{3\alpha-2}.$$

For simplicity, we allow the constant C to depend on $\|Du\|_{L^2(B_2(0))}$, and we will write $2\mu := 3\alpha-2>0$. Clearly

$$\begin{aligned} \fint_{B_{\theta\varrho}(x)} |Dw|^2 &\le \theta^{-m} \fint_{B_\varrho(x)} |Dw|^2 \le C\theta^{-m}\varrho^{2\mu}, \\ |(Dw)_{x,\theta\varrho}|^2 &\le \fint_{B_{\theta\varrho}(x)} |Dw|^2 \le C\theta^{-m}\varrho^{2\mu}. \end{aligned}$$

This gives

$$\begin{aligned} \fint_{B_{\theta\varrho}(x)} |Du-(Du)_{x,\theta\varrho}|^2 &\le 2\fint_{B_{\theta\varrho}(x)} |Dv-(Dv)_{x,\theta\varrho}|^2 + 2\fint_{B_{\theta\varrho}(x)} |Dw|^2 + 2|(Dw)_{x,\theta\varrho}|^2 \\ &\le 2\fint_{B_{\theta\varrho}(x)} |Dv-(Dv)_{x,\theta\varrho}|^2 + C\theta^{-m}\varrho^{2\mu}. \end{aligned}$$

It remains to estimate for the harmonic function v. Using the Poincaré inequality and the standard bounds from above we obtain, for $A \in \mathbb{R}^{n\times m}$ arbitrary,

$$\begin{aligned} \fint_{B_{\theta\varrho}(x)} |Dv-(Dv)_{x,\theta\varrho}|^2 &\le C(\theta\varrho)^2 \fint_{B_{\theta\varrho}(x)} |D^2 v|^2 \\ &\le C(\theta\varrho)^2\varrho^{-2} \fint_{B_\varrho(x)} |Dv-A|^2 \\ &= C\theta^2 \fint_{B_\varrho(x)} |D(v-\ell)|^2 \quad \text{where } \ell(y)=Ay \\ &\le C\theta^2 \fint_{B_\varrho(x)} |Du-A|^2. \end{aligned}$$

In the last step the minimizing property of $v-\ell$ was used. Taking $A=(Du)_{x,\varrho}$ and combining we arrive at

$$\phi(x,\theta\varrho) \le C\theta^2\phi(x,\varrho) + C\theta^{-m}\varrho^{2\mu} \quad \text{where } \phi(x,\varrho) = \fint_{B_\varrho(x)} |Du-(Du)_{x,\varrho}|^2.$$

Using induction, we see that

$$\begin{aligned} \phi(x,\theta^k) &\le (C\theta^2)^k\phi(x,1) + C\theta^{-m}\theta^{(k-1)2\mu}\sum_{j=0}^{k-1}(C\theta^{2-2\mu})^j \\ &\le C\theta^{2\mu k}\Big((C\theta^{2-2\mu})^k + \theta^{-m-2\mu}\sum_{j=0}^{k-1}(C\theta^{2-2\mu})^j\Big). \end{aligned}$$

Now fix $\alpha \in (\frac{2}{3}, 1)$ or equivalently $\mu \in (0, \frac{1}{2})$, and chose $\theta \in (0, \frac{1}{2}]$ with $C\theta^{2-2\mu} \leq \frac{1}{2}$. Then for $\theta^{k+1} < \varrho \leq \theta^k$ we infer, using the best approximating property of the mean value,

$$\begin{aligned}\phi(x, \varrho) &= \fint_{B_\varrho(x))} |Du - (Du)_{x,\varrho}|^2 \\ &\leq \fint_{B_\varrho(x))} |Du - (Du)_{x,\theta^k}|^2 \\ &\leq \theta^{-m} \fint_{B_{\theta^k}(x)} |Du - (Du)_{x,\theta^k}|^2 \\ &\leq C\varrho^{2\mu}.\end{aligned}$$

Campanato's lemma implies that Du is μ-Hölder continuous on $B_1(0)$. □

Corollary 3.3.4. *Let $u \in W^{1,2} \cap L^\infty(U, M)$ be a harmonic map on the open set $U \subset \mathbb{R}^m$ into the smooth submanifold $M \subset \mathbb{R}^n$. Assume that $u(x)$ is continuous at $x_0 \in U$, more precisely*

$$\lim_{\varrho \searrow 0} \|u - p\|_{L^\infty(B_\varrho(x_0))} = 0 \quad \textit{for some } p \in M.$$

Then $u(x)$ is smooth in a full neighborhood of x_0.

Proof. By translations we may assume $x_0 = 0$ and $p = 0$. The $u_\lambda : B \to M$, $u_\lambda(x) = u(\lambda x)$, are harmonic and satisfy

$$\|u_\lambda\|_{L^\infty(B_2(0))} = \|u\|_{L^\infty(B_{2\lambda}(0))} \to 0 \quad \text{as } \lambda \searrow 0.$$

For fixed $\alpha \in (0, 1)$, Theorem 3.3.3 yields $u_\lambda \in C^{1,\mu}(B_1(0), \mathbb{R}^n)$ for some $\mu > 0$. Thus $A(u)(Du, Du)$ is of class $C^{0,\mu}$ near the origin, which means that its Newtonian potential and hence $u(x)$ are locally $C^{2,\mu}$ on a neighborhood of the origin. Repeated application of the Schauder estimates shows that $u(x)$ is smooth on that neighborhood. □

Chapter 4

Hardy space

In this chapter we discuss applications involving estimates in Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.

4.1 Higher integrability of Jacobi determinants

We start by collecting some basic results about the maximal function.

Lemma 4.1.1. *Let $f : \mathbb{R}^n \to [0, \infty)$ be measurable and $0 < p < \infty$. Then we have*

$$\int_{\mathbb{R}^n} f^p(x)\,dx = p \int_0^\infty \alpha^{p-1} |\{x : f(x) > \alpha\}|\,d\alpha. \tag{4.1}$$

Proof. Let χ_f be the characteristic function of the set $\{(x, \alpha) \in \mathbb{R}^n \times [0, \infty) : f(x) > \alpha\}$. Then χ_f is $\mathcal{L}^n \times \mathcal{L}^1$ measurable, and

$$f(x)^p = p \int_0^{f(x)} \alpha^{p-1}\,d\alpha = p \int_0^\infty \alpha^{p-1} \chi_f(x, \alpha)\,d\alpha.$$

Integrating we get by Fubini's theorem

$$\int_{\mathbb{R}^n} f(x)^p\,dx = p \int_0^\infty \alpha^{p-1} \int_{\mathbb{R}^n} \chi_f(x, \alpha)\,dx\,d\alpha = p \int_0^\infty \alpha^{p-1} |\{x : f(x) > \alpha\}|\,d\alpha.$$

□

Definition 4.1.2. *For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define its maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ by*

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x)|\,dx. \tag{4.2}$$

When dealing with the maximal function one needs two basic principles, the Vitali covering lemma and the Calderon Zygmund decomposition. In Vitali's lemma the following notation is used: for any ball B we write $5B$ for the concentric ball with 5 times the radius.

Theorem 4.1.3 (Vitali)**.** *Let $\mathcal{B}$ be a family of nondegenerate closed balls in $\mathbb{R}^n$ with diameter bounded by a constant $d < \infty$. Then there exists a disjoint subfamily $\mathcal{B}'$ with the following property: for any $B \in \mathcal{B}$ there is a $B' \in \mathcal{B}'$ such that*

$$B \cap B' \neq \emptyset \quad \textit{and} \quad \operatorname{diam} B' > \operatorname{diam} B/2,$$

in particular $B \subset 5B'$. *Thus if* $\mathcal{B}$ *is a covering of a set* E *then*

$$|E| \leq 5^n \sum_{B' \in \mathcal{B}'} |B'|. \tag{4.3}$$

Proof. Divide $\mathcal{B}$ into the families $\mathcal{B}_k$ with diameter in $(2^{-k-1}d, 2^{-k}d]$ for $k = 0, 1, \ldots$. We define $\mathcal{B}' = \bigcup_{k=0}^{\infty} \mathcal{B}'_k$ inductively as follows: $\mathcal{B}'_0$ is a maximal disjoint subfamily of $\mathcal{B}_0$, and $\mathcal{B}'_k$ is a maximal disjoint subfamily of those balls in $\mathcal{B}_k$, which do not intersect any ball that was previously selected. Now any $B \in \mathcal{B}$ belongs to some $\mathcal{B}_k$. By maximality, it must intersect a ball $B' \in \mathcal{B}'_j$ for some $j \leq k$, which proves the theorem. □

Theorem 4.1.4 (Calderon-Zygmund [7]). *Let* $f \in L^1(\mathbb{R}^n)$ *with* $f \geq 0$. *For any* $\alpha > 0$ *there exists a countable family* $\mathcal{G}$ *of closed cubes with pairwise disjoint interior, such that the following holds:*

(i) $\alpha < ⨍_Q f(x)\,dx \leq 2^n \alpha$ *for any* $Q \in \mathcal{G}$.

(ii) $f(x) \leq \alpha$ *for almost all* $x \in \mathbb{R}^n \backslash G$, *where* $G = \bigcup_{Q \in \mathcal{G}} Q$.

(iii) $|G| \leq \dfrac{C}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$.

Proof. Chose a subdivision of $\mathbb{R}^n$ into congruent cubes P having volume $|P| \geq \frac{1}{\alpha}\|f\|_{L^1(\mathbb{R}^n)}$, and denote this family by $\mathcal{F}_0$. Clearly for $P \in \mathcal{F}_0$

$$\frac{1}{|P|} \int_P f(x)\,dx \leq \alpha. \tag{4.4}$$

Using induction we now define families $\mathcal{F}_k$, $\mathcal{G}_k$ of cubes for $k = 1, 2, \ldots$. For this divide each $P \in \mathcal{F}_{k-1}$ into 2^n congruent subcubes using edge bisection. Then denote by $\mathcal{F}_k$ the subfamily for which (4.4) holds, and by $\mathcal{G}_k$ the subfamily where (4.4) fails. The union of the cubes in $\mathcal{F}_k$, $\mathcal{G}_k$ is denoted by F_k,G_k; we note $\mathbb{R}^n = F_k \cup \bigcup_{j=1}^{k} G_j$. The process is iterated as long as $\mathcal{F}_k$ is nonempty. We prove the result for $\mathcal{G} = \bigcup \mathcal{G}_k$. Two cubes $Q \in \mathcal{G}_k$, $Q' \in \mathcal{G}_\ell$ with $k < \ell$ have disjoint interior, since Q' comes from some $P \in \mathcal{F}_k$. If Q belongs to $\mathcal{G}_k$ and comes from $P \in \mathcal{F}_{k-1}$, then

$$\alpha < \frac{1}{|Q|} \int_Q f(x)\,dx \leq \frac{2^n}{|P|} \int_P f(x)\,dx \leq 2^n \alpha.$$

This proves (i). For $x \notin G$, we have $x \in P_k$ for a sequence $P_k \in \mathcal{F}_k$, thus

$$\frac{1}{|P_k|} \int_{P_k} f(x)\,dx \leq \alpha \quad \text{ where } |P_k| \to 0.$$

By the Lebesgue differentiation theorem, see Cor. 2, Sect. 1.7 of [14], the left hand side converges to $f(x)$ a.e. which proves (ii). Finally (iii) follows since

$$|G| = \sum_{Q \in \mathcal{G}} |Q| \leq \frac{1}{\alpha} \sum_{Q \in \mathcal{G}} \int_Q |f(x)|\,dx = \frac{1}{\alpha} \int_G |f(x)|\,dx.$$

□

Our aim is to compare the function with its maximal function in terms of integrability. The following are the key inequalities.

Lemma 4.1.5. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $f \geq 0$, and $\alpha > 0$. Then for a constant $C = C(n) < \infty$*

$$|\{x : Mf(x) > \alpha\}| \quad \leq \quad \frac{C}{\alpha} \int_{\{x : f(x) > \frac{\alpha}{2}\}} f(x)\, dx. \tag{4.5}$$

Reversely, there are constants $C = C(n) < \infty$, $\lambda = \lambda(n) > 0$, such that

$$\frac{1}{\alpha} \int_{\{x : f(x) > \alpha\}} f(x)\, dx \quad \leq \quad C\, |\{x : Mf(x) > \lambda\alpha\}|. \tag{4.6}$$

Proof. To prove (4.5) we chose for each $x \in \mathbb{R}^n$ with $Mf(x) > \alpha$ a radius $r_x > 0$ such that

$$\int_{B^x} f(y)\, dy > \alpha |B^x| \qquad \text{where } B^x = \overline{B_{r_x}(x)}.$$

By Vitali, Theorem 4.1.3, there are disjoint B^{x_k}, $k \in \mathbb{N}$, such that the set $\{x : Mf(x) > \alpha\}$ is covered by the enlarged balls $5B^{x_k}$. Hence

$$|\{x : Mf(x) > \alpha\}| \leq 5^n \sum_{k=1}^{\infty} |B^{x_k}| \leq \frac{5^n}{\alpha} \sum_{k=1}^{\infty} \int_{B^{x_k}} |f(y)|\, dy \leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f(x)|\, dx. \tag{4.7}$$

The trick to obtain the improved inequality (4.5) is to consider

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) \geq \alpha/2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f(x) \leq f_1(x) + \frac{\alpha}{2}$ for all $x \in \mathbb{R}^n$, which implies $Mf(x) \leq Mf_1(x) + \frac{\alpha}{2}$ and thus

$$\{x : Mf(x) > \alpha\} \subset \{x : Mf_1(x) > \frac{\alpha}{2}\}.$$

Applying the previous estimate (4.7) to f_1 yields

$$|\{x : Mf(x) > \alpha\}| \leq |\{x : Mf_1(x) > \frac{\alpha}{2}\}| \leq \frac{2 \cdot 5^n}{\alpha} \int_{\{x : f(x) > \alpha/2\}} f(x)\, dx.$$

We now prove (4.6), first assuming $f \in L^1(\mathbb{R}^n)$. Let $Q \in \mathcal{G}$ be as in Theorem 4.1.4 by Calderon-Zygmund. Then for any $x \in Q$ we have, putting $d = \operatorname{diam} Q$,

$$\alpha < \frac{1}{|Q|} \int_Q f(y)\, dy \leq \frac{|B_d(x)|}{|Q|} \frac{1}{|B_d(x)|} \int_{B_d(x)} f(y)\, dy \leq 2^n n^{n/2} Mf(x).$$

Using (ii) and (i) from Theorem 4.1.4 we infer, recalling that cubes in $\mathcal{G}$ have disjoint interior,

$$\begin{aligned} \frac{1}{\alpha} \int_{\{x : f(x) > \alpha\}} f(x)\, dx \quad &\leq \quad \frac{1}{\alpha} \sum_{Q \in \mathcal{G}} \int_Q f(x)\, dx \\ &\leq \quad 2^n \sum_{Q \in \mathcal{G}} |Q| \\ &\leq \quad 2^n |\{x : Mf(x) > 2^{-n} n^{-n/2} \alpha\}|. \end{aligned}$$

To get (4.6) for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we apply the result to $f_k = \chi_{\{|x|<k\}} f \nearrow f$, this yields

$$\frac{1}{\alpha}\int_{\{x:f_k(x)>\alpha\}} f_k(x)\,dx \le C\,|\{x : Mf_k(x) > \lambda\alpha\}| \le C\,|\{x : Mf(x) > \lambda\alpha\}|.$$

The left hand side passes to the limit by monotone convergence. □

As first consequence, we show that for $1 < p < \infty$ there is no difference between f and Mf when it comes to L^p integrability.

Theorem 4.1.6 (Hardy-Littlewood)**.** *For $f \in L^p(\mathbb{R}^n)$ with $1 < p \le \infty$ we have*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \text{ where } C = C(n,p) < \infty. \tag{4.8}$$

Proof. The case $p = \infty$ holds with constant $C(n,\infty) = 1$. For $1 < p < \infty$ we estimate using (4.1) and (4.5)

$$\begin{aligned}
\int_{\mathbb{R}^n} |Mf(x)|^p\,dx &= p\int_0^\infty \alpha^{p-1}|\{x : |Mf(x)| > \alpha\}\,d\alpha \\
&\le C(n)p\int_0^\infty \alpha^{p-2}\int_{\{x\in\mathbb{R}^n:|f(x)|>\alpha/2\}} |f(x)|\,dx\,d\alpha \\
&= C(n)p\int_{\mathbb{R}^n} |f(x)|\int_0^{2|f(x)|} \alpha^{p-2}\,d\alpha\,dx \\
&\le \frac{C(n)2^{p-1}p}{p-1}\int_{\mathbb{R}^n} |f(x)|^p\,dx.
\end{aligned}$$

□

Remark 4.1.7. *The function Mf is never in $L^1(\mathbb{R}^n)$ unless $f \equiv 0$. In fact, for any $R < \infty$ we have $B_R(0) \subset B_{2|x|}(x)$ for $|x| \ge R$, yielding the lower bound*

$$Mf(x) \ge \frac{1}{|B_{2|x|}(x)|}\int_{B_{2|x|}(x)} |f(y)|\,dy \ge \frac{c}{|x|^n}\int_{B_R(0)} |f(y)|\,dy.$$

If $\|f\|_{L^1(B_R(0))} > 0$ then the right hand side is not integrable. To get an example where the maximal function is locally not integrable consider $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \frac{1}{|x|^n \log^2|x|}\chi_{B_{1/e}(0)} \ge 0 \quad \text{where } e = 2.718\ldots.$$

We compute substituting $r = e^{-t}$ for $0 < \varrho \le 1/e$

$$\int_{B_\varrho(0)} f(x)\,dx = \int_0^\varrho \frac{dr}{r\log^2 r} = \int_{-\log\varrho}^\infty \frac{dt}{t^2} = -\frac{1}{\log\varrho}.$$

In particular $f \in L^1(\mathbb{R}^n)$. On the other hand as $B_{2|x|}(x) \supset B_{|x|}(0)$, we estimate for $|x| \le 1/e$

$$Mf(x) \ge \frac{c(n)}{|x|^n}\int_{B_{2|x|}(x)} f(y)\,dy \ge \frac{c(n)}{|x|^n}\int_{B_{|x|}(0)} f(y)\,dy \ge -\frac{c(n)}{|x|^n\log|x|}.$$

The right hand side is not integrable near the origin (for the integrals see also example 2.4.2).

Contrary to the case $p > 1$, the L^1 integrability of the maximal function Mf implies an improved integrability of f. This was discovered by E. Stein.

Theorem 4.1.8 (Stein [55]). *Let $B \subset \mathbb{R}^n$ be a round ball. Then for any $f \in L^1(B)$ we have*

$$\fint_B |f| \log^+ |f| \leq C\big(\|Mf\|_{L^1(B)}\big) \tag{4.9}$$

where $\log^+ := \max(1, \log)$.

Proof. We assume $f \geq 0$ and $B = B_1(0)$ by scaling. By (4.6) we estimate, using $\lambda = \lambda(n) > 0$,

$$\begin{aligned}
\int_B f(x) \log^+ f(x)\, dx &= \int_B f(x) \int_1^\infty \chi_f(x,\alpha)\, \frac{d\alpha}{\alpha}\, dx \\
&= \int_1^\infty \frac{1}{\alpha} \int_{\{x : f(x) > \alpha\}} f(x)\, dx\, d\alpha \\
&\leq C \int_1^\infty |\{x : Mf(x) > \lambda\alpha\}|\, d\alpha \\
&\leq C \int_\lambda^\infty |\{x : Mf(x) > \beta\}|\, d\beta \\
&= C \int_{\{x : Mf(x) > \lambda\}} Mf(x)\, dx \\
&\leq C\Big(\|Mf\|_{L^1(B)} + \int_{\{|x|>1,\, Mf(x)>\lambda\}} Mf(x)\, dx\Big).
\end{aligned}$$

From $\operatorname{spt} f \subset B$ we see that for $|x| > 1$

$$Mf(x) \leq \frac{C}{(|x|-1)^n} \|f\|_{L^1},$$

in particular

$$Mf(x) > \lambda \quad \Rightarrow \quad |x| < 1 + \Big(\frac{C\|f\|_{L^1}}{\lambda}\Big)^{1/n} =: R.$$

Now for $|x| \geq \frac{3}{2}$ we have $Mf(x) \leq 2^n C \|f\|_{L^1}$. The inequality $(1+t)^n \leq C(1+t^n)$ yields

$$\begin{aligned}
\int_{\{x : |x| > \frac{3}{2},\, Mf(x) > \lambda\}} Mf(x)\, dx &\leq C\|f\|_{L^1} R^n \\
&\leq C\|f\|_{L^1}(1 + \|f\|_{L^1}) \\
&\leq C\big(1 + \|Mf\|^2_{L^1(B)}\big).
\end{aligned}$$

On the remaining annulus, consider the reflection

$$\phi : B_1 \backslash B_{\frac{1}{2}} \to B_{\frac{3}{2}} \backslash B_1,\ \phi(x) = (2 - |x|)\frac{x}{|x|}.$$

Given $x \in B_1 \backslash B_{\frac{1}{2}}$, any $y \in B_1$ decomposes as $y = s\frac{x}{|x|} + y^\perp$ where $-1 \leq s \leq 1$. Then $|s - |x|| \leq |s - (2 - |x|)|$ which implies

$$|y - x|^2 = (s - |x|)^2 + |y^\perp|^2 \leq (s - (2 - |x|))^2 + |y^\perp|^2 = |y - \phi(x)|^2.$$

Thus $B_r(\phi(x)) \cap B \subset B_r(x) \cap B$ for any $r > 0$. Substituting $y = \phi(x)$ we conclude

$$\int_{\{1<|x|<\frac{3}{2}\}} Mf(y)\,dy \le C \int_{\{\frac{1}{2}<|x|<1\}} Mf(\phi(x))\,dx \le C \int_{\{\frac{1}{2}<|x|<1\}} Mf(x)\,dx.$$

The theorem follows by combining the estimates. □

Remark 4.1.9. *The set of functions for which $|f|\log^+|f|$ is integrable is called the $L\log L$ class. As noted by Stein the above theorem is sharp, in the sense that the $L\log L$ property implies the local integrability of Mf. To see this we write for any set $E \subset \mathbb{R}^n$*

$$\begin{aligned}\int_E Mf\,dx &= 2\int_0^\infty |\{x \in E : Mf(x) > 2\alpha\}|\,d\alpha \\ &\le 2|E| + 2\int_1^\infty |\{x : Mf(x) > 2\alpha|\,d\alpha.\end{aligned}$$

Now (4.5) yields, with $\chi_f(x,\alpha)$ the characteristic function of $\{|f(x)| > \alpha\}$,

$$\begin{aligned}\int_1^\infty |\{x : Mf(x) > 2\alpha\}|\,d\alpha &\le \int_1^\infty \Big(\frac{C}{\alpha}\int_{\mathbb{R}^n} \chi_{\{x:|f(x)|>\alpha\}}|f(x)|\,dx\Big)\,d\alpha \\ &= C\int_{\mathbb{R}^n} |f(x)| \int_1^\infty \chi_f(x,\alpha)\frac{d\alpha}{\alpha}\,dx \\ &= C\int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|\,dx.\end{aligned}$$

Thus for an arbitrary set E we obtain

$$\int_E Mf(x)\,dx \le C\Big(|E| + \int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|\,dx\Big). \tag{4.10}$$

Next we review some facts about degree theory and Jacobi determinants. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1. For a map $u \in C^2(\overline{\Omega}, \mathbb{R}^n)$ the oriented multiplicity function $i_u : \mathbb{R}^n \to \mathbb{Z}$ is given by

$$i_u(y) = \begin{cases} \sum_{u(x)=y} \operatorname{sign}\det Du(x) & \text{if } y \notin u(\partial\Omega) \text{ is a regular value,} \\ 0 & \text{else.} \end{cases} \tag{4.11}$$

Here $y \notin u(\partial\Omega)$ is a regular value if and only if $\det Du(x) \neq 0$ for all $x \in u^{-1}\{y\}$. By the inverse function theorem and compactness, each regular value has only finitely many preimages so that the sum is defined. Our main tool in the following is the transformation formula: for any $g \in L^1(\Omega)$, the function $y \mapsto \sum_{u(x)=y} g(x)$ is integrable on $\mathbb{R}^n$ and

$$\int_\Omega g(x)|\det Du(x)|\,d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \Big(\sum_{u(x)=y} g(x)\Big)\,d\mathcal{L}^n(y).$$

In particular, the set of points in $\mathbb{R}^n \backslash u(\partial\Omega)$ which are not regular has Lebesgue measure zero. This is actually a step in the proof of the transformation formula, see [14, Section 3.3]. As $u(\partial\Omega)$ is also a null set, the first alternative in the definition of i_u applies almost everywhere.

Now let $h \in C_c^0(\mathbb{R}^n)$. By the transformation formula, we calculate

$$\begin{aligned}\int_{\mathbb{R}^n} h(y)\, i_u(y)\, d\mathcal{L}^n(y) &= \int_\Omega h(u(x))\,\mathrm{sign}\det Du(x)|\det Du(x)|\, d\mathcal{L}^n(x)\\ &= \int_\Omega h(u(x))\det Du(x)\, d\mathcal{L}^n(x)\\ &= \int_\Omega u^*(h(y)\,dy),\end{aligned}$$

where $dy = dy^1 \wedge \ldots \wedge dy^n$. Inserting $h = \operatorname{div}\phi$ where $\phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ we infer

$$\int_{\mathbb{R}^n} (\operatorname{div}\phi)\, i_u\, d\mathcal{L}^n = \int_\Omega u^*(\operatorname{div}\phi\, dy) = \int_\Omega u^* d(\phi \llcorner dy) = \int_\Omega du^*(\phi \llcorner dy) = \int_{\partial\Omega} u^*(\phi \llcorner dy).$$

Taking $\operatorname{spt}\phi \subset \mathbb{R}^n \backslash u(\partial B)$ yields $Di_u = 0$ on $\mathbb{R}^n \backslash u(\partial B)$, hence i_u is constant on the components of that set. For general ϕ we compute further

$$\begin{aligned}u^*(\phi \llcorner dy)(e_1, \ldots, \widehat{e_j}, \ldots, e_n) &= \det(\phi \circ u, \partial_1 u, \ldots, \widehat{\partial_j u}, \ldots, \partial_n u)\\ &= (\phi^i \circ u)\,\det(e_i, \partial_1 u, \ldots, \widehat{\partial_j u}, \ldots, \partial_n u)\\ &= (-1)^{j-1}(\phi^i \circ u)\,\mathrm{cof}(Du)_{ij}.\end{aligned}$$

Here $\mathrm{cof}_{ij}(Du)$ equals $(-1)^{i+j}$ times the ij-minor, i.e. the subdeterminant when the i-th row and j-th column of Du is omitted. Now assume $|\phi| \leq 1$, so that by Cauchy-Schwarz

$$\Big|\int_{\partial\Omega} u^*(\phi \llcorner dy)\Big| \leq \int_{\partial\Omega} |u^*(\phi \llcorner dy)|\, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} |\mathrm{cof}(Du)|\, d\mathcal{H}^{n-1}.$$

Recalling the definition of the variation measure $|Di_u|$ we arrive at

$$|Di_u|(\mathbb{R}^n) \leq \int_{\partial\Omega} |\mathrm{cof}(Du)|\, d\mathcal{H}^{n-1}. \tag{4.12}$$

The following is Lemma 1.3 in [39], see also Theorem 2.10 in [54] for the case $n = 3$.

Lemma 4.1.10. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. For any $x \in \Omega$ and almost all $r \in (0, \mathrm{dist}(x, \partial\Omega))$, we have for a constant $C = C(n) < \infty$*

$$\Big|\int_{B_r(x)} \det Du\, d\mathcal{L}^n\Big|^{\frac{n-1}{n}} \leq C \int_{\partial B_r(x)} |\mathrm{cof}(Du)|\, d\mathcal{H}^{n-1}. \tag{4.13}$$

Proof. We first assume that $u \in C^2(\Omega, \mathbb{R}^n)$. Using once more the transformation formula, and the fact that $i_{u,B_r(x)}$ is integer-valued, we have

$$\int_{B_r(x)} \det Du\, d\mathcal{L}^n = \int_{\mathbb{R}^n} i_{u,B_r(x)}\, d\mathcal{L}^n \leq \int_{\mathbb{R}^n} |i_{u,B_r(x)}|^{\frac{n}{n-1}}\, d\mathcal{L}^n.$$

Further, the Sobolev embedding theorem, see [14, Sec. 5.6], and (4.12) yield

$$\Big(\int_{\mathbb{R}^n} |i_{u,B_r(x)}|^{\frac{n}{n-1}}\, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \leq C\,|Di_{u,B_r(x)}|(\mathbb{R}^n) \leq C \int_{\partial B_r(x)} |\mathrm{cof}(Du)|\, d\mathcal{H}^{n-1}.$$

This proves the lemma for maps $u \in C^2(\Omega, \mathbb{R}^n)$. Now let $u \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, and chose $u_k \in C^2(\Omega, \mathbb{R}^n)$ with $u_k \to u$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. By Fatou we have for $R < \operatorname{dist}(x, \partial\Omega)$

$$\begin{aligned} &\int_0^R \liminf_{k\to\infty} \int_{\partial B_r(x)} |\operatorname{cof}(Du_k) - \operatorname{cof}(Du)|\, d\mathcal{H}^{n-1} dr \\ \leq\; &\lim_{k\to\infty} \int_{B_R(x)} |\operatorname{cof}(Du_k) - \operatorname{cof}(Du)|\, d\mathcal{L}^n = 0. \end{aligned}$$

Thus for almost all $r \in (0, \operatorname{dist}(x, \partial\Omega))$ inequality (4.13) follows by approximation. □

We have now collected all ingredients to prove Müller's higher integrability theorem. The main idea is to estimate the maximal function of the Jacobi determinant by the maximal function of the $(n-1) \times (n-1)$ minors using (4.13). The advantage is that the minors come with a power $\frac{n}{n-1}$, so that Theorem 4.1.6 by Hardy-Littlewood can be applied.

Theorem 4.1.11 ([39]). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open and bounded. If $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ has $\det Du \geq 0$ almost everywhere, then for any compact set $K \subset \Omega$*

$$\int_K \det Du \, \log^+(\det Du) \leq C(d, \delta, \|u\|_{W^{1,n}(\Omega)}), \tag{4.14}$$

where $d = \operatorname{diam} K$ and $\delta = \operatorname{dist}(K, \partial\Omega)$.

Proof. Let B be a ball of radius d containing K, and put $g = \chi_K \det Du$. The result follows from Theorem 4.1.8 once we have the estimate

$$||Mg||_{L^1(B)} \leq C(|B|, \delta, ||u||_{W^{1,n}(\Omega)}). \tag{4.15}$$

We show an improved version where the right hand side depends only on the $L^{\frac{n}{n-1}}$-norm of $\operatorname{cof}(Du)$. For $r \geq \delta/4$ and all $x \in \mathbb{R}^n$ we have the trivial inequality

$$⨍_{B_r(x)} |g|\, d\mathcal{L}^n \leq \frac{C}{\delta^n} \int_\Omega |\det Du|\, d\mathcal{L}^n \leq \frac{C}{\delta^n} \int_\Omega |\operatorname{cof}(Du)|^{\frac{n}{n-1}}\, d\mathcal{L}^n. \tag{4.16}$$

In the last step we used $Du \cdot \operatorname{cof}(Du)^{\mathrm{T}} = (\det Du)\,\mathrm{Id}$, which implies

$$|\det Du|^n = |\det Du|\, |\det \operatorname{cof}(Du)| \leq |\det Du|\, |\operatorname{cof}(Du)|^n,$$

thus $|\det Du| \leq |\operatorname{cof}(Du)|^{n/(n-1)}$. Now for $r \leq \delta/4$ we may assume $\operatorname{dist}(x, \partial\Omega) > \delta/2$. As $\det Du \geq 0$, Lemma 4.1.10 implies for almost all $\varrho \in (r, 2r)$

$$\Big(\int_{B_r(x)} |g|\, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \leq \Big(\int_{B_\varrho(x)} \det Du\, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \leq C \int_{\partial B_\varrho(x)} |\operatorname{cof}(Du)|\, d\mathcal{H}^{n-1}.$$

Integrating on $(r, 2r)$ and dividing by r^n gives, putting $\operatorname{cof}(Du) = 0$ on $\mathbb{R}^n \backslash \Omega$,

$$\Big(⨍_{B_r(x)} |g|\, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \leq C ⨍_{B_{2r}(x)} |\operatorname{cof}(Du)|\, d\mathcal{L}^n \leq CM(\operatorname{cof}(Du))(x). \tag{4.17}$$

Combining (4.16) and (4.17) yields

$$Mg(x) \leq C\, M\big(\operatorname{cof}(Du)\big)^{\frac{n}{n-1}}(x) + \frac{C}{\delta^n} \int_\Omega |\operatorname{cof}(Du)|^{\frac{n}{n-1}}\, d\mathcal{L}^n,$$

and by integrating

$$\|Mg\|_{L^1(B)} \le C\,\|M(\mathrm{cof}(Du))\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(B)} + \frac{C|B|}{\delta^n}\|\mathrm{cof}(Du)\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Omega)}.$$

Now as $\frac{n}{n-1} > 1$ we can apply Theorem 4.1.6 by Hardy-Littlewood to get

$$\|M(\mathrm{cof}(Du))\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C\|\mathrm{cof}(Du)\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = C\|\mathrm{cof}(Du)\|_{L^{\frac{n}{n-1}}(\Omega)},$$

Combining gives (4.15), and hence the theorem. □

In [39] the estimate is stated for $\det Du\,\log(2+\det Du)$. This follows easily from the version above, using $\log(2+s) \le \log s + 1$ for $s \ge 2$. In Section 7 of [39] a counterexample is given, showing that the condition $\det Du \ge 0$ cannot be dropped.

4.2 The Hardy space

As is well-known a bounded sequence f_k in $L^1(\mathbb{R}^n)$ may have a weak limit which is not representable by an $L^1(\mathbb{R}^n)$ function, but only by a signed Radon measure. The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is continuously embedded into $L^1(\mathbb{R}^n)$ and has a norm which scales like the $L^1(\mathbb{R}^n)$ norm. However, as opposed to $L^1(\mathbb{R}^n)$ the unit ball in $\mathcal{H}^1(\mathbb{R}^n)$ is weakly sequentially compact.

To start we recall the notion of convergence in $C^0_c(\mathbb{R}^n)$, i.e. $\phi_k \to \phi$ if and only if

$$\bigcup_{k=1}^{\infty} \mathrm{spt}\,\phi_k \subset\subset \mathbb{R}^n \quad \text{and} \quad \|\phi_k - \phi\|_{C^0(\mathbb{R}^n)} \to 0.$$

It is possible to construct an underlying topology, however this is omitted for reasons of simplicity. For a linear form $\Lambda : C^0_c(\mathbb{R}^n) \to \mathbb{R}$ and $U \subset \mathbb{R}^n$ open, we define

$$|\Lambda|(U) := \sup\{\Lambda(\phi) : \phi \in C^0_c(\mathbb{R}^n),\, \mathrm{spt}\,\phi \subset U,\, |\phi| \le 1\} \in [0,\infty]. \tag{4.18}$$

For arbitrary sets $E \subset \mathbb{R}^n$ we then put $|\Lambda|(E) = \inf\{|\Lambda|(U) : E \subset U \text{ open}\}$. We denote by $C^0_c(\mathbb{R}^n)'$ the set of those Λ for which $|\Lambda|(U) < \infty$ whenever $U \subset\subset \mathbb{R}^n$. Clearly, any such Λ is sequentially continuous on $C^0_c(\mathbb{R}^n)$. Moreover, the Riesz representation theorem asserts that $|\Lambda|$ is a Radon measure, the so-called variation measure of Λ, and that there is a $|\Lambda|$-measurable function $\sigma : \mathbb{R}^n \to \{\pm 1\}$ such that

$$\Lambda(\phi) = \int_{\mathbb{R}^n} \phi\,\sigma d|\Lambda| \quad \text{for any } \phi \in C^0_c(\mathbb{R}^n).$$

The convolution of $\Lambda \in C^0_c(\mathbb{R}^n)'$ with $\phi \in C^0_c(\mathbb{R}^n)$ is the function

$$\phi * \Lambda : \mathbb{R}^n \to \mathbb{R},\ (\phi * \Lambda)(x) = \Lambda(\phi^x) \quad \text{where } \phi^x(y) = \phi(x-y).$$

For example $(\phi * \delta_0)(x) = \phi(x)$. We note that $\phi * \Lambda \in C^0(\mathbb{R}^n)$ since the map $\mathbb{R}^n \to C^0_c(\mathbb{R}^n)$, $x \mapsto \phi^x$, is sequentially continuous, as is $\Lambda : C^0_c(\mathbb{R}^n) \to \mathbb{R}$. Any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ defines canonically a functional $\Lambda_f \in C^0_c(\mathbb{R}^n)'$ by

$$\Lambda_f(\phi) = \int_{\mathbb{R}^n} \phi(y) f(y)\,dy,$$

and the two notions of convolution are consistent, in the sense that

$$(\phi * \Lambda_f)(x) = \Lambda_f(\phi^x) = \int_{\mathbb{R}^n} \phi(x-y) f(y)\, dy = (\phi * f)(x).$$

Now we introduce a general class of test functions, namely

$$\mathcal{T} = \{\phi \in C_c^\infty(\mathbb{R}^n) : \operatorname{spt}\phi \subset B_1(0) \text{ and } \|D\phi\|_{L^\infty} \leq 1\}. \tag{4.19}$$

Clearly $\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq 1$ for $\phi \in \mathcal{T}$. For the rescalings we use the notation

$$\phi_t(x) = t^{-n}\phi\Big(\frac{x}{t}\Big) \quad \text{for } t > 0, \tag{4.20}$$

thus $\operatorname{spt}\phi_t \subset B_t(0)$ and $\|D\phi_t\|_{L^\infty(\mathbb{R}^n)} \leq t^{-(n+1)}$. In the following definition the term *grand* refers to the fact that the maximimum over all kernels in $\mathcal{T}$ is considered, rather than working with a specific one; this makes the application more flexible.

Definition 4.2.1. *The grand maximal function of $\Lambda \in C_c^0(\mathbb{R}^n)'$ is defined by*

$$\Lambda^*(x) = \sup_{\phi\in\mathcal{T}} \sup_{t>0} |\phi_t * \Lambda(x)|. \tag{4.21}$$

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt}\phi \subset B_R(0)$ and $\|D\phi\|_{L^\infty(\mathbb{R}^n)} = \alpha > 0$. Then the function $\psi(x) = \frac{1}{R\alpha}\phi(Rx)$ belongs to $\mathcal{T}$, and we calculate

$$(\phi * \Lambda)(x) = \Lambda\big(y \mapsto \phi(x-y)\big) = \Lambda\big(y \mapsto R^{n+1}\alpha\psi_R(x-y)\big) = R^{n+1}\alpha(\psi_R * \Lambda)(x).$$

Thus for any $\phi \in C_c^\infty(\mathbb{R}^n)$ we have the inequality

$$|(\phi * \Lambda)(x)| \leq R^{n+1}\|D\phi\|_{L^\infty(\mathbb{R}^n)}\Lambda^*(x) \quad \text{if } \operatorname{spt}\phi \subset B_R(0). \tag{4.22}$$

There are several characterizations of Hardy space, whose equivalence is by no means obvious, see [15] or [57]. A nice introduction is due to Semmes [52].

Definition 4.2.2. *$\mathcal{H}^1(\mathbb{R}^n)$ is the set of all $\Lambda \in C_c^0(\mathbb{R}^n)'$ for which $\Lambda^* \in L^1(\mathbb{R}^n)$. We put*

$$\|\Lambda\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|\Lambda^*\|_{L^1(\mathbb{R}^n)}. \tag{4.23}$$

As $(\Lambda_1 + \Lambda_2)^* \leq \Lambda_1^* + \Lambda_2^*$ and $(\alpha\Lambda)^* = |\alpha|\Lambda^*$, the Hardy space is a normed vector space. The following lemma will allow us to consider its elements as L^1 functions.

Lemma 4.2.3. *The space $\mathcal{H}^1(\mathbb{R}^n)$ is continuously embedded into $L^1(\mathbb{R}^n)$.*

Proof. We use approximation by smoothing. Choose a fixed kernel $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt}\phi \subset B_1(0)$ and $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$. Writing $\Lambda = |\Lambda|\llcorner\sigma$ by the Riesz representation theorem and putting $\check{\phi}(x) = \phi(-x)$, we compute using Fubini's theorem, recalling $\phi_t^x(y) = \phi_t(x-y)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} (\phi_t * \Lambda)(x)\,\eta(x)\,dx &= \int_{\mathbb{R}^n} \Lambda(\phi_t^x)\,\eta(x)\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi_t(x-y)\sigma(y)\,d|\Lambda|(y)\,\eta(x)\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi_t(x-y)\eta(x)\,dx\,\sigma(y)\,d|\Lambda|(y) \\
&= \Lambda\big(\check{\phi}_t * \eta\big) \;\to\; \Lambda(\eta) \quad \text{as } t \searrow 0.
\end{aligned}$$

To justify this we note that $(x,y) \mapsto \phi_t(x-y)\,\eta(x)\,\sigma(y)$ is integrable with respect to $\mathcal{L}^n \times |\Lambda|$. Namely, the function is measurable and we have, putting $K_t = \{y \in \mathbb{R}^n : \operatorname{dist}(y, \operatorname{spt}\eta) \leq t\}$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\phi_t(x-y)|\,|\eta(x)|\,dx\,d|\Lambda|(y) \leq \|\eta\|_{L^\infty}\|\phi\|_{L^1}|\Lambda|(K_t) < \infty.$$

Now from (4.22) we have the bound

$$|\phi_t * \Lambda(x)| \leq t^{n+1}\|D\phi_t\|_{L^\infty(\mathbb{R}^n)}\Lambda^*(x) = \|D\phi\|_{L^\infty(\mathbb{R}^n)}\Lambda^*(x) \in L^1(\mathbb{R}^n).$$

As the $\phi_t * \Lambda$ are equiintegrable they converge in $C^0_c(\mathbb{R}^n)'$ subsequentially to some $f \in L^1(\mathbb{R}^n)$. But $\Lambda_f = \Lambda$ by the above, hence the sublimit improves to a limit. Finally

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \liminf_{t \searrow 0} \|\phi_t * \Lambda\|_{L^1(\mathbb{R}^n)} \leq \|D\phi\|_{L^\infty(\mathbb{R}^n)}\|\Lambda^*\|_{L^1(\mathbb{R}^n)}.$$

□

From now on the elements of $\mathcal{H}^1(\mathbb{R}^n)$ are regarded as $L^1(\mathbb{R}^n)$ functions, in particular we write f^* instead of Λ^*_f. As pointed out at the beginning, the following weak compactness theorem distinguishes the space $\mathcal{H}^1(\mathbb{R}^n)$ from $L^1(\mathbb{R}^n)$.

Theorem 4.2.4 (weak compactness in $\mathcal{H}^1(\mathbb{R}^n)$)**.** *Let f_k be a bounded sequence in $\mathcal{H}^1(\mathbb{R}^n)$. Then there exists an $f \in \mathcal{H}^1(\mathbb{R}^n)$, such that for a subsequence $f_k \to f$ in $C^0_c(\mathbb{R}^n)'$, and*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq \operatorname{liminf}_{k\to\infty} \|f_k\|_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.24}$$

Proof. By Lemma 4.2.3 we have $\|f_k\|_{L^1(\mathbb{R}^n)} \leq C$, so that $f_k \to \Lambda$ in $C^0_c(\mathbb{R}^n)'$ after passing to a subsequence. Now for $\phi \in \mathcal{T}$ and $t > 0$ we have

$$(\phi_t * f_k)(x) = \int_{\mathbb{R}^n} \phi_t^x(y) f_k(y)\,dy \overset{k\to\infty}{\longrightarrow} \Lambda(\phi_t^x) = (\phi_t * \Lambda)(x),$$

which implies

$$(\phi_t * \Lambda)(x) = \lim_{k\to\infty}(\phi_t * f_k)(x) \leq \liminf_{k\to\infty} f_k^*(x).$$

Take the supremum with respect to $\phi \in \mathcal{T}$ and $t > 0$. Then by Fatou's lemma

$$\|\Lambda\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|\Lambda^*\|_{L^1(\mathbb{R}^n)} \leq \liminf_{k\to\infty}\|f_k^*\|_{L^1(\mathbb{R}^n)} = \liminf_{k\to\infty}\|f_k\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

Finally $\Lambda = \Lambda_f$ where $f \in L^1(\mathbb{R}^n)$ by Lemma 4.2.3, which finishes the proof. □

Corollary 4.2.5. *$\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space.*

Proof. Let $f_k \in \mathcal{H}^1(\mathbb{R}^n)$ be a Cauchy sequence. By Lemma 4.2.3 and Fischer-Riesz, f_k converges in $L^1(\mathbb{R}^n)$ to some $f \in L^1(\mathbb{R}^n)$. Theorem 4.2.4 implies that $f \in \mathcal{H}^1(\mathbb{R}^n)$, and that

$$\|f - f_\ell\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq \liminf_{k\to\infty}\|f_k - f_\ell\|_{\mathcal{H}^1(\mathbb{R}^n)} < \varepsilon \quad \text{for } \ell > K(\varepsilon).$$

□

Corollary 4.2.6 (Cancellation in $\mathcal{H}^1(\mathbb{R}^n)$). *For every $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} f(y)\,dy = 0. \tag{4.25}$$

Proof. We first check the scaling of the Hardy norm. For $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $t > 0$ let $f_t(y) = t^{-n} f(\frac{y}{t})$. Compute for $\phi \in \mathcal{T}$, $t > 0$, by substituting $y = tz$,

$$\begin{aligned}(\phi_s * f_t)(x) &= \int_{\mathbb{R}^n} \phi_s(x-y) t^{-n} f\big(\frac{y}{t}\big)\,dy \\ &= \int_{\mathbb{R}^n} \phi_s\big(t\,(\frac{x}{t} - z)\big)\, f(z)\,dz \\ &= t^{-n}\big(\phi_{\frac{s}{t}} * f\big)\big(\frac{x}{t}\big).\end{aligned}$$

We estimate on the right with $f^*(\frac{x}{t})$, and then take the supremum over $\phi \in \mathcal{T}$, $s > 0$, to get

$$f_t^*(x) = t^{-n} f^*\big(\frac{x}{t}\big) \qquad \text{for } x \in \mathbb{R}^n,\ t > 0.$$

We first get the inequality, for equality we use $(f_t)_{\frac{1}{t}} = f$. In particular we obtain

$$\|f_t\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \qquad \text{for all } t > 0.$$

Now for $f \in L^1(\mathbb{R}^n)$ the f_t converge in $C_c^0(\mathbb{R}^n)$ to a multiple of the Dirac measure δ_0 as $t \searrow 0$, in fact dominated convergence yields

$$\int_{\mathbb{R}^n} \phi(x) f_t(x)\,dx = \int_{\mathbb{R}^n} \phi(ty) f(y)\,dy \to \phi(0) \int_{\mathbb{R}^n} f(y)\,dy.$$

On the other hand we must have $f_t \to \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n)$ for a subsequence by Theorem 4.2.4. As the Dirac measure is not in $\mathcal{H}^1(\mathbb{R}^n)$ we conclude that $\int_{\mathbb{R}^n} f(y)\,dy = 0$. □

Alternatively, we can argue more directly: for a given sequence $x_k \to \infty$, we pass to a subsequence with $\frac{x_k}{|x_k|} \to z$. For $\phi \in \mathcal{T}$ we have putting $R_k = |x_k|$

$$|x_k|^n f^*(x_k) \geq |x_k|^n |\phi_{2R_k} * f(x_k)| = 2^{-n} \int_{\mathbb{R}^n} \phi\big(\frac{x_k - y}{2R_k}\big) f(y)\,dy \to 2^{-n} \phi\big(\frac{z}{2}\big) \int_{\mathbb{R}^n} f(y)\,dy.$$

Choosing ϕ appropriately we obtain

$$\liminf_{x \to \infty} |x|^n f^*(x) \geq c \left| \int_{\mathbb{R}^n} f(y)\,dy \right| \qquad \text{for some } c > 0.$$

Thus f^* integrable implies that the integral of f is zero.

Next we compare the grand maximal function to the maximal function of Hardy-Littlewood.

Lemma 4.2.7. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The for any $x \in \mathbb{R}^n$ we have*

(1) $f^*(x) \leq C\, Mf(x)$,

(2) $Mf(x) \leq C f^*(x)$, *if $f \geq 0$.*

Proof. To prove the first statement we calculate for $\phi \in \mathcal{T}$, $t > 0$,

$$\begin{aligned} |(\phi_t * f)(x)| &= t^{-n}\Big|\int_{\mathbb{R}^n} \phi(\frac{x-y}{t}) f(y)\, dy\Big| \\ &\le \|\phi\|_{L^\infty(\mathbb{R}^n)} t^{-n} \int_{B_t(x)} |f(y)|\, dy \\ &\le C\, Mf(x). \end{aligned}$$

Taking the supremum over $\phi \in \mathcal{T}$, $t > 0$ shows claim (1). On the other hand, choosing $\phi \in \mathcal{T}$ such that $\phi \ge \frac{1}{4}\chi_{B_{1/2}(0)}$ we can estimate, for $f \ge 0$,

$$(\phi_t * f)(x) = t^{-n} \int_{\mathbb{R}^n} \phi(\frac{x-y}{t}) f(y)\, dy \ge t^{-n}\frac{1}{4}\int_{B_{t/2}(x)} f(y)\, dy \ge c \fint_{B_{t/2}(x)} f(y)\, dy.$$

This implies

$$Mf(x) = \sup_{t>0} \fint_{B_t(x)} f(y)\, dy \le \frac{1}{c}\sup_{t>0}(\phi_{2t} * f)(x) \le \frac{1}{c} f^*(x).$$

□

Theorem 4.2.8. *Let $f \in L^1(\mathbb{R}^n)$ such that $\operatorname{spt} f \subset B_R(0)$ and $\int_{\mathbb{R}^n} f(x)dx = 0$. Then*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\left(\|Mf\|_{L^1(B_{2R}(0))} + \|f\|_{L^1(\mathbb{R}^n)}\right).$$

Proof. We estimate the L^1 integral of f^* by splitting into the regions $B_{2R}(0)$ and $\mathbb{R}^n \backslash B_{2R}(0)$. For $|x| \le 2R$ the inequality $f^*(x) \le C\, Mf(x)$ from Lemma 4.2.7 yields

$$\int_{B_{2R}(0)} f^*(x)\, dx \le C\, \|Mf\|_{L^1(B_{2R}(0))}. \tag{4.26}$$

For $|x| \ge 2R$ we have $\operatorname{dist}(x, B_R(0)) = |x| - R \ge \frac{1}{2}|x|$, hence

$$\phi_t * f(x) = \int_{B_1(0)} \phi(z) f(x - tz)\, dz = 0 \quad \text{when } 0 < t < \frac{1}{2}|x|.$$

For $t \ge \frac{1}{2}|x|$ we estimate using $\int_{\mathbb{R}^n} f(y)\, dy = 0$

$$\begin{aligned} |\phi_t * f(x)| &= \Big|\int_{B_R(0)} (\phi_t(x-y) - \phi_t(x)) f(y)\, dy\Big| \\ &\le \|D\phi_t\|_{L^\infty(\mathbb{R}^n)} R \int_{\mathbb{R}^n} |f(y)|\, dy \\ &\le \frac{CR}{|x|^{n+1}} \int_{\mathbb{R}^n} |f(y)|\, dy. \end{aligned}$$

Integrating shows

$$\int_{\mathbb{R}^n \backslash B_{2R}(0)} f^*(x)\, dx \le C \int_{\mathbb{R}^n} |f(y)|\, dy. \tag{4.27}$$

The theorem follows by combining (4.26) and (4.27). □

Corollary 4.2.9. *Let $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, with $\operatorname{spt} f \subset B_R(0)$ and $\int_{\mathbb{R}^n} f(x)dx = 0$. Then $f \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C(p)\, R^{n-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The Hardy-Littlewood inequality, see Theorem 4.1.6, implies

$$\|Mf\|_{L^1(B_{2R}(0))} \leq C\, R^{n-\frac{n}{p}} \|Mf\|_{L^p(\mathbb{R}^n)} \leq C(p)\, R^{n-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Since also $\|f\|_{L^1(\mathbb{R}^n)} \leq C\, R^{n-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$, the claim follows from Theorem 4.2.8. □

Remark 4.2.10. *We give an example of a function $f \in \mathcal{H}^1(\mathbb{R})$ with compact support, for which Mf is not locally integrable. Consider*

$$f(x) = \sum_{k=2}^{\infty} \frac{a_k(x)}{k(\log k)^2} \qquad \text{where } a_k = -k\chi_{[-\frac{1}{k},0)} + k\chi_{(0,\frac{1}{k}]}.$$

The function $a_1(x)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$ by Corollary 4.2.9. As $a_k(x) = ka_1(kx) = (a_1)_{\frac{1}{k}}(x)$, we have $\|a_k\|_{\mathcal{H}^1(\mathbb{R})} = \|a_1\|_{\mathcal{H}^1(\mathbb{R})}$ for all k, and

$$\|f\|_{\mathcal{H}^1(\mathbb{R})} \leq C \sum_{k=2}^{\infty} \frac{1}{k(\log k)^2} < \infty.$$

On the other hand, for $x \in (0, \frac{1}{2}]$ chose $n \geq 1$ with $\frac{1}{2(n+1)} < x \leq \frac{1}{2n}$, and estimate

$$f(x) \geq \sum_{k=2}^{2n} \frac{1}{(\log k)^2} \geq \frac{n+1}{(\log 2n)^2} \geq \frac{1}{2x(\log 1/x)^2}.$$

It follows that

$$f(x) \log f(x) \geq \frac{1}{2x \log \frac{1}{x}} \Big(1 - \frac{2 \log\big(\log \frac{1}{x}\big) + \log 2}{\log \frac{1}{x}}\Big).$$

Thus f does not belong to the $L \log L$ class, and Stein's theorem 4.1.8 shows that Mf is not locally integrable. The integrability of f^* is due to a cancellation effect.

In 1993 Coifman, Lions, Meyer & Semmes [10] gave a number of applications of Hardy space to partial differential equations. Some of them were previously known by other methods, among them is the so-called div-curl lemma from Murat and Tartar [40, 61]. In fact, this is a classical result in compensated compactness. We give a version involving differential forms, for which we now recall some basic facts.

The exterior derivative $d\omega$ and its adjoint $d^*\omega$ of a differential form ω are

$$d\omega = \sum_{i=1}^{n} dx^i \wedge \partial_i \omega \quad \text{and} \quad d^*\omega = -\sum_{i=1}^{n} e_i \llcorner \partial_i \omega.$$

Using coordinates one easily checks that $\langle \zeta \wedge \omega, \eta \rangle = \langle \omega, z \llcorner \eta \rangle$ where $\zeta = \langle \cdot\,, z \rangle$. In particular we have as claimed, for forms with compact support,

$$\int_{\mathbb{R}^n} \langle d\omega, \eta \rangle \, dx = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \langle dx^i \wedge \partial_i \omega, \eta \rangle \, dx = -\int_{\mathbb{R}^n} \sum_{i=1}^{n} \langle \omega, e_i \llcorner \partial_i \eta \rangle \, dx = \int_{\mathbb{R}^n} \langle \omega, d^*\eta \rangle \, dx.$$

Lemma 4.2.11. *We have $d^*d + dd^* = -\Delta$.*

Proof. For $z \in \mathbb{R}^n$ we denote by $I(z)\omega = z \llcorner \omega$ the interior multiplication and by $E(z)\omega = \zeta \wedge \omega$, $\zeta = \langle \cdot, z \rangle$, the exterior multiplication. Then

$$d^* d\omega = -\sum_{i,j=1}^{n} I(e_i)E(e_j)\partial^2_{ij}\omega, \quad \text{and} \quad dd^*\omega = -\sum_{i,j=1}^{n} E(e_j)I(e_i)\partial^2_{ji}\omega.$$

As $\partial^2_{ij}\omega = \partial^2_{ji}\omega$, the claim follows by proving that

$$\mathrm{Sym}\big(I(e_i)E(e_j) + E(e_j)I(e_i)\big) = \delta_{ij}\,\mathrm{Id}.$$

By polarization with respect to e_i, e_j, it is in fact sufficient to show

$$I(z)E(z) + E(z)I(z) = \mathrm{Id} \quad \text{for any } z \in \mathbb{R}^n,\ |z| = 1.$$

Now $I(z) = E(z)^*$, thus $\Lambda^*(\mathbb{R}^n) = \operatorname{im} E(z) \oplus \ker I(z)$ and any ω decomposes as

$$\omega = E(z)\omega' + \omega'' \quad \text{where } I(z)\omega'' = 0,\ I(z)\omega' = 0.$$

Using $E(z)^2 = 0$ and $I(z)\omega'' = 0$ we have

$$I(z)E(z)\omega = I(z)E(z)\omega'' = (\zeta \wedge \omega'')(z, \cdot) = \zeta(z)\omega'' = \omega''.$$

We compute further

$$E(z)I(z)\omega = E(z)I(z)E(z)\omega' = E(z)\omega'.$$

The claim follows by adding the two equations. □

Lemma 4.2.12. *Let $\beta \in L^q(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$ where $1 < q < \infty$. If $d\beta = 0$ in the sense of distributions, then there exists a form $\gamma \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n, \Lambda^{k-1}(\mathbb{R}^n))$ such that $d\gamma = \beta$, and*

$$\|D\gamma\|_{L^q(\mathbb{R}^n)} \le C\,\|\beta\|_{L^q(\mathbb{R}^n)} \quad \textit{where } C = C(n, q). \tag{4.28}$$

Proof. For any $\beta \in L^q(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$ there exists a $\phi \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$ such that

$$\Delta\phi = \beta \quad \text{and} \quad \|D^2\phi\|_{L^q(\mathbb{R}^n)} \le C\|\beta\|_{L^q(\mathbb{R}^n)} \quad \text{where } C = C(n, q) < \infty.$$

Namely, if $\beta \in C^\infty_c(\mathbb{R}^n)$ we take the Newtonian potential

$$\phi(x) = \int_{\mathbb{R}^n} \Gamma(x - y)\beta(y)\,dy \quad \text{where } \Gamma(z) = \begin{cases} \frac{1}{(2-n)\omega_n}|z|^{2-n} & \text{for } n \ge 3, \\ \frac{1}{2\pi}\log|z| & \text{for } n = 2. \end{cases}$$

The L^q estimate of $D^2\phi$ is then the Calderon-Zygmund inequality, see for instance [2]. For general $\beta \in L^q(\mathbb{R}^n)$ we approximate by $\beta_j \in C^\infty_c(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$. Then $\beta_j = \Delta\phi_j$ where

$$\limsup_{j\to\infty} \|D^2\phi_j\|_{L^q(\mathbb{R}^n)} \le C\|\beta\|_{L^q(\mathbb{R}^n)}.$$

Subtracting a linear function, we can arrange that

$$\int_{B_1(0)} D\phi_j\,dx = 0 \quad \text{and} \quad \int_{B_1(0)} \phi_j\,dx = 0.$$

A standard contradiction argument using Rellich's theorem yields

$$\|\phi_j\|_{W^{1,q}(B_R(0))} \leq C(R,q)\|D^2\phi_j\|_{L^q(\mathbb{R}^n)}.$$

After passing to a subsequence we have $\phi_j \to \phi$ in $W^{1,q}_{\text{loc}}(\mathbb{R}^n)$, and $\|D^2\phi\|_{L^q(\mathbb{R}^n)} \leq C\|\beta\|_{L^q(\mathbb{R}^n)}$. Thus ϕ is the desired solution of $\Delta\phi = \beta$.

Now if $d\beta = 0$ then $d^*d\phi$ is harmonic. In fact for any $\zeta \in C^\infty_c(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$ we compute, using $d^2 = 0$ as well as $(d^*)^2 = 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \langle d^*d\phi, \Delta\zeta\rangle \, dx &= -\int_{\mathbb{R}^n} \langle d^*d\phi, (d^*d + dd^*)\zeta\rangle \, dx \\
&= -\int_{\mathbb{R}^n} \langle d\phi, d(d^*d\zeta)\rangle \, dx - \int_{\mathbb{R}^n} \langle d^*\phi, d^*(d^*d\zeta)\rangle \, dx \\
&= -\int \langle D\phi, D(d^*d\zeta)\rangle \, dx \\
&= \int_{\mathbb{R}^n} \langle \beta, d^*d\zeta\rangle \, dx.
\end{aligned}$$

As $d^*d\phi$ belongs to $L^q(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$, the mean value inequality implies that $d^*d\phi$ vanishes identically. This in turn implies $-dd^*\phi = \beta$, and the lemma is proved by taking $\gamma = -d^*\phi$. $\square$

The following is the key observation of Coifmann, Lions, Meyer and Semmes.

Theorem 4.2.13 ([10]). *Let $\alpha \in L^p(\mathbb{R}^n, \Lambda^k(\mathbb{R}^n))$, $\beta \in L^q(\mathbb{R}^n, \Lambda^{n-k}(\mathbb{R}^n))$, where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $d\alpha = 0$ and $d\beta = 0$ weakly, then $\alpha \wedge \beta \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|\alpha \wedge \beta\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\,\|\alpha\|_{L^p(\mathbb{R}^n)}\|\beta\|_{L^q(\mathbb{R}^n)}. \tag{4.29}$$

Proof. By Lemma 4.2.12 there exists $\gamma \in W^{1,q}_{\text{loc}}(\mathbb{R}^n, \Lambda^{n-k-1}(\mathbb{R}^n))$ such that

$$d\gamma = \beta \quad \text{and} \quad \|D\gamma\|_{L^q(\mathbb{R}^n)} \leq C\|\beta\|_{L^q(\mathbb{R}^n)}. \tag{4.30}$$

On the other hand, the equation $d\alpha = 0$ has the weak formulation

$$\int_{\mathbb{R}^n} \alpha \wedge d\zeta = 0 \quad \text{for all } \zeta \in C^\infty_c(\mathbb{R}^n, \Lambda^{n-k-1}(\mathbb{R}^n)). \tag{4.31}$$

To see this we need the formula, for $\omega \in \Lambda^k(\mathbb{R}^n)$, $\eta \in \Lambda^{n-k}(\mathbb{R}^n)$ and $*$ the Hodge star operator,

$$\omega \wedge \eta = (-1)^{k(n-k)}\langle \omega, *\eta\rangle \, dx^1 \wedge \ldots \wedge dx^n.$$

Using this we compute

$$\begin{aligned}
\alpha \wedge d\zeta &= \alpha \wedge \sum_{i=1}^n dx^i \wedge \partial_i\zeta \\
&= (-1)^{(k+1)(n-(k+1))}\,(-1)^k \sum_{i=1}^n \left\langle dx^i \wedge \alpha, *\partial_i\zeta\right\rangle dx^1 \wedge \ldots \wedge dx^n \\
&= (-1)^{n(k+1)-1}\left\langle \alpha, \sum_{i=1}^n e_i \llcorner \partial_i * \zeta\right\rangle dx^1 \wedge \ldots \wedge dx^n \\
&= (-1)^{n(k+1)}\left\langle \alpha, d^*(*\zeta)\right\rangle dx^1 \wedge \ldots \wedge dx^n.
\end{aligned}$$

This shows (4.31). Now by approximation, any form $\zeta = \eta(\gamma - \gamma_0)$ with $\eta \in C_c^\infty(\mathbb{R}^n)$ and $\gamma_0 \in \Lambda^{n-k-1}(\mathbb{R}^n)$ constant, is admissible in (4.31), thus we get

$$\int_{\mathbb{R}^n} \eta\, \alpha \wedge \beta = -\int_{\mathbb{R}^n} \alpha \wedge d\eta \wedge (\gamma - \gamma_0). \tag{4.32}$$

For given $\phi \in \mathcal{T}$ and $t > 0$, we take $\eta(y) = \phi_t^x(y) = \phi_t(x-y)$. Then $|d\eta| \leq t^{-n-1}$ and

$$|\phi_t * (\alpha \wedge \beta)(x)| = \int_{\mathbb{R}^n} \phi_t^x \alpha \wedge \beta \leq \frac{C}{t} \fint_{B_t(x)} |\alpha|\, |\gamma - \gamma_0|\, dy.$$

We now use Hölder's inequality with exponent $r \in (1,p]$. The second factor, which gets the power $s = \frac{r}{r-1} \in [q, \infty)$, is estimated by the Sobolev-Poincaré inequality. More precisely

$$\begin{aligned}
|\phi_t * (\alpha \wedge \beta)(x)| &\leq \frac{C}{t}\Big(\fint_{B_t(x)} |\alpha|^r\, dy\Big)^{\frac{1}{r}} \Big(\fint_{B_t(x)} |\gamma - \gamma_0|^s\, dy\Big)^{\frac{1}{s}} \\
&\leq C\Big(\fint_{B_t(x)} |\alpha|^r\, dy\Big)^{\frac{1}{r}} \Big(\fint_{B_t(x)} |D\gamma|^\lambda\, dy\Big)^{\frac{1}{\lambda}} \\
&\leq CM(|\alpha|^r)(x)^{\frac{1}{r}}\, M(|D\gamma|^\lambda)(x)^{\frac{1}{\lambda}}.
\end{aligned}$$

Here we need $\frac{1}{\lambda} \leq 1 + \frac{1}{n} - \frac{1}{r}$. Take the supremum over $t > 0$ and integrate, then use Hölder with exponents p, q to get

$$\begin{aligned}
\int_{\mathbb{R}^n} (\alpha \wedge \beta)^*\, dx &= C \int_{\mathbb{R}^n} M(|\alpha|^r)^{\frac{1}{r}}\, M(|D\gamma|^\lambda)^{\frac{1}{\lambda}}\, dx \\
&\leq C\, \|M(|\alpha|^r)\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}^{\frac{1}{r}}\, \|M(|D\gamma|^\lambda)\|_{L^{\frac{q}{\lambda}}(\mathbb{R}^n)}^{\frac{1}{\lambda}} \\
&\leq C\, \||\alpha|^r\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}^{\frac{1}{r}}\, \||D\gamma|^\lambda\|_{L^{\frac{q}{\lambda}}(\mathbb{R}^n)}^{\frac{1}{\lambda}} \\
&= C\, \|\alpha\|_{L^p(\mathbb{R}^n)} \|D\gamma\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

To apply the Hardy-Littlewood theorem 4.1.6 we needed that $r < p$, $\lambda < q$. We eventually fix the parameters: we can chose $r > 0$ such that

$$\frac{1}{p} < \frac{1}{r} < \min\big(1, 1 + \frac{1}{n} - \frac{1}{q}\big). \tag{4.33}$$

Then $r \in (1,p)$, and we can chose $\lambda > 0$ such that

$$\frac{1}{q} < \frac{1}{\lambda} \leq \min\big(1, 1 + \frac{1}{n} - \frac{1}{r}\big). \tag{4.34}$$

Thus $\lambda \in [1,q)$, and the Sobolev-Poincaré inequality applies. Recalling the L^q estimate from Lemma 4.2.12, we arrive at the desired bound

$$\|\alpha \wedge \beta\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|\alpha\|_{L^p(\mathbb{R}^n)} \|\beta\|_{L^q(\mathbb{R}^n)}.$$

□

The paper [10] states the theorem for vector fields $E \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $B \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{curl} E = 0$ and $\operatorname{div} B = 0$, claiming that

$$\|\langle E, B\rangle\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\,\|E\|_{L^p(\mathbb{R}^n)}\|B\|_{L^q(\mathbb{R}^n)}.$$

This result is also included in our formulation by considering the forms

$$\alpha = \sum_{i=1}^n E_i\,dx^i \quad \text{and} \quad \beta = B\llcorner dx^1 \wedge \ldots \wedge dx^n = \sum_{i=1}^n (-1)^i B_i\,dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n.$$

We then have $*(\alpha \wedge \beta) = \langle E, B\rangle$. One possible application is to Jacobi determinants of maps $u : \mathbb{R}^n \to \mathbb{R}^n$ with $Du \in L^n(\mathbb{R}^n, \mathbb{R}^{n\times n})$. The theorem then implies

$$\det Du = *(du^1 \wedge \ldots \wedge du^n) \in \mathcal{H}^1(\mathbb{R}^n).$$

In the case $\det Du \geq 0$ this yields another proof of Müller's theorem 4.1.11: one combines Stein's theorem 4.1.8 with Lemma 4.2.7 to obtain, for any ball $B \subset \mathbb{R}^n$,

$$\begin{aligned}
\|\det Du\,\log^+ \det Du\|_{L^1(B)} &\leq C\big(B, \|M(\det Du)\|_{L^1(B)}\big)\\
&\leq C(B, \|(\det Du)^*\|_{L^1(B)})\\
&\leq C\big(B, \|\det Du\|_{\mathcal{H}^1(\mathbb{R}^n)}\big)\\
&\leq C(B, \|Du\|_{L^n(\mathbb{R}^n)}).
\end{aligned}$$

The $\mathcal{H}^1$-estimate for $\det Du$ can be combined with the following regularity result by Fefferman and Stein [15], thereby proving a certain generalization of Wente's theorem 3.2.1. Note that functions in $W^{2,1}$ are continuous in dimension $n = 2$.

Theorem 4.2.14 ([15]). *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$, and assume that $u : \mathbb{R}^n \to \mathbb{R}$ is a solution of*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

Then $u = u_0 + h$ where $h : \mathbb{R}^n \to \mathbb{R}$ is harmonic and $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\|D^2 u_0\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

4.3 Atomic decomposition

In this section we prove that every element in $\mathcal{H}^1$ can be decomposed into so-called atoms.

Definition 4.3.1. *A function $a \in L^\infty(\mathbb{R}^n)$ is called an $\mathcal{H}^1$-atom (with admissible ball B), if the following holds:*

$$\operatorname{spt} a \subset B, \tag{4.35}$$

$$||a||_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{|B|}, \tag{4.36}$$

$$\int_{\mathbb{R}^n} a(x)dx = 0. \tag{4.37}$$

This notion is invariant under recalings, more precisely $B = B_1(0)$ is admissible for $a(z)$ if and only if $B_t(x)$ is admissible for $a_t^x(y) = t^{-n}a\big(\frac{x-y}{t}\big)$. For example, the functions a_k in example 4.2.10 are $\mathcal{H}^1$-atoms on $\mathbb{R}$.

Lemma 4.3.2. *For any $\mathcal{H}^1$-atom a we have $\|a\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C$, for $C < \infty$ universal.*

Proof. By translation we can assume that a has admissible ball $B = B_R(0)$. Applying Corollary 4.2.9 we obtain

$$\|a\|_{\mathcal{H}^1(\mathbb{R}^n)} \le CR^n \|a\|_{L^\infty(\mathbb{R}^n)} \le C. \tag{4.38}$$

□

As a consequence of this bound, we can build series of atoms as follows.

Lemma 4.3.3. *Let $a_k \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$, be a sequence of $\mathcal{H}^1$-atoms, and let $\lambda_k \in \mathbb{R}$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$. Then $f = \sum_{k=1}^\infty \lambda_k a_k$ converges in $\mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C \sum_{k=1}^\infty |\lambda_k|. \tag{4.39}$$

Proof. The series converges absolutely since

$$\sum_{k=1}^\infty \|\lambda_k a_k\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C \sum_{k=1}^\infty |\lambda_k| < \infty.$$

The claim follows since $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space, see Corollory 4.2.5. □

The goal of this section is to prove a converse to Lemma 4.3.3, namely to decompose a given $f \in \mathcal{H}^1(\mathbb{R}^n)$ into a sum of $\mathcal{H}^1$-atoms. For this we need the following Whitney decomposition, see also [56].

Lemma 4.3.4. *Let $\Omega \subset \mathbb{R}^n$ be open, $\Omega \ne \mathbb{R}^n$, and $d(x) = \operatorname{dist}(x, F)$ where $F = \mathbb{R}^n \backslash \Omega$. There exists a collection $\mathcal{G}$ of closed cubes P with the following properties, where $\hat{P}$ denotes the concentric cube scaled by factor 2:*

(i) $\operatorname{diam} P \le \operatorname{dist}(P, F) < 4 \operatorname{diam} P$,

(ii) $\bigcup_{P \in \mathcal{G}} P = \Omega$ *and* $\hat{P} \subset \Omega$.

(iii) $\operatorname{int}(P \cap P') = \emptyset$ *for any* $P, P' \in \mathcal{G}$,

Furthermore let $\mathcal{G}_x = \{P \in \mathcal{G} : \hat{P} \cap B_{d(x)/2}(x) \ne \emptyset\}$ for $x \in \Omega$. Then we have, for constants $c = c(n) > 0$ and $C = C(n) < \infty$,

(iv) $c\, d(x) \le \operatorname{diam} P \le C\, d(x)$ *for any* $P \in \mathcal{G}_x$, *and* $\#\mathcal{G}_x \le C$.

We note that cubes in $\mathcal{G}$ have positive volume by (i), hence $\mathcal{G}$ is countable by (iii).

Proof. For given $x \in \Omega$ choose $\ell \in \mathbb{Z}$ maximal with $\sqrt{n}\, 2^{\ell+1} \leq d(x)$. Let P be a cube with vertices in the grid $2^\ell \mathbb{Z}^n$ and sidelength 2^ℓ, such that $x \in P$. Then the following holds:

$$\begin{aligned} \operatorname{dist}(P, F) &\geq d(x) - \operatorname{diam} P \geq \sqrt{n}\, 2^{\ell+1} - \sqrt{n}\, 2^\ell = \operatorname{diam} P, \\ \operatorname{dist}(P, F) &\leq d(x) < \sqrt{n}\, 2^{\ell+2} = 4 \operatorname{diam} P. \end{aligned}$$

Thus if $\mathcal{G}_0$ is the set of all dyadic cubes P with property (i), then Ω is exhausted by $\mathcal{G}_0$. Moreover any $P \in \mathcal{G}_0$ satisfies

$$\operatorname{dist}(\hat{P}, F) \geq \operatorname{dist}(P, F) - \operatorname{diam} P/2 \geq \operatorname{diam} P/2 > 0.$$

We take $\mathcal{G}$ as the set of maximal cubes P in $\mathcal{G}_0$, in the sense that P is not contained in any bigger cube of $\mathcal{G}_0$. As F is nonempty, any cube in $\mathcal{G}_0$ is contained in a maximal cube, hence $\mathcal{G}$ satisfies (i) and (ii). Now consider two dyadic intervals $I_{1,2}$ with lengths $2^{\ell_1} \leq 2^{\ell_2}$ and common interior. Then $I_1 \subset I_2$, since the endpoints of I_2 are also vertices of the 2^{ℓ_1}-grid. Therefore by maximality $\mathcal{G}$ has also property (iii).

For $y \in \hat{P} \cap B_{d(x)/2}(x)$ and any $z \in P$ we estimate

$$\operatorname{dist}(P, F) \leq \operatorname{dist}(z, F) \leq |z - y| + |y - x| + d(x) \leq |z - y| + \frac{3}{2} d(x).$$

Using $\operatorname{diam} P \leq \operatorname{dist}(P, F)$ and $\inf_{z \in P} |z - y| \leq \frac{1}{2} \operatorname{diam} P$, we obtain after rearranging

$$\operatorname{diam} P \leq 3\, d(x).$$

On the other hand, also for $y \in \hat{P} \cap B_{d(x)/2}(x)$ and $z \in P$ arbitrary, we have

$$d(x) \leq |x - y| + |y - z| + \operatorname{dist}(z, F) \leq \frac{d(x)}{2} + \frac{3}{2} \operatorname{diam} P + \operatorname{dist}(z, F).$$

Rearranging and taking the infimum among $z \in P$, we see that

$$d(x) \leq 3 \operatorname{diam} P + 2 \operatorname{dist}(P, F) \leq 11 \operatorname{diam} P.$$

Hence the first statement in (iv) settled. Now $\operatorname{dist}(x, P) \leq \operatorname{dist}(x, \hat{P}) + \frac{1}{2} \operatorname{diam} P \leq 2\, d(x)$, thus $P \subset B_{5d(x)}(x)$ and $|P| \geq c(n)\, d(x)^n$. As the cubes in $\mathcal{G}$ have disjoint interior, property (iv) follows by volume comparison. □

Lemma 4.3.5. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in C^\infty_c(B_\varrho(x_0))$. Then for given $z \in \mathbb{R}^n$ we have*

$$\Big| \int_{\mathbb{R}^n} \phi(y) f(y)\, dy \Big| \leq (|z - x_0| + \varrho)^{n+1} \|D\phi\|_{L^\infty} f^*(z). \tag{4.40}$$

Proof. We write

$$\int_{\mathbb{R}^n} \phi(y) f(y)\, dy = \int_{\mathbb{R}^n} \phi(z - (z - y)) f(y)\, dy = \int_{\mathbb{R}^n} \phi^z(z - y) f(y)\, dy = (\phi^z * f)(z).$$

Now we have $\operatorname{spt} \phi^z \subset B_\varrho(z - x_0) \subset B_R(0)$ for $R = |z - x_0| + \varrho$. The inequality follows by applying the estimate (4.22). □

The key step in the atomic decomposition is the following Calderon-Zygmund type argument. For its statement, we note that f^* is lower semicontinuous and hence superlevel sets $\{x \in \mathbb{R}^n : f^*(x) > \alpha\}$ are open, as f^* is defined as a supremum over continuous functions.

Lemma 4.3.6. *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $\alpha > 0$, such that $\Omega := \{x \in \mathbb{R}^n : f^*(x) > \alpha\} \neq \mathbb{R}^n$. Let P_k, $k \in \mathbb{N}$, be the family of cubes obtained by Lemma 4.3.4, and put $Q_k := \hat{P}_k$. Then the following holds:*

(i) *For each k there is a function b_k with support in Q_k and integral zero, such that*

$$\|b_k\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \int_{Q_k} f^*(x)\,dx. \tag{4.41}$$

(ii) *For $g = f - b$, where $b = \sum_{k=1}^\infty b_k$, we have $|g(x)| \leq C\alpha$ almost everywhere.*

Proof. Fix $\xi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt}\xi \subset (-1,1)^n$, $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]^n$, and define

$$\xi_k \in C_c^\infty(\mathbb{R}^n),\ \xi_k(x) = \xi\Big(\frac{x - x_k}{\ell_k}\Big),$$

where x_k,ℓ_k are the center and sidelength of P_k, in particular $\operatorname{spt}\xi_k \subset \operatorname{int} Q_k$ and $\xi_k \equiv 1$ on P_k. As $\Omega = \bigcup_{j=1}^\infty P_j$, we get $\sum_{j=1}^\infty \xi_j \geq 1$ on Ω. Moreover, by Lemma 4.3.4 (iv), locally all but finitely many of the ξ_j are zero, thus we obtain the smooth partition of unity

$$\eta_k \in C^\infty(\mathbb{R}^n),\ \eta_k(x) = \frac{\xi_k(x)}{\sum_{j=1}^\infty \xi_j(x)}.$$

Clearly $0 \leq \eta_k \leq 1$ and $\operatorname{spt}\eta_k \subset \operatorname{int} Q_k$. Moroever we compute

$$D\eta_k(x) = \frac{1}{\Big(\sum_{j=1}^\infty \xi_j(x)\Big)^2} \sum_{j=1}^\infty \big(D\xi_k(x)\xi_j(x) - \xi_k(x)D\xi_j(x)\big).$$

According to Lemma 4.3.4(iv) the number of j with $Q_j \cap Q_k \neq \emptyset$ is bounded by $C(n)$, and $|D\xi_j(x)| \leq C/\ell_j \leq C/\ell_k$ for these j. Hence

$$\|D\eta_k\|_{C^0(\mathbb{R}^n)} \leq \frac{C}{\ell_k} \quad \text{for } C = C(n). \tag{4.42}$$

Furthermore

$$c(n)\ell_k^n \leq \int_{P_k} \eta_k(x)\,dx \leq \int_{Q_k} \eta_k(x)\,dx \leq C\ell_k^n. \tag{4.43}$$

We define $b_k \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt} b_k \subset \operatorname{int} Q_k$ by

$$b_k(x) = (f(x) - c_k(f))\,\eta_k \quad \text{where} \quad c_k(f) = \frac{\int_{\mathbb{R}^n} f(y)\eta_k(y)\,dy}{\int_{\mathbb{R}^n} \eta_k(y)\,dy}. \tag{4.44}$$

Note that $c_k(f)$ is the mean value with respect to the measure $\mathcal{L}^n \llcorner \eta_k$, in particular

$$\int_{\mathbb{R}^n} b_k(x)\,dx = \int_{\mathbb{R}^n} f(x)\,\eta_k(x)dx - c_k(f)\int_{\mathbb{R}^n} \eta_k(x)dx = 0. \tag{4.45}$$

We apply Lemma 4.3.5 to estimate $c_k(f)$. As $\operatorname{spt}\eta_k \subset B_{C\ell_k}(x_k)$ and $\|D\eta_k\|_{C^0} \leq \frac{C}{\ell_k}$, we can estimate for any $z \in \mathbb{R}^n$, using also (4.43),

$$|c_k(f)| \leq C\frac{(|z-x_k|+\ell_k)^{n+1}}{\ell_k^{n+1}} f^*(z).$$

By Lemma 4.3.4(i) there is a point $z \in \mathbb{R}^n \backslash \Omega$ with $|z-x_k| \leq C\ell_k$, thus we have

$$|c_k(f)| \leq Cf^*(z) \text{ for } |z-x_k| \leq C\ell_k, \quad \text{in particular } |c_k(f)| \leq C\alpha. \tag{4.46}$$

We now verify statement (ii) of the theorem. For $x \notin \Omega$ we simply have $|g(x)| = |f(x)| \leq f^*(x) \leq \alpha$. If $x \in \Omega$, then we get, since η_k is a partition of unity,

$$g(x) = f(x) - \sum_{k=1}^{\infty} b_k(x) = f(x) - \sum_{k=1}^{\infty} \big(f(x) - c_k(f)\big)\eta_k = \sum_{k=1}^{\infty} c_k(f)\eta_k(x).$$

Using (4.46) we can estimate

$$|g(x)| \leq \sum_{k=1}^{\infty} |c_k(f)|\eta_k(x) \leq C\alpha,$$

so (ii) is proved. We now turn to estimating $b_k^*(x)$, first in the case $x \in Q_k$. Consider

$$\phi_t * (\eta_k f)(x) = \int_{\mathbb{R}^n} \phi_t(x-y)\eta_k(y)f(y)\,dy \quad \text{ where } \phi \in \mathcal{T},\ t > 0.$$

We apply Lemma 4.3.5 with $\phi(y)$ replaced by $\psi(y) := \phi_t(x-y)\eta_k(y)$ and with $z = x$. Using either $\operatorname{spt}\psi \subset B_t(x)$ or alternatively $\operatorname{spt}\psi \subset B_{C\ell_k}(x_k)$, we get

$$|\phi_t * (\eta_k f)(x)| \leq C \min(t, |x-x_k| + C\ell_k)^{n+1} \|D\psi\|_{C^0(\mathbb{R}^n)} f^*(x).$$

By (4.42) we have

$$\|D\psi\|_{C^0(\mathbb{R}^n)} \leq \frac{1}{t^{n+1}} + \frac{C}{t^n \ell_k}.$$

Inserting and taking the supremum over $\phi \in \mathcal{T}$ and $t > 0$, we see that

$$(\eta_k f)^*(x) \leq Cf^*(x) \quad \text{ for } |x-x_k| \leq C\ell_k. \tag{4.47}$$

Furthermore using (4.46) we can also estimate

$$|\phi_t * \big(c_k(f)\eta_k\big)(x)| \leq |c_k(f)| \int_{\mathbb{R}^n} |\phi_t(x-y)\eta_k(y)|\,dy \leq Cf^*(x) \quad \text{ for } |x-x_k| \leq C\ell_k. \tag{4.48}$$

Taking again the supremum over $\phi \in \mathcal{T}$, $t > 0$, and combining yields

$$b_k^*(x) \leq Cf^*(x) \quad \text{ for } |x-x_k| \leq C\ell_k. \tag{4.49}$$

To treat the complemetary case, we now assume $x \in \mathbb{R}^n \backslash Q_k$. By (4.45) we can write

$$(\phi_t * b_k)(x) = \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-x_k))\, b_k(y)\,dy = I_1 - I_2,$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-x_k))\, \eta_k(y) f(y)\, dy, \\ I_2 &= c_k(f) \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-x_k))\, \eta_k(y)\, dy. \end{aligned}$$

We assume that ξ was initially chosen such that $\xi(x) = 0$ for $\|x\|_\infty \geq \frac{2}{3}$. Then

$$\eta_k(y) = 0 \quad \text{ for } \|y - x_k\|_\infty \geq \frac{2\ell_k}{3}.$$

For $\|y - x_k\|_\infty \leq \frac{2\ell_k}{3}$, we can estimate

$$\|x-y\|_\infty \geq \|x-x_k\|_\infty - \|y-x_k\|_\infty \geq \|x-x_k\|_\infty - \frac{2}{3}\|x-x_k\|_\infty \geq \frac{1}{3}\|x-x_k\|_\infty.$$

It follows that

$$|x-y| \geq \|x-y\|_\infty \geq \frac{1}{3}\|x-x_k\|_\infty \geq \frac{1}{3\sqrt{n}}|x-x_k|.$$

Thus if $t \leq \frac{1}{3\sqrt{n}}|x-x_k|$, then $\phi_t(x-y) = 0$ and $\phi_t(x-x_k) = 0$, in particular

$$I_{1,2} = 0 \quad \text{ for } t \leq \frac{1}{3\sqrt{n}}|x-x_k|. \tag{4.50}$$

Now assume that $t \geq \frac{1}{3\sqrt{n}}|x-x_k|$. By the mean value theorem, we have

$$|\phi_t(x-y) - \phi_t(x-x_k)| \leq \|D\phi_t\|_{C^0(\mathbb{R}^n)}|y-x_k| \leq \frac{|y-x_k|}{t^{n+1}}.$$

For $\psi(y) := \big(\phi_t(x-y) - \phi_t(x-x_k)\big)\eta_k(y)$ we get recalling (4.42)

$$|D\psi(y)| \leq \frac{1}{t^{n+1}} + \frac{\sqrt{n}\ell_k}{t^{n+1}}\frac{C}{\ell_k} \leq \frac{C}{|x-x_k|^{n+1}}.$$

We apply Lemma 4.3.5, taking again $z \in \mathbb{R}^n \backslash \Omega$ with $|z - x_k| \leq C\ell_k$. Then $f^*(z) \leq \alpha$, and we obtain using $\operatorname{spt}\psi \subset B_{C\ell_k}(x_k)$

$$|I_1| \leq \frac{C\ell_k^{n+1}\alpha}{|x-x_k|^{n+1}} \quad \text{ for } t \geq \frac{1}{3\sqrt{n}}|x-x_k|. \tag{4.51}$$

The same estimate follows for I_2 using (4.46), namely

$$|I_2| \leq \frac{C\ell_k\alpha}{t^{n+1}}|Q_k| \leq \frac{C\ell_k^{n+1}\alpha}{|x-x_k|^{n+1}} \quad \text{ for } t \geq \frac{1}{3\sqrt{n}}|x-x_k|. \tag{4.52}$$

Combining (4.50), (4.51) and (4.52) we conclude

$$\begin{aligned} \int_{\mathbb{R}^n\backslash Q_k} b_k^*(x)\, dx &\leq C\ell_k^{n+1}\alpha \int_{\mathbb{R}^n\backslash Q_k} \frac{dx}{|x-x_k|^{n+1}} \\ &\leq C\ell_k^{n+1}\alpha \int_{\ell_k}^\infty \frac{dr}{r^2} \\ &\leq C\alpha\, |Q_k| \\ &\leq C\int_{Q_k} f^*(x)\, dx. \end{aligned}$$

Here we used $Q_k \subset \Omega = \{f^* > \alpha\}$. Claim (i) follows from this estimate and (4.49). □

The following characterization of Hardy space is one of the key results in [15].

Theorem 4.3.7 (Fefferman-Stein)**.** *For any $f \in \mathcal{H}^1(\mathbb{R}^n)$ there exists a sequence a_k, $k \in \mathbb{N}$, of $\mathcal{H}^1$-atoms and a sequence $\lambda_k \in \mathbb{R}$, such that*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \tag{4.53}$$

where the convergence is in the $\mathcal{H}^1$-norm, and moreover

$$\sum_{k=1}^{\infty} |\lambda_k| \leq C \, \|f\|_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.54}$$

Proof. For each $\nu \in \mathbb{Z}$ we apply Lemma 4.3.6 with $\alpha = 2^\nu > 0$, obtaining the decomposition

$$f = g^\nu + b^\nu = g^\nu + \sum_{k=1}^{\infty} b_k^\nu. \tag{4.55}$$

We put $\Omega^\nu := \{x \in \mathbb{R}^n : f^*(x) > 2^\nu\}$. By Lemma 4.3.6(i) and Lemma 4.3.4(ii),(iv) we have

$$\sum_{k=1}^{\infty} \|b_k^\nu\|_{\mathcal{H}^1} \leq C \sum_{k=1}^{\infty} \int_{Q_k^\nu} f^*(x)\, dx \leq C \int_{\Omega^\nu} f^*(x) dx = \int_{\mathbb{R}^n} \chi_{\{f^*>2^\nu\}} f^*(x)\, dx.$$

Since $f^* \in L^1(\mathbb{R}^n)$ we therefore conclude that

$$\|f - g^\nu\|_{\mathcal{H}^1} \leq \sum_{k=1}^{\infty} \|b_k^\nu\|_{\mathcal{H}^1} \to 0 \quad \text{as } \nu \nearrow \infty. \tag{4.56}$$

On the other hand, Lemma 4.3.6(ii) says that

$$\|g^\nu\|_{L^\infty(\mathbb{R}^n)} \leq C 2^\nu \to 0 \quad \text{as } \nu \searrow -\infty. \tag{4.57}$$

Combining (4.56) and (4.57) we have in particular

$$\sum_{|\nu| \leq N} (g^{\nu+1} - g^\nu) = g^{N+1} - g^{-N} \to f \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{as } N \to \infty. \tag{4.58}$$

Now using that η_k^ν is a partition of unity on $\Omega^\nu \supset \Omega^{\nu+1}$, we can write

$$\begin{aligned} g^{\nu+1} - g^\nu &= b^\nu - b^{\nu+1} \\ &= \sum_{k=1}^{\infty} \big(f - c_k^\nu(f)\big)\eta_k^\nu - \sum_{\ell=1}^{\infty} \big(f - c_\ell^{\nu+1}(f)\big)\eta_\ell^{\nu+1} \sum_{k=1}^{\infty} \eta_k^\nu \\ &= \sum_{k=1}^{\infty} \big(f - c_k^\nu(f)\big)\eta_k^\nu - \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \big(f - c_\ell^{\nu+1}(f)\big)\eta_\ell^{\nu+1}\eta_k^\nu. \end{aligned}$$

Note that the sums are locally finite, hence interchanging the summation poses no problem. By definition of the $c_k^\nu(f)$, the integral of each term in the first sum is zero. To achieve this also for the second sum, we just subtract the necessary corrections. Define

$$c_{k,\ell}^\nu(f) = c_\ell^{\nu+1}\big((f - c_\ell^{\nu+1}(f))\eta_k^\nu\big) = ⨍_{\mathbb{R}^n} (f - c_\ell^{\nu+1}(f))\eta_k^\nu \, d(\mathcal{L}^n \llcorner \eta_\ell^{\nu+1}). \tag{4.59}$$

Note that $c^\nu_{k,\ell}(f) = 0$ if $Q^{\nu+1}_\ell \cap Q^\nu_k = \emptyset$. We now claim that $g^{\nu+1} - g^\nu = \sum_{k=1}^\infty A^\nu_k$ where

$$A^\nu_k = \big(f - c^\nu_k(f)\big)\eta^\nu_k - \sum_{\ell=1}^\infty \Big(\big(f - c^{\nu+1}_\ell(f)\big)\eta^\nu_k - c^\nu_{k,\ell}(f)\Big)\eta^{\nu+1}_\ell.$$

In fact, the corrections cancel when summing over k, since by linearity of $c^{\nu+1}_\ell$ we get

$$\begin{aligned}\sum_{k=1}^\infty c^\nu_{k,\ell}(f) &= \sum_{k=1}^\infty c^{\nu+1}_\ell\big((f - c^{\nu+1}_\ell(f))\eta^\nu_k\big) \\ &= c^{\nu+1}_\ell\Big(\big(f - c^{\nu+1}_\ell(f)\big)\sum_{k=1}^\infty \eta^\nu_k\Big) \\ &= c^{\nu+1}_\ell\big(f - c^{\nu+1}_\ell(f)\big) = 0.\end{aligned}$$

By definition, the A^ν_k integrate to zero and have support in the union of Q^ν_k with those $Q^{\nu+1}_\ell$ intersecting Q^ν_k. Choosing some $x \in Q^\nu_k \cap Q^{\nu+1}_\ell$ we have by Lemma 4.3.4(iv), as $F^\nu \subset F^{\nu+1}$,

$$\operatorname{diam} Q^{\nu+1}_\ell \leq C \operatorname{dist}(x, F^{\nu+1}) \leq C \operatorname{dist}(x, F^\nu) \leq C \operatorname{diam} Q^\nu_k.$$

Therefore A^ν_k has support in a ball B^ν_k with $\operatorname{diam} B^\nu_k \leq C\ell^\nu_k$. We finally claim that

$$\|A^\nu_k\|_{L^\infty(\mathbb{R}^n)} \leq C\, 2^\nu. \tag{4.60}$$

For this we reorder the terms in A^ν_k as follows.

$$A^\nu_k = f\eta^\nu_k\big(1 - \sum_{\ell=1}^\infty \eta^{\nu+1}_\ell\big) - c^\nu_k(f)\eta^\nu_k + \sum_{\ell=1}^\infty \big(c^{\nu+1}_\ell(f)\eta^\nu_k + c^\nu_{k,\ell}(f)\big)\,\eta^{\nu+1}_\ell.$$

As $\eta^{\nu+1}_\ell$ is a partition of unity, the first term vanishes on $\Omega^{\nu+1}$, while on $\mathbb{R}^n \backslash \Omega^{\nu+1}$ we have $|f(x)| \leq f^*(x) \leq 2^{\nu+1}$. For the second term, we recall $|c^\nu_k(f)| \leq C2^\nu$ from (4.46). The constant $c^\nu_{k,\ell}(f)$ is estimated similarly by Lemma 4.3.5, replacing f by $(f - c^{\nu+1}_\ell(f))\eta^\nu_k$, and taking $z \in F^{\nu+1}$ such that $|z - x^{\nu+1}_\ell| \leq C\ell^{\nu+1}_k \leq C\ell^\nu_k$. This yields

$$|c^\nu_{k,\ell}(f)| \leq C\,\big((f - c^{\nu+1}_\ell(f))\eta^\nu_k\big)^*(z) \leq Cf^*(z) + C|c^{\nu+1}_\ell(f)| \leq C\,2^\nu.$$

Here we used (4.47) and (4.48). Since the overlap of the $\eta^{\nu+1}_\ell$ is estimated by Lemma 4.3.4(iv), the bound (4.60) is established. Now put $a^\nu_k = A^\nu_k/\lambda^\nu_k$ where $\lambda^\nu_k = C2^\nu|B^\nu_k|$ with $C < \infty$ as in (4.60). It is immediate that the a^ν_k are $\mathcal{H}^1$-atoms with admissible ball B^ν_k. Moreover

$$\begin{aligned}\sum_{|\nu|\leq N}\sum_{k=1}^\infty |\lambda^\nu_k| &\leq C \sum_{|\nu|\leq N} 2^\nu \sum_{k=1}^\infty |Q^\nu_k| \\ &\leq C \sum_{|\nu|\leq N} 2^\nu |\{f^* > 2^\nu\}| \\ &\leq C \sum_{|\nu|\leq N} \int_{2^{\nu-1}}^{2^\nu} |\{f^* > t\}|\, dt \\ &\leq C \int_{\mathbb{R}^n} f^*(x)\, dx = \|f\|_{\mathcal{H}^1(\mathbb{R}^n)}.\end{aligned}$$

By Lemma 4.3.3 the series

$$\sum_{\nu\in\mathbb{Z}}(g^{\nu+1}-g^{\nu}) = \sum_{\nu\in\mathbb{Z}}\sum_{k=1}^{\infty} A_k^{\nu} = \sum_{\nu\in\mathbb{Z}}\sum_{k=1}^{\infty}\lambda_k^{\nu}a_k^{\nu}$$

converges absolutely in $\mathcal{H}^1(\mathbb{R}^n)$ to some function $\tilde{f}\in\mathcal{H}^1(\mathbb{R}^n)$. But the convergence is also in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by Lemma 4.2.3, and we conclude $\tilde{f}=f$ by (4.58). Thus we have constructed an atomic decomposition as desired. □

The rest of this section presents an application to the Poisson equation $\Delta u = f$. For f belonging to $\mathcal{H}^1(\mathbb{R}^n)$, we show that all second derivatives are in $L^1(\mathbb{R}^n)$. We start by recalling some facts about the Newtonian potential. For $\omega_{n-1}=|\mathbb{S}^{n-1}|$ the fundamental solution of the Laplace operator is

$$\Gamma:\mathbb{R}^n\backslash\{0\}\to\mathbb{R},\ \Gamma(x)=\begin{cases}\dfrac{1}{(2-n)\omega_{n-1}}|x|^{2-n} & \text{for } n\geq 3,\\ \dfrac{1}{2\pi}\log|x| & \text{for } n=2.\end{cases}$$

The Newtonian potential of a function f is given by the formula (whenever defined)

$$Nf:\mathbb{R}^n\to\mathbb{R},\ Nf(x)=\int_{\mathbb{R}^n}\Gamma(x-y)f(y)\,dy.$$

For $f\in C_c^{\infty}(\mathbb{R}^n)$ we know that Nf is well-defined, smooth and solves $\Delta(Nf)=f$.

Lemma 4.3.8. *For $f\in L^{\infty}(\mathbb{R}^n)$ with compact support, we have $u=Nf\in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\|D^2u\|_{L^2(\mathbb{R}^n)}=\|f\|_{L^2(\mathbb{R}^n)}.$$

Proof. We first assume $f\in C_c^{\infty}(\mathbb{R}^n)$. Putting $d=\mathrm{diam}(\mathrm{spt}\,f)$ we have

$$\int_{\mathrm{spt}\,f}|\Gamma(x-y)|\,dy\leq\int_{\{|z|\leq|x|+d\}}|\Gamma(z)|\,dz\leq C(d,R)<\infty\quad\text{for } |x|\leq R.$$

This implies

$$|u(x)|\leq C(d,R)\,\|f\|_{L^{\infty}(\mathbb{R}^n)}\quad\text{for } |x|\leq R.$$

Differentiation under the integral and integration by parts yields

$$\partial_i u(x)=\int_{\mathbb{R}^n}\partial_i\Gamma(x-y)f(y)\,dy.$$

Repeating the argument above then also implies

$$|Du(x)|\leq C(d,R)\,\|f\|_{L^{\infty}(\mathbb{R}^n)}\quad\text{for } |x|\leq R.$$

Finally we have

$$\partial_{ij}^2u(x)=\int_{\mathbb{R}^n}\partial_{ij}^2\Gamma(x-y)\,f(y)\,dy\quad\text{for } x\in\mathbb{R}^n\backslash\mathrm{spt}\,f.$$

Now for $y \in \operatorname{spt} f$ and $|x|$ large, we have $|x-y| \geq |x|/2$ and see from the kernel representations

$$|Du(x)| \leq \frac{C}{|x|^{n-1}},\ |D^2u(x)| \leq \frac{C}{|x|^n} \quad \text{for } |x| \text{ large.}$$

We calculate

$$\begin{aligned}\int_{B_R(0)} |D^2u|^2\,dx &= \int_{B_R(0)} \partial_i(\partial_j u\,\partial^2_{ij}u)\,dx - \int_{B_R(0)} \partial_j u\,\partial_j \Delta u\,dx\\ &= \int_{B_R(0)} \partial_i\big(\partial_j u\,\partial^2_{ij}u - \partial_i u \Delta u\big)\,dx + \int_{B_R(0)} |\Delta u|^2\,dx.\end{aligned}$$

Letting $R \to \infty$ we conclude

$$\|D^2u\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} < \infty.$$

For general f we consider the smoothings $f_\varepsilon = \eta_\varepsilon * f$. Dominated convergence implies that $Nf_\varepsilon(x) \to Nf(x)$ for all $x \in \mathbb{R}^n$. From the bound in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ we see that $u_\varepsilon \to u$ uniformly, this implies in particular $\Delta u = f$ in the sense of distributions. By weak* compactness we also have $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. But now $\partial^2_{ij}u_\varepsilon$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and thus $\partial^2_{ij}u_\varepsilon \to \partial^2_{ij}u \in L^2(\mathbb{R}^n)$. In particular we have

$$\|D^2u\|_{L^2(\mathbb{R}^n)} = \lim_{\varepsilon \searrow 0} \|D^2u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \lim_{\varepsilon \searrow 0} \|f_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

This finishes the proof of the lemma. □

Theorem 4.3.9. *For any $f \in \mathcal{H}^1(\mathbb{R}^n)$ there exists $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^n)$ solving $\Delta u = f$, such that*

$$\|D^2u\|_{L^1(\mathbb{R}^n)} \leq C\,\|f\|_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.61}$$

We remark that if $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is any other solution of $\Delta v = f$ with $D^2v \in L^1(\mathbb{R}^n)$, then by Liouville $D^2(v-u) = 0$ and $v-u$ is affine-linear. In particular (4.61) holds also for v.

Proof. Let $a \in L^1(\mathbb{R}^n)$ be an $\mathcal{H}^1$-atom with admissible ball $B = B_R(0)$. By the previous lemma, the Newtonian potential $u^a = Na$ belongs to $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\int_{B_{2R}(0)} |D^2u^a|\,dx \leq \|D^2u^a\|_{L^2(\mathbb{R}^n)}|B|^{\frac{1}{2}} = \|a\|_{L^2(\mathbb{R}^n)}|B|^{\frac{1}{2}}.$$

Now let $K_{ij} = \partial^2_{ij}\Gamma$. For $|x| \geq 2R$ we can differentiate the kernel to get

$$\partial^2_{ij}u^a(x) = \int_{\mathbb{R}^n} K_{ij}(x-y)a(y)\,dy = \int_{\mathbb{R}^n} \big(K_{ij}(x-y) - K(x)\big)a(y)\,dy.$$

In the last step we used that a has integral zero. For $t \in [0,1]$ and $y \in B_R(0)$ we have $|x-ty| \geq |x| - R \geq \frac{1}{2}|x|$, which yields

$$|K_{ij}(x-y) - K_{ij}(x)| \leq \int_0^1 |DK_{ij}(x-ty)\cdot y|\,dt \leq C|y||x|^{-(n+1)}.$$

Using $|x| \geq 2R \geq 2|y|$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_{2R}(0)} |K_{ij}(x-y) - K_{ij}(x)|\, dx &\leq C|y| \int_{|x| \geq 2|y|} |x|^{-(n+1)}\, dx \\ &\leq C|y| \int_{2|y|}^{\infty} t^{-2}\, dt \leq C.\end{aligned}$$

Inserting we find

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_{2R}(0)} |\partial^2_{ij} u^a(x)|\, dx &\leq C \int_{B_R(0)} |a(y)| \int_{\mathbb{R}^n \setminus B_{2R}(0)} |K_{ij}(x-y) - K_{ij}(x)|\, dx\, dy \\ &\leq C\, \|a\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Combining the two estimates using the L^∞-norm, we finally arrive at

$$\|D^2 u^a\|_{L^1(\mathbb{R}^n)} \leq C\, \|a\|_{L^\infty(\mathbb{R}^n)} |B| \leq C.$$

Now let $f = \sum_{j=1}^\infty \lambda_j a_j$ be the atomic decomposition of f given by Theorem 4.3.7. Put

$$f_k = \sum_{j=1}^k \lambda_j a_j \quad \text{and} \quad v_k = \sum_{j=1}^k \lambda_j u^{a_j} \in W^{2,2}_{\text{loc}}(\mathbb{R}^n).$$

We have $\Delta v_k = f_k \to f$ in $\mathcal{H}^1(\mathbb{R}^n)$. Furthermore our estimates and Theorem 4.3.7 give

$$\sum_{j=1}^\infty \|D^2(\lambda_j u^{a_j})\|_{L^1(\mathbb{R}^n)} \leq C \sum_{j=1}^\infty |\lambda_j| \leq C \|f\|_{\mathcal{H}^1(\mathbb{R}^n)} < \infty.$$

Thus $D^2 v_k$ converges in $L^1(\mathbb{R}^n)$ to some $W \in L^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$, which satisfies

$$\operatorname{tr} W = f \quad \text{and} \quad \|W\|_{L^1(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

Now we pass to $u_k(x) = v_k(x) - (A_k \cdot x + b_k)$ where

$$A_k = \int_{B_1(0)} Dv_k(x)\, dx \quad \text{and} \quad b_k = \int_{B_1(0)} v_k(x)\, dx.$$

By a standard contradiction argument involving Rellich's theorem, compare Lemma 4.2.12, we get after passing to subsequence

$$u_k \to u \text{ in } W^{1,1}_{\text{loc}}(\mathbb{R}^n).$$

It follows that $D^2 u = W$, and $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n)$ is the desired solution. □

One can in fact prove the optimal regularity $D^2 u \in \mathcal{H}^1(\mathbb{R}^n)$, as noted by Stein [57]. In [52] there are some remarks about localizing the concept of Hardy space.

Chapter 5

Lorentz spaces

In this chapter we study Lorentz spaces, which have been introduced by George Lorentz around 1950. These spaces can be viewed as interpolations of the classical L^p-spaces, and they are particularly relevant in connection with optimal Sobolev embeddings.

5.1 Definition and basic properties

Theorem 5.1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then there is a unique non-increasing, right continuous function $f^* : [0,\infty) \to [0,\infty]$ such that*

$$|\{f^* > s\}| = |\{|f| > s\}| \quad \text{for all } s \in [0,\infty). \tag{5.1}$$

Moreover f^* has the property

$$\int_0^\infty f^*(t)\,dt = \int_0^\infty f_*(s)\,ds = \int_{\mathbb{R}^n} |f(x)|\,dx. \tag{5.2}$$

The function $f_* : [0,\infty) \to [0,\infty]$, $f_*(s) = |\{|f| > s\}|$, is called the distribution function of f; it is nonincreasing and continuous from the right. The theorem asserts the existence and uniqueness of a non-increasing, right continuous function f^* on $[0,\infty)$ having the same distribution function as f. We remark that if $f : [0,\infty) \to [0,\infty]$ is non-increasing and continuous from the right, then we have $f^* = f$ by uniqueness.

Proof. It is easy to check that a function $f : [0,\infty) \to [0,\infty]$ is non-increasing and continuous from the right if and only if the set $\{f > r\}$ is an interval of the form $[0,b)$, for all $r \geq 0$. Now assume that $f^* : [0,\infty) \to [0,\infty]$ has all required properties. We claim that

$$\{f^* > s\} = [0, f_*(s)) \quad \text{for all } s \geq 0. \tag{5.3}$$

In fact, the set on the left is an interval $[0,b)$ where $b = |\{|f| > s\}| = f_*(s)$ by (5.1). A statement equivalent to (5.3) but more symmetric is

$$f^*(t) > s \quad \Leftrightarrow \quad f_*(s) > t. \tag{5.4}$$

In particular we obtain uniqueness since we then also have

$$\{f_* > t\} = [0, f^*(t)) \quad \text{for all } t \geq 0. \tag{5.5}$$

For existence we define $f^*(t)$ by (5.5). Then f^* is nonincreasing and continuous from the right, and (5.3) follows reversely. We note from the above that

$$f^*(t) = |\{f_* > t\}| \quad \text{and} \quad f_*(s) = |\{f^* > s\}|. \tag{5.6}$$

Finally we compute

$$\int_0^\infty f^*(t)\,dt = \int_0^\infty |\{f^* > s\}|\,ds = \int_0^\infty \underbrace{|\{|f| > s\}|}_{=f_*(s)}\,ds = \int_{\mathbb{R}^n} |f(x)|\,dx.$$

□

In the next lemma we prove the Hardy-Littlewood-Polya inequality.

Lemma 5.1.2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \int_0^\infty f^*(t)g^*(t)\,dt. \tag{5.7}$$

Proof. Using Fubini‘s Theorem on $\mathbb{R}^n \times (0,\infty)^2$ we calculate for $f, g \ge 0$

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)g(x)\,dx &= \int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty \chi_{\{r<f(x)\}}\chi_{\{s<g(x)\}}\,drds\,dx \\
&= \int_0^\infty\int_0^\infty |\{f > r\} \cap \{g > s\}|\,drds \\
&\le \int_0^\infty\int_0^\infty \min\big(|\{f > r\}|, |\{g > s\}|\big)\,drds \\
&= \int_0^\infty\int_0^\infty \min\big(f_*(r), g_*(s)\big)\,drds \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \chi_{\{r<f^*(t)\}}\chi_{\{s<g^*(t)\}}\,dt\,drds \\
&= \int_0^\infty f^*(t)g^*(t)\,dt.
\end{aligned}$$

We used that $f^*(t) > r$, $g^*(t) > s$ if and only if $t < \min(f_*(s), g_*(s))$ by (5.4). □

To introduce the Lorentz spaces we further define, for $f : \mathbb{R}^n \to \mathbb{R}$ measurable,

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\,du \quad \text{for } t > 0. \tag{5.8}$$

f^{**} is non-increasing and $f^{**} \ge f^*$. For $f \in L^1(\mathbb{R}^n)$ we have $f^* \in L^1([0,\infty))$ by (5.6), therefore f^{**} is continuous on $(0,\infty)$. Moreover $f^{**}(t) \to f^*(0)$ as $t \searrow 0$, since f^* is right continuous, and $f^{**}(t) \to 0$ as $t \nearrow \infty$.

Definition 5.1.3. *Let $1 \le p \le \infty$, $1 \le q \le \infty$. A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to the Lorentz space $L^{p,q}(\mathbb{R}^n)$ if and only if the following integral is finite:*

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \begin{cases} \Big(\int_0^\infty \big(t^{\frac{1}{p}} f^{**}(t)\big)^q \dfrac{dt}{t}\Big)^{\frac{1}{q}}, & \textit{for } 1 \le q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \textit{for } q = \infty. \end{cases} \tag{5.9}$$

For $p=1$, $1\le q<\infty$, the definition just reduces to $L^{1,q}(\mathbb{R}^n)=\{0\}$. We emphasize that the $L^{p,q}(\mathbb{R}^n)$ norm has the same scaling as the usual $L^p(\mathbb{R}^n)$ norm, that is

$$\|f_\lambda\|_{L^{p,q}(\mathbb{R}^n)} = \lambda^{-\frac{n}{p}}\|f\|_{L^{p,q}(\mathbb{R}^n)} \quad \text{for any } \lambda>0. \tag{5.10}$$

Lemma 5.1.4 (Hardy's inequality). *Let $1\le q<\infty$, $r>0$ and $g:(0,\infty)\to[0,\infty)$, then*

$$\int_0^\infty \Big(\int_0^t g(u)\,du\Big)^q t^{-r-1}\,dt \;\le\; \Big(\frac{q}{r}\Big)^q \int_0^\infty (ug(u))^q\, u^{-r-1}\,du \tag{5.11}$$

$$\int_0^\infty \Big(\int_t^\infty g(u)\,du\Big)^q t^{r-1}\,dt \;\le\; \Big(\frac{q}{r}\Big)^q \int_0^\infty (ug(u))^q\, u^{r-1}\,du. \tag{5.12}$$

Proof. The function $\varphi(x)=|x|^q$ is convex. Putting $\mu=\mathcal{L}^1\llcorner u^{\alpha-1}$ for $\alpha>0$ to be chosen, we get by Jensen's inequality

$$\begin{aligned}
\Big(\int_0^t g(u)\,du\Big)^q &= \Big(\frac{t^\alpha}{\alpha}\Big)^q\Big(⨍_0^t g(u)u^{1-\alpha}\,d\mu\Big)^q\\
&\le \Big(\frac{t^\alpha}{\alpha}\Big)^q ⨍_0^t g(u)^q\,u^{(1-\alpha)q}\,d\mu\\
&= \Big(\frac{t^\alpha}{\alpha}\Big)^{q-1}\int_0^t (g(u)u)^q\,u^{\alpha(1-q)-1}\,du.
\end{aligned}$$

Inserting we obtain using Fubini, provided that $\alpha(q-1)<r$,

$$\begin{aligned}
\int_0^\infty \Big(\int_0^t g(u)\,du\Big)^q t^{-r-1}\,dt &\le \Big(\frac{1}{\alpha}\Big)^{q-1}\int_0^\infty t^{\alpha(q-1)-r-1}\int_0^t (g(u)u)^q\,u^{\alpha(1-q)-1}\,du\,dt\\
&= \Big(\frac{1}{\alpha}\Big)^{q-1}\int_0^\infty (g(u)u)^q\,u^{\alpha(1-q)-1}\int_u^\infty t^{\alpha(q-1)-r-1}\,dt\,du\\
&\le \Big(\frac{1}{\alpha}\Big)^{q-1}\frac{1}{r-\alpha(q-1)}\int_0^\infty (g(u)u)^q\,u^{-r-1}\,du.
\end{aligned}$$

(5.11) follows by taking $\alpha=\frac{r}{q}$ (which is in fact optimal). We deduce (5.12) from (5.11), applied to $g_1(u)=g(1/u)/u^2$ Substituting $s=1/t$ and then $v=1/u$ we get

$$\begin{aligned}
\int_0^\infty \Big(\int_s^\infty g(v)\,dv\Big)^q s^{r-1}ds &= \int_0^\infty \Big(\int_{\frac{1}{t}}^\infty g(v)\,dv\Big)^q t^{-r-1}\,dt\\
&= \int_0^\infty \Big(\int_0^t g_1(u)\,du\Big)^q t^{-r-1}\,dt\\
&\le \Big(\frac{q}{r}\Big)^q\int_0^\infty (ug_1(u))^q\,u^{-r-1}\,du\\
&= \Big(\frac{q}{r}\Big)^q\int_0^\infty (vg(v))^q\,v^{r-1}\,dv.
\end{aligned}$$

□

Defining $\|f\|_{L^{p,q}_*(\mathbb{R}^n)}$ just as $\|f\|_{L^{p,q}(\mathbb{R}^n)}$, but with f^* instead of f^{**}, we have

$$\|f\|_{L^{p,q}_*(\mathbb{R}^n)} \le \|f\|_{L^{p,q}(\mathbb{R}^n)} \le \frac{p}{p-1}\|f\|_{L^{p,q}_*(\mathbb{R}^n)} \quad \text{for all } 1<p\le\infty,\ 1\le q\le\infty. \tag{5.13}$$

The first inequality is obvious as $f^* \leq f^{**}$. For $q < \infty$, the second inequality follows from Hardy's inequality putting $r = q - \frac{q}{p} > 0$:

$$\begin{aligned}
\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t} &= \int_0^\infty \Big(\int_0^t f^*(u)\,du\Big)^q t^{\frac{q}{p}-q-1}\,dt \\
&\leq \left(\frac{p}{p-1}\right)^q \int_0^\infty (uf^*(u))^q u^{\frac{q}{p}-q-1}\,du \\
&= \left(\frac{p}{p-1}\right)^q \int_0^\infty \left(u^{\frac{1}{p}} f^*(u)\right)^q \frac{du}{u}.
\end{aligned}$$

For $q = \infty$ we compute by hand that

$$t^{\frac{1}{p}} f^{**}(t) = t^{\frac{1}{p}-1} \int_0^t f^*(s)\,ds \leq t^{\frac{1}{p}-1} \sup_{s>0}\left(s^{\frac{1}{p}} f^*(s)\right) \int_0^t s^{-\frac{1}{p}} ds = \frac{p}{p-1} \sup_{s>0}\left(s^{\frac{1}{p}} f^*(s)\right).$$

The definition of the spaces $L_*^{p,q}(\mathbb{R}^n)$ with norms $\|f\|_{L_*^{p,q}(\mathbb{R}^n)}$ makes sense also for $p = 1$, a special case is $L_*^{1,1}(\mathbb{R}^n) = L^1(\mathbb{R}^n) = L^{1,\infty}(\mathbb{R}^n)$. By contrast for $p = \infty$ we have $L_*^{\infty,q}(\mathbb{R}^n) = \{0\}$ for $1 \leq q < \infty$, while $L_*^{\infty,\infty}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) = L^{1,\infty}(\mathbb{R}^n)$.

The choice of f^{**} in definition 5.1.3 is motivated by the fact that the triangle inequality holds, see [36]. Here we will only show that taking f^* yields a quasinorm. We compute

$$(f+g)_*(2s) = |\{|f+g| > 2s\}| \leq |\{|f| > s\}| + |\{|g| > s\}| = f_*(s) + g_*(s).$$

Using (5.5) we get

$$\begin{aligned}
(f+g)^*(2t) &= 2|\{s \geq 0 : (f+g)_*(2s) > 2t\}| \\
&\leq 2\left(|\{s \geq 0 : f_* > t\}| + |\{s \geq 0 : g_* > t\}|\right) \\
&= 2\big(f^*(t) + g^*(t)\big).
\end{aligned}$$

From here we easily see that

$$\|f+g\|_{L^{p,q}(\mathbb{R}^n)} \leq 2^{1+\frac{1}{p}} \left(\|f\|_{L^{p,q}(\mathbb{R}^n)} + \|g\|_{L^{p,q}(\mathbb{R}^n)}\right). \tag{5.14}$$

In particular $L^{p,q}(\mathbb{R}^n)$ is a vector space. We now show that the classical L^p spaces are included in the Lorentz family as the special case $q = p$.

Lemma 5.1.5. *For $1 < p < \infty$ the $L^{p,p}(\mathbb{R}^n)$-norm is equivalent to the $L^p(\mathbb{R}^n)$-norm, and the $L^{1,\infty}(\mathbb{R}^n)$-norm is equal to the $L^1(\mathbb{R}^n)$-norm.*

Proof. By (5.13) the norm $\|f\|_{L^{p,p}(\mathbb{R}^n)}$ is bounded above and below by the integral

$$\Big(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^p \frac{dt}{t}\Big)^{\frac{1}{p}} = \|f^*\|_{L^p([0,\infty))}.$$

But $\|f^*\|_{L^p([0,\infty)} = \|f\|_{L^p(\mathbb{R}^n)}$ by Lemma 4.1.1. That lemma also yields that

$$\|f\|_{L^{1,\infty}(\mathbb{R}^n)} = \sup_{t>0}\left(tf^{**}(t)\right) = \|f^*\|_{L^1([0,\infty))} = \|f\|_{L^1(\mathbb{R}^n)}.$$

□

The next Lemma proves a duality result for Lorentz spaces.

Lemma 5.1.6. *Let $f \in L^{p,q}(\mathbb{R}^n)$, $g \in L^{p',q'}(\mathbb{R}^n)$ with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$ $(1<p<\infty)$. Then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le C\,\|f\|_{L^{p,q}(\mathbb{R}^n)}\|g\|_{L^{p',q'}(\mathbb{R}^n)}. \tag{5.15}$$

Proof. Using Lemma 5.1.2 and Hölder's inequlity we estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\,dx &\le \int_0^\infty t^{\frac{1}{p}} f^*(t)\, t^{\frac{1}{p'}} g^*(t)\,\frac{dt}{t}\\ &\le \Big(\int_0^\infty \big(t^{\frac{1}{p}} f^*(t)\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \Big(\int_0^\infty \big(t^{\frac{1}{p'}} f^*(t)\big)^{q'} \frac{dt}{t}\Big)^{\frac{1}{q'}}.\end{aligned}$$

The result now follows from (5.13). □

Next we deal with relations among the spaces $L^{p,q}(\mathbb{R}^n)$.

Lemma 5.1.7. *Let $\Omega \subset \mathbb{R}^n$ be measurable with $|\Omega|<\infty$. Then for any $f:\mathbb{R}^n \to \mathbb{R}$ measurable we have, with a constant C depending only on the parameters,*

$$\|f\|_{L^{p,q'}(\mathbb{R}^n)} \le C\,\|f\|_{L^{p,q}(\mathbb{R}^n)} \quad \text{for } p>1 \text{ and } q\le q', \tag{5.16}$$

$$\|f\chi_\Omega\|_{L^{p,q}(\mathbb{R}^n)} \le C\,|\Omega|^{\frac{1}{p}-\frac{1}{p'}}\,\|f\chi_\Omega\|_{L^{p',q'}(\mathbb{R}^n)} \quad \text{for } p<p' \text{ and } q,q' \text{ abitrary.} \tag{5.17}$$

Proof. To show (5.16) in the case $q'=\infty$ we compute

$$(t^{\frac{1}{p}} f^*(t))^q = \frac{q}{p} f^*(t)^q \int_0^t s^{\frac{q}{p}-1}\,ds \le \frac{q}{p}\int_0^t \big(s^{\frac{1}{p}} f^*(s)\big)^q \frac{ds}{s}.$$

Using this and (5.13), we estimate further for $q'<\infty$

$$\int_0^\infty (t^{\frac{1}{p}} f^*(t))^{q'} \frac{dt}{t} \le \sup_{s>0}(s^{\frac{1}{p}} f^*(s))^{q'-q}\int_0^\infty \big(s^{\frac{1}{p}} f^*(t)\big)^q \frac{dt}{t} \le C\|f\|^{q'}_{L^{p,q}}.$$

For (ii) note first that $(f\chi_\Omega)_*(s)=|\{|f|\chi_\Omega>s\}|\le|\Omega|$. Recalling (5.6) this implies $f^*(t)=|\{f_*>t\}|=0$ for $t\ge|\Omega|$. Now we estimate

$$\begin{aligned}\|f\chi_\Omega\|_{L^{p,1}(\mathbb{R}^n)} &\le C\int_0^{|\Omega|} t^{\frac{1}{p}-1}(f\chi_\Omega)^*(t)\,dt\\ &\le C\sup_{s>0}\big(s^{\frac{1}{p'}}(f\chi_\Omega)^*(s)\big)\int_0^{|\Omega|} t^{\frac{1}{p}-\frac{1}{p'}-1}\,dt\\ &\le C\,|\Omega|^{\frac{1}{p}-\frac{1}{p'}}\|f\chi_\Omega\|_{L^{p',\infty}}(\mathbb{R}^n).\end{aligned}$$

Finally we conclude

$$\|f\chi_\Omega\|_{L^{p,q}(\mathbb{R}^n)} \le C\,\|f\chi_\Omega\|_{L^{p,1}(\mathbb{R}^n)} \le C\,|\Omega|^{\frac{1}{p}-\frac{1}{p'}}\|f\chi_\Omega\|_{L^{p',\infty}(\mathbb{R}^n)} \le \|f\chi_\Omega\|_{L^{p',q'}(\mathbb{R}^n)}.$$

The lemma is proved. □

Lemma 5.1.8. *For any $0<\lambda<n$ the function $I_\lambda(x)=|x|^{-\lambda}$ belongs to $L^{\frac{n}{\lambda},\infty}(\mathbb{R}^n)$.*

Proof. We compute using again (5.6)

$$\begin{aligned}(I_\lambda)_*(s) &= |\{I_\lambda > s\}| = \alpha_n\Big(\frac{1}{s}\Big)^{\frac{n}{\lambda}},\\ (I_\lambda)^*(t) &= |\{(I_\lambda)_* > t\}| = \Big(\frac{\alpha_n}{t}\Big)^{\frac{\lambda}{n}}.\end{aligned}$$

Thus we have $t^{\frac{\lambda}{n}}(I_\lambda)^*(t) = (\alpha_n)^{\frac{\lambda}{n}}$ for all $t > 0$, and the result follows from (5.13). □

The next lemma is a technical result which is needed later on.

Lemma 5.1.9. *For $1 < p \le \infty$ and $1 \le q < \infty$, we have*

$$\|f\|^q_{L^{p,q}_*} = \int_0^\infty t^{\frac{q}{p}-1}(t^{\frac{1}{p}} f^*(t))^q \,\frac{dt}{t} = \int_0^\infty (f_*(s))^{\frac{q}{p}} s^{q-1}\, ds. \tag{5.18}$$

In particular, the $L^{p,q}(\mathbb{R}^n)$ norm is bounded by the right hand side from above and from below.

Proof. By substituting $t = r^p$, using Fubini and (5.2), we calculate

$$\begin{aligned}\|f\|^q_{L^{p,q}(\mathbb{R}^n)} &\ge \int_0^\infty t^{\frac{q}{p}-1}(f^*(t))^q\, dt\\ &= p\int_0^\infty r^{q-p}(f^*(r^p))^q\, r^{p-1}\, dr\\ &= pq\int_0^\infty\int_0^\infty \chi_{\{s<f^*(r^p)\}} r^{q-1}s^{q-1}\, dsdr\\ &= pq\int_0^\infty\int_0^\infty \chi_{\{r<f_*(s)^{\frac{1}{p}}\}} r^{q-1}s^{q-1}\, drds\\ &= p\int_0^\infty f_*(s)^{\frac{q}{p}} s^{q-1}\, ds.\end{aligned}$$

□

Lemma 5.1.10. *Let $f \in L^1(\mathbb{R}^n)$ and $s > 0$. Then for a constant $C = C(n)$ we have*

$$(Mf)_*(Cs) \le \frac{1}{s}\int_{\{|f|>s\}} |f(x)|\, dx. \tag{5.19}$$

Proof. We may assume $f \ge 0$. Let $M = \{x \in \mathbb{R}^n : Mf(x) > Cs\}$. For any $x \in M$ there exists a ball $B^x = B_{r^x}(x)$ such that

$$Cs|B^x| < \int_{B^x} f(y)\, dy \le \int_{B^x\cap\{f>s\}} f(y)\, dy + s|B^x|.$$

By absorbing we see that

$$|B^x| < \frac{1}{(C-1)s}\int_{B^x\cap\{f>s\}} f(y)\, dy.$$

In particular we get $|B^x| \le \frac{1}{(C-1)s}\|f\|_{L^1(\mathbb{R}^n)} < \infty$. By Vitali's covering lemma, Theorem 4.1.3, there is a subset $M' \subset M$ such that the balls B^x, $x \in M'$, are pairwise disjoint, and such that

$$M \subset \bigcup_{x\in M'} 5B^x.$$

Here $5B^x$ means the concentric ball scaled by factor 5. We conclude

$$\begin{aligned}(Mf)_*(Cs) = |M| &\le 5^n \sum_{x\in M'} |B^x| \\ &\le \frac{5^n}{(C-1)s} \sum_{x\in M'} \int_{B^x\cap\{f>s\}} f(y)\,dy \\ &\le \frac{5^n}{(C-1)s} \int_{\{f>s\}} f(y)\,dy.\end{aligned}$$

The result follows by taking $C = 5^n + 1$. □

Lemma 5.1.11. *Let $f \in L^1(\mathbb{R}^n)$. Then for a constant $C = C(n)$*

$$(Mf)^*(t) \le C\, f^{**}(t) \quad \text{for all } t > 0. \tag{5.20}$$

Proof. We assume $f \ge 0$. Choosing $s = f^{**}(t)$ in Lemma 5.1.10 we obtain

$$\begin{aligned}f^{**}(t)\,(Mf)_*\big(Cf^{**}(t)\big) &\le \int_{\{f>f^{**}(t)\}} f(x)\,dx \\ &\le \int_{\{f>f^*(t)\}} f(x)\,dx \\ &\le \int_0^t f^*(t')\,dt' = t\,f^{**}(t).\end{aligned}$$

Here we used

$$\begin{aligned}\int_{\{f>f^*(t)\}} f(x)\,dx &= \int_{f^*(t)}^\infty |\{f>s\}|\,ds + f^*(t)|\{f>f^*(t)\}| \\ &= \int_{f^*(t)}^\infty |\{f^*>s\}|\,ds + f^*(t)|\{f^*>f^*(t)\}| \\ &= \int_{\{f^*>f^*(t)\}} f^*(t')\,dt' \le \int_0^t f^*(t')\,dt'.\end{aligned}$$

Finally the claim of the lemma follows, namely we have

$$(Mf)^*(t) = |\{(Mf)_* > t\}| \le Cf^{**}(t).$$

□

Theorem 5.1.12. *Let $1 < p < \infty$ and $f \in L^{p,q}(\mathbb{R}^n)$. Then we have*

$$\|Mf\|_{L^{p,q}(\mathbb{R}^n)} \le C\,\|f\|_{L^{p,q}(\mathbb{R}^n)}. \tag{5.21}$$

Proof. This follows directly from Lemma 5.1.11 and (5.13). An alternative proof is by combining the Hardy-Littlewood Theorem 4.1.6 and Marcinkiewicz interpolation, see Theorem 5.2.1. □

Lemma 5.1.13. *Let $f \in L^{p,q}(\mathbb{R}^n)$ where $1 \le p, q < \infty$. If $\eta \in C^0_c(B_1(0))$ is a kernel with $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$, then*

$$\eta_\varrho * f \to f \text{ in } L^{p,q}(\mathbb{R}^n) \quad \text{where } \eta_\varrho(x) = \varrho^{-n}\eta\big(\frac{x}{\varrho}\big).$$

Proof. By splitting $f = f^+ - f^-$ we can assume $f \geq 0$. We first show that if $0 \leq f_k \nearrow f$ pointwise, then the convergence is in $L^{p,q}(\mathbb{R}^n)$. Before entering this argument, we note that $|\{f > s\}| < \infty$ for any $s > 0$. Otherwise $f^*(s) = \infty$ on some interval $[0, \delta)$, which implies $f^*(t) = |\{f_* > t\}| \geq \delta$ for all $t > 0$. But then also $f^{**}(t) \geq \delta$ for all $t > 0$, and hence

$$\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \, \frac{dt}{t} \geq \delta^q \int_0^\infty t^{\frac{q}{p}-1} \, dt = \infty.$$

Our claim follows by repeated use of dominated convergence. First $\{f - f_k > s\} \searrow \emptyset$, and we have $\{f - f_k > s\} \subset \{f > s\}$ where $|\{f > s\}| < \infty$, therefore

$$(f - f_k)_*(s) = |\{f - f_k > s\}| \searrow 0.$$

It follows that $\{(f - f_k)_* > t\} \searrow \emptyset$, and $\{(f - f_k)_* > t\} \subset \{f_* > t\}$ where $|\{f_* > t\}| = f^*(t) < \infty$. This implies

$$(f - f_k)^*(t) = |\{(f - f_k)_* > t\}| \searrow 0.$$

Now $(f - f_k)^* \leq f^* \in L^1((0, t)$ for any $t > 0$, and we get

$$(f - f_k)^{**}(t) = \frac{1}{t} \int_0^t (f - f_k)^*(t') \, dt' \searrow 0.$$

Finally $t^{\frac{1}{p}} (f - f_k)^{**}(t) \leq t^{\frac{1}{p}} f^{**}(t) \in L^q(\frac{dt}{t})$, and we conclude

$$\|f - f_k\|^q_{L^{p,q}(\mathbb{R}^n)} = \int_0^\infty \Big(t^{\frac{1}{p}} (f - f_k)^{**}(t)\Big)^q \, \frac{dt}{t} \searrow 0.$$

As any measurable function $f \geq 0$ can be approximated from below monotonically by step functions, it is now sufficient to prove the theorem for $f = \chi_E$ where $E \subset \mathbb{R}^n$ is bounded and measurable. The $L^{p,q}$ norm of χ_E is easily computed:

$$(\chi_E)_*(s) \leq \begin{cases} |E| & \text{for } 0 < s < 1 \\ 0 & \text{for } s \geq 1. \end{cases} \qquad \chi_E^*(t) \leq \begin{cases} 1 & \text{for } 0 < t < |E|, \\ 0 & \text{for } t \geq |E|. \end{cases}$$

This implies further

$$\chi_E^{**}(t) = \begin{cases} 1 & \text{for } 0 < t < |E|, \\ \frac{1}{t}|E| & \text{for } t \geq |E|. \end{cases}$$

Using this one obtains

$$\|\chi_E\|_{L^{p,q}(\mathbb{R}^n)} = \Big(\frac{p^2}{q(p-1)}\Big)^{\frac{1}{q}} |E|^{\frac{1}{p}}. \tag{5.22}$$

To bound $\eta_\varrho * f$ in $L^{p,q}(\mathbb{R}^n)$, we use

$$|(\eta_\varepsilon * f)(x)| = \varepsilon^{-n} \Big| \int_{\mathbb{R}^n} \eta\Big(\frac{x - y}{\varepsilon}\Big) f(y) \, dy \Big| \leq C \, \|\eta\|_{C^0(\mathbb{R}^n)} Mf(x).$$

Applying Theorem 5.1.12 we obtain

$$\|\eta_\varrho * f\|_{L^{p,q}(\mathbb{R}^n)} \leq C \, \|\eta\|_{C^0(\mathbb{R}^n)} \|f\|_{L^{p,q}(\mathbb{R}^n)}. \tag{5.23}$$

Now let $\varepsilon > 0$ be given. By Lusin's theorem, there exists a function $\tilde{\chi} \in C_c^0(\mathbb{R}^n)$ such that

$$0 \leq \tilde{\chi} \leq 1, \quad \text{and} \quad |\{\tilde{\chi} \neq \chi_E\}| < \varepsilon.$$

Using the triangle inequlity, (5.23) and then (5.22), we see that

$$\begin{aligned} \|\eta_\varrho * \chi_E - \chi_E\|_{L^{p,q}(\mathbb{R}^n)} &\leq C\|\eta_\varrho * (\chi_E - \tilde{\chi})\|_{L^{p,q}(\mathbb{R}^n)} + C\|\eta_\varrho * \tilde{\chi} - \tilde{\chi}\|_{L^{p,q}(\mathbb{R}^n)} \\ &\quad + C\|\tilde{\chi} - \chi_E\|_{L^{p,q}(\mathbb{R}^n)} \\ &\leq C\|\eta_\varrho * \tilde{\chi} - \tilde{\chi}\|_{L^{p,q}(\mathbb{R}^n)} + C(\eta)\,\varepsilon^{\frac{1}{p}}. \end{aligned}$$

But by Lemma 5.1.7 the $L^{p,q}$ norm is estimated by the $L^{\infty,\infty}$ norm, which is just the L^∞ norm. Therefore $\eta_\varrho * \tilde{\chi} \to \tilde{\chi}$ in $L^{p,q}(\mathbb{R}^n)$, and the lemma is proved. □

Lemma 5.1.14. *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable. Then for a constant $C = C(n)$*

$$\int_E |x|^{\lambda - n} \leq \frac{C}{\lambda} |E|^{\frac{\lambda}{n}} \quad \text{for } 0 < \lambda < n. \tag{5.24}$$

Proof. First of all we compute for any ball $B = B_R(0)$ that

$$\int_B |x|^{\lambda-n}\,dx = \alpha_n \int_0^R r^{\lambda-1}\,dr = \frac{\alpha_n}{\lambda} R^\lambda \leq \frac{C(n)}{\lambda} |B|^{\frac{\lambda}{n}} \tag{5.25}$$

Choose $R > 0$ such that $|B| = |E|$. Then we have

$$|B \backslash E| = |B| - |B \cap E| = |E| - |E \cap B| = |E \backslash B|.$$

This implies

$$\int_{E\backslash B} |x|^{\lambda-n}\,dx \leq R^{\lambda-n}|E\backslash B| = R^{\lambda-n}|B\backslash E| \leq \int_{B\backslash E} |x|^{\lambda-n}\,dx.$$

Adding $\int_{E\cap B} |x|^{\lambda-n}\,dx$ to both sides we get, recalling $|B| = |E|$,

$$\int_E |x|^{\lambda-n}\,dx \leq \int_B |x|^{\lambda-n}\,dx \leq \frac{C(n)}{\lambda}|E|^{\frac{\lambda}{n}}.$$

□

We are now in the position to prove the first main application of Lorentz spaces. Namely we show that functions whose gradient is in $L^{n,1}$ are continuous.

Theorem 5.1.15. *Let $Du \in L^{n,1}(\mathbb{R}^n)$. Then u is continuous, and for any ball $B \subset \mathbb{R}^n$*

$$\Big| u(x) - \fint_B u \Big| \leq C\,\|\chi_B Du\|_{L^{n,1}(\mathbb{R}^n)}, \quad \text{for any point } x \in B. \tag{5.26}$$

Proof. We have $Du \in L^{n,n}(\mathbb{R}^n) = L^n(\mathbb{R}^n)$ by Lemma 5.1.7 and Lemma 5.1.5. By scaling we can assume $B = B_1(0)$. We have for $y = x + \varrho\omega \in B$

$$|u(x) - u(y)| = \Big| \int_0^\varrho (\partial_r u)(x + r\omega)\,dr \Big| \leq \int_0^\infty (\chi_B |Du|)(x + r\omega)\,dr.$$

Integrating we obtain

$$\begin{aligned}|u(x) - \fint_B u(y)\,dy| &\leq C\int_0^2\int_{\mathbb{S}^{n-1}} \chi_B(x+\varrho\omega)|u(x)-u(x+\varrho\omega)|\,\varrho^{n-1}d\omega\,d\varrho \\ &\leq C\int_0^2\int_{\mathbb{S}^{n-1}} \chi_B(x+\varrho\omega)\int_0^\infty (\chi_B|Du|)(x+r\omega)\,dr\,\varrho^{n-1}d\omega\ d\varrho \\ &\leq C\int_0^\infty\int_{\mathbb{S}^{n-1}} r^{1-n}(\chi_B|Du|)(x+r\omega)\,r^{n-1}d\omega\,dr \\ &= C\int_B |x-y|^{1-n}|Du(y)|\,dy.\end{aligned}$$

The right hand side is estimated using Lemma 5.24:

$$\begin{aligned}\int_B |z-x|^{1-n}|Du(x)|\,dx &= \int_B |z-x|^{1-n}\int_0^\infty \chi_{\{s<|Du(x)|\}}\,ds\,dx \\ &= \int_0^\infty\int_{B\cap\{|Du|>s\}} |z-x|^{1-n}\,dx\,ds \\ &\leq C\int_0^\infty (\chi_B Du)_*(s)^{\frac{1}{n}}\,ds \\ &\leq C\,\|\chi_B Du\|_{L^{n,1}(\mathbb{R}^n)}.\end{aligned}$$

In the last step we used Lemma 5.1.9 with $p=n$ and $q=1$. Now approximate u in $L^{p,q}(\mathbb{R}^n)$ by smooth functions u_ϱ using Lemma 5.1.13. Then $u_\varrho \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|u_\varrho - u_{\varrho'}\|_{C^0(B)} \leq \Big|\fint_B (u_\varrho - u_{\varrho'})\Big| + C\,\|\chi_B(Du_\varrho - Du_{\varrho'})\|_{L^{n,1}(\mathbb{R}^n)} \to 0 \quad \text{for } \varrho,\ \varrho' \to 0.$$

This shows $u \in C^0(B)$. □

We now show that the standard Sobolev embedding in $\mathbb{R}^n$ can be improved.

Theorem 5.1.16 (Poornima [44]). *Let $f \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$, $n \geq 2$, have $Df \in L^1(\mathbb{R}^n)$. Then f belongs to $L^{\frac{n}{n-1},1}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} \leq C\,\|Df\|_{L^1(\mathbb{R}^n)}. \tag{5.27}$$

Proof. By approximation, it is enough to prove the estimate for $f \in C^1_c(\mathbb{R}^n)$. Fix a nondecreasing cutoff function $\varphi \in C^1(\mathbb{R})$ with $\varphi(t)=0$, $\varphi(t)=1$ for $t \geq 1$, and consider

$$\varphi^\lambda_\varepsilon \in C^1(\mathbb{R}),\ \varphi^\lambda_\varepsilon(s) = \varphi\Big(\frac{s-\lambda}{\varepsilon}\Big) \quad \text{for } \lambda \geq 0,\ \varepsilon > 0.$$

We have $D(\varphi^\lambda_\varepsilon \circ f) = (\varphi^\lambda_\varepsilon)' \circ f\,Df$, in particular

$$|D(\varphi^\lambda_\varepsilon \circ f)| \leq \frac{C}{\varepsilon}\chi_{\{\lambda<f<\lambda+\varepsilon\}}|Df|.$$

This yields the estimate

$$\begin{aligned}\int_0^\infty \|D(\varphi^\lambda_\varepsilon \circ f)\|_{L^1(\mathbb{R}^n)}\,d\lambda &\leq \frac{C}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^n} \chi_{\{\lambda<f(x)<\lambda+\varepsilon\}}|Df(x)|\,dxd\lambda \\ &= \frac{C}{\varepsilon}\int_{\mathbb{R}^n}|Df(x)|\int_{f(x)-\varepsilon}^{f(x)} d\lambda \\ &= C\,\|Df\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{5.28}$$

We will now prove the inequality

$$\int_0^\infty f_*(s)^{\frac{n-1}{n}}\,ds \le C\,\|Df\|_{L^1(\mathbb{R}^n)}. \tag{5.29}$$

For $s>0$ let $f_*^\pm(s) = |\{f^\pm > s\}|$. Then $f_*(s) = f_*^+(s) + f_*^-(s)$, thus

$$f_*(s)^{\frac{n-1}{n}} \le \big(2\max(f_*^+(s), f_*^-(s))\big)^{\frac{n-1}{n}} \le 2^{\frac{n-1}{n}}\big(f_*(s)^{\frac{n-1}{n}} + f_*^-(s)^{\frac{n-1}{n}}\big).$$

Thus it suffices to prove (5.29) with f_*^+ on the left. Now by the usual Sobolev inequality

$$\|\varphi_\varepsilon^\lambda \circ f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C\,\|D(\varphi_\varepsilon^\lambda \circ f)\|_{L^1(\mathbb{R}^n)}.$$

Integrating over $[0,\infty)$ we obtain using (5.28)

$$\int_0^\infty \|\varphi_\varepsilon^\lambda \circ f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C\|Df\|_{L^1(\mathbb{R}^n)}.$$

Finally we take the limit as $\varepsilon \searrow 0$. It is convenient to introduce

$$\psi_\varepsilon : [0,\infty) \to \mathbb{R},\ \psi_\varepsilon(\lambda) = \int_{\mathbb{R}^n} |\varphi_\varepsilon^\lambda \circ f|^{\frac{n}{n-1}}\,dx.$$

Clearly $\varphi_\varepsilon^\lambda \circ f \to \chi_{\{f>\lambda\}}$ pointwise in $\mathbb{R}^n$, and $|\varphi_\varepsilon \circ f| \le \chi_{\operatorname{spt} f}$. Therefore by dominated convergence

$$\lim_{\varepsilon\searrow 0} \psi_\varepsilon(\lambda) = |\{f>\lambda\}| = f_*^+(\lambda).$$

Applying Fatou's lemma we finally conclude

$$\begin{aligned}\int_0^\infty f_*^+(\lambda)^{\frac{n-1}{n}}\,d\lambda &\le \liminf_{\varepsilon\searrow 0} \psi_\varepsilon(\lambda)^{\frac{n-1}{n}}\,d\lambda \\ &= \liminf_{\varepsilon\searrow 0} \|\varphi_\varepsilon^\lambda \circ f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}\,d\lambda \\ &\le C\,\|Df\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

The desired estimate now follows from Lemma 5.18, choosing $p = \frac{n}{n-1}$ and $q=1$ there. □

More generally we deduce the continuous embedding, for $f \in L^{\frac{np}{n-p}}(\mathbb{R}^n)$ with $Df \in L^p(\mathbb{R}^n)$,

$$\|f\|_{L^{q,r}} \le C(n,p)\,\|Df\|_{L^p(\mathbb{R}^n)} \quad \text{where } q = \frac{np}{n-p},\ r = \frac{(n-1)p}{n-p}. \tag{5.30}$$

To see this we first note $(f^r)_*(\sigma) = f_*(\sigma^{\frac{1}{r}})$ for any $\sigma>0$. Substituting $\sigma = s^r$ yields

$$(f^r)^*(t) = |\{(f^r)_* > t\}| = |\{s^r : f_*(s) > t\}| = r\int_{\{f_*>t\}} s^{r-1}\,ds.$$

The $L^{p,q}(\mathbb{R}^n)$ norm of f^r is given by

$$\begin{aligned}\int_0^\infty \big(t^{\frac{1}{p}}(f^r)^*(t)\big)^q\,\frac{dt}{t} &= r^q\int_0^\infty t^{\frac{q}{p}-1}\Big(\int_{\{f_*>t\}} s^{r-1}\,ds\Big)^q\,dt \\ &\le r^q\int_0^\infty t^{\frac{q}{p}-1}\,|\{f_*>t\}|^{q-1}\int_0^\infty \chi_{\{f_*>t\}}s^{(r-1)q}\,dt \\ &\le r^q\int_0^\infty s^{r-1}\int_0^{f_*(s)} t^{\frac{1}{p}-1}\,dtds \\ &= rp\int_0^\infty s^{r-1}f_*(s)^{\frac{1}{p}}\,ds.\end{aligned}$$

By Lemma 5.18 this implies

$$\|f\|_{L^{rq,r}(\mathbb{R}^n)} \leq C \|f^r\|_{L^{q,1}(\mathbb{R}^n)}^{\frac{1}{r}} \quad \text{(and vice versa)}. \tag{5.31}$$

Now for f as above and $1 < p < n$ we take $r = \frac{n-1}{n-p} p > p$ and estimate using Hölder

$$\begin{aligned} \|D(f^r)\|_{L^1(\mathbb{R}^n)} &\leq C \|f^{r-1} Df\|_{L^1(\mathbb{R}^n)} \\ &\leq C \|f^{r-1}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)} \|Df\|_{L^p(\mathbb{R}^n)} \\ &= C \|f\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)}^{\frac{p-1}{n-p} n} \|Df\|_{L^p(\mathbb{R}^n)} \\ &\leq C \|Df\|_{L^p(\mathbb{R}^n)}^r. \end{aligned}$$

Combining with (5.31) for $q = \frac{n}{n-1}$ we see that

$$\|f\|_{L^{\frac{np}{n-p},r}(\mathbb{R}^n)} \leq C \|f^r\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)}^{\frac{1}{r}} \leq \|D(f^r)\|_{L^1(\mathbb{R}^n)}^{\frac{1}{r}} \leq C \|Df\|_{L^p(\mathbb{R}^n)}. \tag{5.32}$$

5.2 Interpolation and PDE estimates

The next Theorem is the so called Marcinkiewicz interpolation theorem for Lorentz spaces and it is taken from [58]. Here we use the notion of a subadditive operator between vector spaces V, W of measurable functions on $\mathbb{R}^n$. A map $T : V \to W$ is called subadditive if for all $f, g \in V$ and $\lambda \in \mathbb{R}$ one has, for almost all $x \in \mathbb{R}^n$,

$$|T(f+g)(x)| \leq |(Tf)(x)| + |Tg(x)| \quad \text{and} \quad |T(\lambda f)(x)| = |\lambda| \, |T(f)(x)|.$$

An example of a non-linear subadditive operator is Mf, the maximal operator.

Theorem 5.2.1. *Let $1 \leq r_0 < r_1 \leq \infty$, $1 \leq p_0 \neq p_1 \leq \infty$, and let T be a subadditive operator satisfying the following estimates:*

$$\begin{array}{ccll} L^{r_0,1}(\mathbb{R}^n) & \xrightarrow{T} & L^{p_0,\infty}(\mathbb{R}^n), & \|Tf\|_{L^{p_0,\infty}}(\mathbb{R}^n) \leq C \|f\|_{L^{r_0,1}}(\mathbb{R}^n), \\ \cup & & \cup\cap & \\ L^{r_1}(\mathbb{R}^n) & \xrightarrow{T} & L^{p_1,\infty}(\mathbb{R}^n), & \|Tf\|_{L^{p_1,\infty}(\mathbb{R}^n)} \leq C \|f\|_{L^{r_1,1}}(\mathbb{R}^n), \end{array}$$

Then for $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ $(0 < \theta < 1)$, and for $1 \leq q \leq \infty$, we have

$$\|Tf\|_{L^{p,q}(\mathbb{R}^n)} \leq C \|f\|_{L^{r,q}(\mathbb{R}^n)}, \tag{5.33}$$

where $C = C(p_i, r_i, \theta)$.

Proof. For $f \in L^{r,q}(\mathbb{R}^n)$ we consider cutoffs at level $f^*(b)$ given by

$$f^b(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(b), \\ 0 & \text{else,} \end{cases} \qquad f_b(x) = \begin{cases} 0 & \text{if } |f(x)| > f^*(b), \\ f(x) & \text{else.} \end{cases}$$

Here $b > 0$ is a variable, it will be specified later. We compute

$$(f^b)_*(s) = \begin{cases} f_*(s) & \text{if } s \geq f^*(b), \\ f_*(f^*(b)) & \text{else.} \end{cases} \qquad (f_b)_*(s) = \begin{cases} 0 & \text{if } s \geq f^*(b), \\ f_*(s) - f_*(f^*(b)) & \text{else.} \end{cases}$$

In the second step we get

$$(f^b)^*(t) = \begin{cases} 0 & \text{if } t \ge f_*(f^*(b)), \\ f^*(t) & \text{else.} \end{cases} \quad (f_b)^*(t) = \begin{cases} 0 & \text{if } t \ge f_*(0) - f_*(f^*(b)), \\ f^*(t + f_*(f^*(b))) & \text{else.} \end{cases}$$

From the definition of f^* we get the inequality

$$f_*(f^*(b)) = |\{f > f^*(b)\}| = |\{f^* > f^*(b)\}| \le b. \tag{5.34}$$

This implies

$$(f^b)^*(t) \le \begin{cases} 0 & \text{if } t \ge b, \\ f^*(t) & \text{else.} \end{cases} \tag{5.35}$$

By monotonicity of f^*, we have $(f_b)^*(t) \le f^*(t)$ for all $t > 0$. On the other hand, we may use

$$f^*(f_*(s)) = |\{f_* > f_*(s)\}| \le s, \quad \text{thus} \quad (f_b)^*(0) \le f^*(f_*(f^*(b))) \le f^*(b).$$

In particular we see that

$$(f_b)^*(t) \le \begin{cases} f^*(t) & \text{if } t \ge b, \\ f^*(b) & \text{else.} \end{cases} \tag{5.36}$$

Now we estimate Tf. By subadditivity we have for almost every $x \in \mathbb{R}^n$

$$|Tf(x)| = |T(f^b + f_b)(x)| \le |Tf^b(x)| + |Tf_b(x)|.$$

It follows that for any $s > 0$ we have

$$\{|Tf| > s_1 + s_2\} \subset \{|Tf^b| > s_1\} \cup \{|Tf_b| > s_2\}.$$

This means

$$(Tf)_*(s_1 + s_2) \le (Tf^b)_*(s_1) + (Tf_b)_*(s_2).$$

Choosing $s_1 = (Tf^b)^*(t)$ and $s_2 = (Tf_b)^*(t)$ and using (5.34) we infer

$$(Tf)^*(s_1 + s_2) \le t + t = 2t.$$

Thus we have shown

$$(Tf)^*(2t) \le s_1 + s_2 = (Tf^b)^*(t) + (Tf_b)^*(t) \quad \text{for all } t > 0. \tag{5.37}$$

As final preparation, we note that by assumption, for all $t > 0$ and all $b > 0$,

$$t^{\frac{1}{p_0}} (Tf^b)^*(t) \le C\|f^b\|_{L^{r_0,1}(\mathbb{R}^n)}, \tag{5.38}$$

$$t^{\frac{1}{p_1}} (Tf_b)^*(t) \le C\|f_b\|_{L^{r_1,1}(\mathbb{R}^n)}. \tag{5.39}$$

The key idea of the proof is to cutoff depending on t, namely we choose $b = t^\gamma$ where $\gamma > 0$ will be fixed appropriately.

We first consider the case $r_1 < \infty$, $q < \infty$. Then we have by (5.37)

$$\begin{aligned}
\|Tf\|^q_{L^{p,q}_*(\mathbb{R}^n)} &= \int_0^\infty \left(t^{\frac{1}{p}}(Tf)^*(t)\right)^q \frac{dt}{t} \\
&\le C\int_0^\infty \left(t^{\frac{1}{p}}(Tf)^*(2t)\right)^q \frac{dt}{t} \\
&\le \underbrace{\int_0^\infty \left(t^{\frac{1}{p}}(Tf^{t^\gamma})^*(t)\right)^q \frac{dt}{t}}_{=:I_1} + \underbrace{\int_0^\infty \left(t^{\frac{1}{p}}(Tf_{t^\gamma})^*(t)\right)^q \frac{dt}{t}}_{=:I_2}.
\end{aligned}$$

In the first integral, we estimate applying (5.38), for $b = t^\gamma$,

$$\begin{aligned}
I_1 &\le C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_0}}\|f^{t^\gamma}\|_{L^{r_0,1}(\mathbb{R}^n)}\right)^q \frac{dt}{t} \\
&\le C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_0}}\int_0^{t^\gamma} \lambda^{\frac{1}{r_0}} f^*(\lambda)\,\frac{d\lambda}{\lambda}\right)^q \frac{dt}{t} \\
&\le C\int_0^\infty \left(u^{\left(\frac{1}{p}-\frac{1}{p_0}\right)\frac{1}{\gamma}}\int_0^{u} \lambda^{\frac{1}{r_0}} f^*(\lambda)\,\frac{d\lambda}{\lambda}\right)^q \frac{du}{u}.
\end{aligned}$$

In the second last step we used (5.35), then in the last step $t = u^\gamma$ was substituted. At this point we chose γ appropriately, namely we take

$$\gamma = \frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{r_0}-\frac{1}{r}} = \frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{r}-\frac{1}{r_1}}. \tag{5.40}$$

Then we continue appplying Hardy's inequality, see (5.11),

$$\begin{aligned}
I_1 &\le C\int_0^\infty \left(u^{\frac{1}{r}-\frac{1}{r_0}}\int_0^{u} \lambda^{\frac{1}{r_0}-1} f^*(\lambda)\,d\lambda\right)^q \frac{du}{u} \\
&\le C\int_0^\infty \left(\lambda^{\frac{1}{r_0}-1} f^*(\lambda)\right)^q \lambda^{\frac{q}{r}-\frac{q}{r_0}-1}\,d\lambda \\
&= C\int_0^\infty \left(\lambda^{\frac{1}{r}} f^*(\lambda)\right)^q \frac{d\lambda}{\lambda} = C\|f\|^q_{L^{r,q}_*(\mathbb{R}^n)}.
\end{aligned}$$

The estimate of the second integral is rather similar. We have

$$\begin{aligned}
I_2 &\le C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_1}}\|f_{t^\gamma}\|_{L^{r_1,1}(\mathbb{R}^n)}\right)^q \frac{dt}{t} \\
&\le C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_1}}\int_0^{t^\gamma} \lambda^{\frac{1}{r_1}} f^*(t^\gamma)\,\frac{d\lambda}{\lambda}\right)^q \frac{dt}{t} + C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_1}}\int_{t^\gamma}^\infty \lambda^{\frac{1}{r_1}} f^*(\lambda)\,\frac{d\lambda}{\lambda}\right)^q \frac{dt}{t} \\
&\le C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_1}} f^*(t^\gamma) t^{\frac{\gamma}{r_1}}\right)^q \frac{dt}{t} + C\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{p_1}}\int_{t^\gamma}^\infty \lambda^{\frac{1}{r_1}} f^*(\lambda)\,\frac{d\lambda}{\lambda}\right)^q \frac{dt}{t}.
\end{aligned}$$

Recalling the choice of γ from (5.40) we continue, substituting $u = t^\gamma$,

$$\begin{aligned}
I_2 &\le C\int_0^\infty \left(u^{\frac{1}{r}} f^*(u)\right)^q \frac{du}{u} + C\int_0^\infty \left(t^{\frac{1}{r}-\frac{1}{r_1}}\int_u^\infty \left(\lambda^{\frac{1}{r_1}} f^*(\lambda)\right)^q \frac{d\lambda}{\lambda}\right) \frac{du}{u} \\
&\le C\|f\|^q_{L^{r,q}(\mathbb{R}^n)} + C\int_0^\infty \left(\lambda^{\frac{1}{r_1}} f^*(\lambda)\right)^q \lambda^{\frac{q}{r}-\frac{q}{r_1}-1}\,d\lambda \\
&\le C\|f\|^q_{L^{r,q}(\mathbb{R}^n)}.
\end{aligned}$$

Here we used the case (5.12) of Hardy's inequality. Having completed the proof for $r_1, q < \infty$, we address next $r_1 < \infty$, $q = \infty$. By the above calculations, we estimate

$$\begin{aligned} t^{\frac{1}{p}}(Tf)^*(t) &\leq Ct^{\frac{1}{p}-\frac{1}{p_0}} \int_0^{t^\gamma} \lambda^{\frac{1}{r_0}} f^*(\lambda)\, \frac{d\lambda}{\lambda} \\ &\quad + Ct^{\frac{1}{p}-\frac{1}{p_1}} \int_0^{t^\gamma} \lambda^{\frac{1}{r_1}} f^*(t^\gamma)\, \frac{d\lambda}{\lambda} + Ct^{\frac{1}{p}-\frac{1}{p_1}} \int_{t^\gamma}^{\infty} \lambda^{\frac{1}{r_1}} f^*(\lambda)\, \frac{d\lambda}{\lambda}. \end{aligned}$$

Using $\lambda^{\frac{1}{r}} f^*(\lambda) \leq \|f\|_{L_*^{r,\infty}(\mathbb{R}^n)}$ we obtain further

$$\begin{aligned} t^{\frac{1}{p}}(Tf)^*(t) &\leq C\,\|f\|_{L^{r,\infty}(\mathbb{R}^n)} t^{\frac{1}{p}-\frac{1}{p_0}} \int_0^{t^\gamma} \lambda^{\frac{1}{r_0}-\frac{1}{r}}\, \frac{d\lambda}{\lambda} + C\, t^{\frac{1}{p}-\frac{1}{p_1}} f^*(t^\gamma) t^{\frac{\gamma}{r_1}} \\ &\quad + C\,\|f\|_{L^{r,\infty}(\mathbb{R}^n)} t^{\frac{1}{p}-\frac{1}{p_1}} \int_{t^\gamma}^{\infty} \lambda^{\frac{1}{r_1}-\frac{1}{r}}\, \frac{d\lambda}{\lambda} \\ &\leq C\,\|f\|_{L^{r,\infty}}. \end{aligned}$$

The proof of the remaining case $r_1, q = \infty$ uses that $(f_b)^*(t) \leq f^*(b)$; it is left to the reader. □

In the following we prove some PDE-estimates involving Lorentz-spaces. Before doing this we need the following Lemma (see [29]).

Lemma 5.2.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded with C^1-boundary, let $g = (g_1, g_2) \in L^1(\Omega, \mathbb{R}^2)$ and let α be a solution of*

$$\begin{aligned} \Delta\alpha &= \operatorname{div} g \quad \text{in } \ \Omega, \\ \alpha &= 0 \quad \text{on } \ \partial\Omega. \end{aligned} \tag{5.41}$$

Then we have that the operator

$$P(g) = \nabla\alpha \tag{5.42}$$

is continuous between $L^{p,q}(\Omega, \mathbb{R}^2)$, $1 < p < \infty$, $1 \leq q \leq \infty$, and itself.

Proof. From standard L^p-theory we know that P is continuous between $L^p(\Omega, \mathbb{R}^2)$, $1 < p < \infty$, and itself. Therefore we can apply Theorem 5.2.1 to get the desired result. □

Lemma 5.2.3. *Let $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$ with $\frac{1}{p_1} + \frac{1}{p_2} > 1$. Then $h = f \star g \in L^{r,s}(\mathbb{R}^n)$ where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and s is a number such that $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. Moreover we have*

$$||h||_{L^{r,s}(\mathbb{R}^n)} \leq c||f||_{L^{p_1,q_1}(\mathbb{R}^n)}||g||_{L^{p_2,q_2}(\mathbb{R}^n)}. \tag{5.43}$$

Proof. See [67]. □

The next three theorems are taken from [29] (see also [1] and [18]).

Theorem 5.2.4. *Let $\Omega \subset \mathbb{R}^2$ be open with $\partial\Omega \in C^1$. Let $f \in L^1(\Omega)$ and let φ be a solution of*

$$\begin{aligned} -\Delta\varphi &= f \quad \text{in } \ \Omega, \\ \varphi &= 0 \quad \text{on } \ \partial\Omega, \end{aligned} \tag{5.44}$$

then $\nabla\varphi \in L^{2,\infty}(\Omega, \mathbb{R}^2)$ and

$$||\nabla\varphi||_{L^{2,\infty}(\Omega)} \leq c(\Omega)||f||_{L^1(\Omega)}. \tag{5.45}$$

Proof. We consider $\overline{f} \in L^1(\mathbb{R}^2)$ which we obtain by extending f by 0 outside of Ω and we define

$$\psi(x) = \int_{\mathbb{R}^2} K(x-y)\overline{f}(y)dy,$$

where $K(x) = \frac{1}{2\pi}\ln(\frac{1}{|x|})$. Then we know that

$$-\Delta\psi = \overline{f}$$

in $\mathbb{R}^2$ and

$$\nabla\psi = \int_{\mathbb{R}^2} \nabla K(x-y)\overline{f}(y)dy.$$

Since $\overline{f} \in L^1(\mathbb{R}^2) = L^{1,\infty}(\mathbb{R}^2)$ and $|\nabla K|(x-y) \leq \frac{c}{|x-y|} \in L^{2,\infty}$ (see Lemma 5.1.8) we can apply Lemma 5.2.3 to get that $\nabla\psi \in L^{2,\infty}$ and

$$||\nabla\psi||_{L^{2,\infty}} \leq c||\overline{f}||_{L^1} = c||f||_{L^1(\Omega)}.$$

Since $\nabla\varphi = P(\nabla\psi|_\Omega)$ we can apply Lemma 5.2.2 to conclude the proof of the Theorem. □

In the following Theorem we improve Wente‘s inequality.

Theorem 5.2.5. *Let $\Omega \subset \mathbb{R}^2$ be open with C^1-boundary, let $f \in \mathcal{H}^1(\mathbb{R}^2)$ and let φ be a solution of*

$$\begin{aligned} -\Delta\varphi &= f \quad in \quad \Omega, \\ \varphi &= 0 \quad on \quad \partial\Omega, \end{aligned} \tag{5.46}$$

then $\nabla\varphi \in L^{2,1}(\Omega, \mathbb{R}^2)$ and

$$||\nabla\varphi||_{L^{2,1}(\Omega)} \leq c(\Omega)||f||_{\mathcal{H}^1(\mathbb{R}^2)}. \tag{5.47}$$

Proof. We let

$$\psi(x) = \int_{\mathbb{R}^2} K(x-y)f(y)dy,$$

where $K(x) = \frac{1}{2\pi}\ln(\frac{1}{|x|})$. Then we know that

$$-\Delta\psi = f$$

on $\mathbb{R}^2$. By Theorem 4.3.9 we know that $\psi \in W^{2,1}(\mathbb{R}^2)$ with

$$||\nabla^2\psi||_{L^1} \leq c||f||_{\mathcal{H}^1}.$$

Hence we get from Theorem 5.1.16 that $\nabla\psi \in L^{2,1}(\mathbb{R}^2)$ with

$$||\nabla\psi||_{L^{2,1}} \leq c||f||_{\mathcal{H}^1}.$$

Using again that $\nabla\varphi = P(\nabla\psi|_\Omega)$ we can apply Lemma 5.2.2 to conclude the proof of the Theorem. □

The next theorem is also an improvement of Wente‘s inequality. This theorem has recently been used T. Rivière [48] in his study of Willmore surfaces.

Theorem 5.2.6. *Let $\Omega \subset \mathbb{R}^2$ be open with C^2-boundary, let a, b be functions such that $\nabla a \in L^{2,\infty}(\Omega)$, $b \in W^{1,2}(\Omega)$ and let φ be a solution of*

$$\begin{aligned} -\Delta\varphi &= \{a, b\} \quad \text{in} \ \ \Omega, \\ \varphi &= 0 \quad \text{on} \ \ \partial\Omega, \end{aligned} \tag{5.48}$$

then $\varphi \in W^{1,2}(\Omega)$ and

$$||\nabla\varphi||_{L^2(\Omega)} \leq c||\nabla a||_{L^{2,\infty}(\Omega)}||\nabla b||_{L^2(\Omega)}. \tag{5.49}$$

Proof. We first prove (5.49) for $a, b \in W^{1,2}(\Omega)$. Let $U \supset \Omega$ be smooth and bounded. Moreover we let $\tilde{a}, \tilde{b} \in W_0^{1,2}(U)$ be the extensions of a, b and we let ψ be the solution of

$$\begin{aligned} -\Delta\psi &= \{\tilde{a}, \tilde{b}\} \quad \text{in} \ \ U, \\ \psi &= 0 \quad \text{on} \ \ \partial U. \end{aligned}$$

Then we he have

$$\begin{aligned} ||\nabla\psi||_{L^2}^2 &= -\int_U \psi\Delta\psi \\ &= \int_U \psi\{\tilde{a}, \tilde{b}\} \\ &= \int_U \tilde{a}\{\psi, \tilde{b}\} \\ &= -\int_U \tilde{a}\Delta\Psi \\ &= \int_U \nabla\tilde{a}\nabla\Psi \\ &\leq c||\nabla\tilde{a}||_{L^{2,\infty}}||\nabla\Psi||_{L^{2,1}} \\ &\leq c||\nabla\tilde{a}||_{L^{2,\infty}}||\nabla\tilde{b}||_{L^2}||\nabla\psi||_{L^2}, \end{aligned}$$

where we used Lemma 5.1.6, Theorem 5.2.5 and where Ψ is a solution of

$$\begin{aligned} -\Delta\Psi &= \{\psi, \tilde{b}\} \quad \text{in} \ \ U, \\ \Psi &= 0 \quad \text{on} \ \ \partial U. \end{aligned}$$

Since the extension operator is continuous from $W^{1,p}$ to $W^{1,p}$ for every $1 < p < \infty$ we can apply Theorem 5.2.1 to get

$$||\nabla\psi||_{L^2(U)} \leq c||\nabla a||_{L^{2,\infty}(\Omega)}||\nabla b||_{L^2(\Omega)}.$$

Moreover, since

$$\begin{aligned} -\Delta\varphi &= -\Delta\psi \quad \text{in} \ \ \Omega, \\ \varphi &= 0 \quad \text{on} \ \ \partial\Omega, \end{aligned}$$

we get that

$$\begin{aligned}||\nabla\varphi||_{L^2(\Omega)} &\leq ||\nabla\psi||_{L^2(U)} \\ &\leq c||\nabla a||_{L^{2,\infty}(\Omega)}||\nabla b||_{L^2(\Omega)}.\end{aligned}$$

This proves (5.49) for $a, b \in W^{1,2}(\Omega)$.
If we now only have a such that $\nabla a \in L^{2,\infty}(\Omega)$ we choose a sequence $a_k \in \cap_{1\leq p<2} W^{1,p}(\Omega)$ such that $a_k \to a$ in $W^{1,p}(\Omega)$ for every $p < 2$ and $||\nabla a_k||_{L^{2,\infty}} \leq c||a||_{L^{2,\infty}}$ (note that you can't find a sequence $a_k \in W^{1,2}$ with the above properties). Indeed you can just consider the convolution of a with a sequence of mollifiers which are in L^1 and then you can apply Lemma 5.2.3 to get the desired properties. Then we have by (5.49) that the solution φ_k of

$$\begin{aligned}-\Delta\varphi_k &= \{a_k, b\} \quad \text{in} \quad \Omega, \\ \varphi_k &= 0 \quad \text{on} \quad \partial\Omega,\end{aligned}$$

is bounded in $W^{1,2}(\Omega)$ and therefore we have that $\varphi_k \to \eta$ weakly in $W^{1,2}(\Omega)$ with

$$||\nabla\eta||_{L^2} \leq c||\nabla a||_{L^{2,\infty}}||\nabla b||_{L^2}.$$

Additionally, since $\{a_k, b\} = \partial_x(a_k\partial_y b) - \partial_y(a_k\partial_x b)$, we have that

$$\{a_k, b\} \to \{a, b\}$$

in $W^{-1,p}$ for every $1 \leq p < 2$ and therefore

$$\varphi_k \to \varphi$$

in $W^{1,p}$ for every $1 \leq p < 2$. This shows that $\varphi = \eta$ and finishes the proof of the Theorem. □

Chapter 6

Regularity of geometric variational problems

This chapter addresses the regularity of critical points of two-dimensional, conformally invariant variational integrals. The case of harmonic maps was settled by Hélein [26], whereas the general result including surfaces of prescribed, variable mean curvature is due to T. Rivière [46]. It is his proof that is presented here.

6.1 Gauge transformations

Lemma 6.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^2. Then for any $\omega \in W^{1,2}(\Omega, \Lambda^1)$ we have the identity*

$$\int_\Omega (|d\omega|^2 + |d^*\omega|^2 - |D\omega|^2) = \int_{\partial\Omega} \Big(h(\omega^\top, \omega^\top) + H|\omega(\nu)|^2 + d(\omega(\nu))(\omega^\top) + \omega(\nu) d^*_\top \omega^\top \Big). \quad (6.1)$$

Here h, H denote the second fundamental form and mean curvature of $\partial\Omega$ with respect to the inner normal ν, and $\omega^\top$ is the projection of ω to $T(\partial\Omega)$ using $\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$.

Proof. For a one-form ω, the exterior derivative and the divergence are given by

$$d\omega = \sum_{1 \le i < j \le n} (\partial_i \omega_j - \partial_j \omega_i) dx^i \wedge dx^j \quad \text{and} \quad d^*\omega = \sum_{i=1}^n \partial_i \omega_i.$$

$$|d\omega|^2 = \frac{1}{2} \sum_{i,j=1}^n (\partial_i \omega_j - \partial_j \omega_i)^2.$$

We note that d^* is formally L^2 adjoint to the derivative d on functions. We compute

$$\begin{aligned}
|D\omega|^2 &= \sum_{i,j=1}^n (\partial_i \omega_j)^2, \\
|d\omega|^2 &= \frac{1}{2}\sum_{i,j=1}^n (\partial_i\omega_j - \partial_j\omega_i)^2 = \sum_{i,j=1}^n (\partial_i\omega_j)^2 - \sum_{i,j=1}^n \partial_j(\omega_i\partial_i\omega_j) + \sum_{i,j=1}^n \omega_i\, \partial^2_{ji}\omega_j, \\
|d^*\omega|^2 &= \sum_{i,j=1}^n (\partial_i\omega_i)(\partial_j\omega_j) = \sum_{i,j=1}^n \partial_i(\omega_i\partial_j\omega_j) - \sum_{i,j=1}^n \omega_i\, \partial^2_{ij}\omega_j
\end{aligned}$$

Combining and integrating by parts we obtain, recalling that ν denotes the inner unit normal,

$$\int_\Omega (|d\omega|^2 + |d^*\omega|^2 - |D\omega|^2) = \int_{\partial\Omega} \big((D_\omega\omega)(\nu) + \omega(\nu)d^*\omega\big)\, d\mu.$$

For given $p \in \partial\Omega$, we now choose a local tangent frame $\tau_1, \ldots, \tau_{n-1}$, such that at p

$$D_{\tau_\alpha}\tau_\beta = h(\tau_\alpha, \tau_\beta)\nu, \quad \text{and hence} \quad D_{\tau_\alpha}\nu = -\sum_{\beta=1}^{n-1} h(\tau_\alpha,\tau_\beta)\tau_\beta.$$

Furthermore, we extend ν such that $D_\nu\nu = 0$ at p. Then we have at that point p

$$\begin{aligned}
(D_\omega\omega)(\nu) &= \partial_\omega(\omega(\nu)) - \sum_{\alpha=1}^{n-1} \omega(\tau_\alpha)\omega(D_{\tau_\alpha}\nu) \\
&= \partial_\omega(\omega(\nu)) + \sum_{\alpha,\beta=1}^{n-1} h(\tau_\alpha,\tau_\beta)\omega(\tau_\alpha)\omega(\tau_\alpha) \\
&= \partial_\omega(\omega(\nu)) + h(\omega^\top, \omega^\top),
\end{aligned}$$

Secondly, we get

$$\begin{aligned}
d^*\omega &= -(D_{\tau_\alpha}\omega)(\tau_\alpha) - (D_\nu\omega)(\nu) \\
&= -\partial_{\tau_\alpha}(\omega^\top(\tau_\alpha)) + \omega(D_{\tau_\alpha}\tau_\alpha) - \partial_\nu(\omega(\nu)) \\
&= d^*_\top\omega^\top + H\omega(\nu) - \partial_\nu(\omega(\nu)).
\end{aligned}$$

Here $\omega^\top$ is the pullback to $\partial\Omega$ by the inclusion map, $d^*_\top$ is the intrinsic divergence on $\partial\Omega$. We used that the at p. Combining we find

$$\begin{aligned}
(D_\omega\omega)(\nu) + \omega(\nu)d^*\omega &= h(\omega^\top,\omega^\top) + H\omega(\nu)^2 \\
&\quad + \sum_{\alpha=1}^{n-1} \omega(\tau_\alpha)\partial_{\tau_\alpha}(\omega(\nu)) + \omega(\nu)d^*_\top\omega^\top \\
&= h(\omega^\top,\omega^\top) + H\omega(\nu)^2 + d(\omega(\nu))(\omega^\top) + \omega(\nu)d^*_\top\omega^\top.
\end{aligned}$$

The claim of the lemma follows by integrating. □

In this section we prove an existence result for Coulomb gauges due to Uhlenbeck [62]. The issue is to construct a preferred gauge for a connection on a vector bundle over a Riemannian manifold. More precisely the theorem deals with a local situation $B \times \mathbb{R}^m$, where B is the unit ball in $\mathbb{R}^n$ and the $\mathbb{R}^m$ factor represents the coordinates with respect to a given frame. A connection is then given by a matrix-valued one-form $A = A_i(x)\,dx^i$ with $A_i(x) \in \mathbb{R}^{m\times m}$. It induces a notion of parallel vector fields along curves $\gamma : [a,b] \to B$ by the linear ordinary differential equation

$$\frac{\nabla_A v}{dt} = v' + A(\gamma')\,v = 0 \quad \text{ where } v : [a,b] \to \mathbb{R}^m. \tag{6.2}$$

We should really write $(A \circ \gamma)(\gamma')$, however it is customary to omit the basepoint. The connection is often denoted by its local form $\nabla_A = d + A$. For simplicity we restrict to SO_m bundles; this means that the bundle is oriented and carries a Riemannian metric. The $\mathbb{R}^m$ factor represents the coordinates with respect to some oriented orthonormal frame, thus the bundle metric becomes the standard scalar product $\langle\cdot,\cdot\rangle$. The connections are required to be compatible in the sense of the product rule, for any vector fields ϕ, ψ along γ,

$$\frac{d}{dt}\langle v, w\rangle = \Big\langle \frac{\nabla_A v}{dt}, w\Big\rangle + \Big\langle v, \frac{\nabla_A w}{dt}\Big\rangle \quad \Leftrightarrow \quad \big\langle A(\gamma')\,e_i, e_j\big\rangle + \big\langle e_i, A(\gamma')\,e_j\big\rangle = 0.$$

Thus A is an $\mathfrak{so}_m$-valued one-form. Any oriented orthonormal frame $\mathcal{F} = \{v_1, \ldots, v_m\}$ over B induces new coordinates $v_{\mathcal{F}}$, such that $v = P v_{\mathcal{F}}$ for some $P : B \to \mathrm{SO}_m$. It follows that

$$\begin{aligned} \Big(\frac{\nabla_A v}{dt}\Big)_{\mathcal{F}} &= P^{-1}\frac{\nabla_A}{dt}(P v_{\mathcal{F}}) \\ &= P^{-1}(P v_{\mathcal{F}})' + P^{-1}A(\gamma')P v_{\mathcal{F}} \\ &= v_{\mathcal{F}}' + \big(P^{-1}dP(\gamma') + P^{-1}A(\gamma')P\big)v_{\mathcal{F}}. \end{aligned}$$

The map $P : B \to \mathrm{SO}_m$ is called a gauge transformation, and the one-form $P^{-1}dP + P^{-1}AP$ is the transformed connection. The group of gauge transformations acts isometrically on the space of connections with respect to the L^2 distance

$$\mathrm{dist}(\nabla_A, \nabla_B)^2 = \int_B |\nabla_A - \nabla_B|^2\,dx = \int_B |A - B|^2\,dx.$$

In fact we have

$$\int_B |P^{-1}\nabla_A P - P^{-1}\nabla_B P|^2\,dx = \int_B |P^{-1}(\nabla_A - \nabla_B)P|^2\,dx = \int_B |\nabla_A - \nabla_B|^2\,dx.$$

It is therefore natural to ask whether any gauge orbit contains an element which minimizes the distance to the trivial connection $d = \nabla_0$. If ∇_A is the desired minimizer, then we get by chosing $P = \exp(t\chi)$ with $\chi : B \to \mathfrak{so}_m$

$$0 = \frac{1}{2}\frac{d}{dt}\int_B |e^{-t\chi}d(e^{t\chi}) + e^{-t\chi}Ae^{t\chi}|\,dx|_{t=0} = \int_B \langle A, d\chi\rangle.$$

This means that the minimizer is a weak solution to the equations

$$d * A = 0 \text{ in } B, \qquad \nu \llcorner A = 0 \text{ on } \partial B.$$

These are called the Coulomb or Hodge gauge conditions. The following result is due to Uhlenbeck [62].

Theorem 6.1.2. *Let $A_0 \in L^2(B, \Lambda^1 \otimes \mathfrak{so}_m)$ be a connection on $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Then there exists a gauge transformation $P \in W^{1,2}(B, \mathrm{SO}_m)$, such that $A = P^{-1}dP + P^{-1}A_0P \in L^2(B, \Lambda^1 \otimes \mathfrak{so}_m)$ has the following properties:*

(1) *A solves the system $d^*A = 0$ in B, $\nu \llcorner A = 0$ on ∂B.*

(2) $\|dP\|_{L^2(B)} + \|A\|_{L^2(B)} \leq C\|A_0\|_{L^2(B)}$.

(3) *There is a $\xi \in W^{1,2}(B, \Lambda^2 \otimes \mathfrak{so}_m)$ with $i^*_{\partial B}(d * \xi) = 0$ on ∂B, such that*

$$d^*\xi = A \quad \text{and} \quad \|\xi\|_{W^{1,2}(B)} \leq C\|A\|_{L^2(B)}.$$

Proof. As outlined we consider a minimizing sequence $P_k \in W^{1,2}(B, \mathrm{SO}_m)$ for the functional

$$E(P) = \int_B |P^{-1}dP + P^{-1}A_0P|^2\, dx = \int_B |dP + A_0P|^2\, dx. \tag{6.3}$$

We have the inequality

$$E(P) \geq (1-\varepsilon)\int_B |dP|^2\, dx - C_\varepsilon \int_B |A_0|^2. \tag{6.4}$$

Moreover $|P_k| = n$, thus we can assume $P_k \to P \in W^{1,2}(B, \mathrm{SO}_m)$ weakly in $W^{1,2}$, strongly in L^2 and pointwise almost everywhere. As $|A_0P_k| = |A_0|$ we get $A_0P_k \to A_0P$ in $L^2(B)$ by Vitali's convergence theorem. Thus the infimum is attained by P, and

$$E(P) \leq \liminf_{k\to\infty} E(P_k) \leq E(\mathrm{Id}) = \int_B |A_0|^2.$$

Put $A = P^{-1}dP + P^{-1}A_0P$. Then $\|A\|_{L^2(B)} \leq \|A_0\|_{L^2(B)}$ by the minimizing property, and from $dP = PA - A_0P$ we see that

$$\|dP\|_{L^2(B)} \leq \|A\|_{L^2(B)} + \|A_0\|_{L^2(B)} \leq 2\|A_0\|_{L^2(B)}.$$

Moeoever, as explained above, A satisfies the weak Hodge gauge conditions

$$\int_B \langle d\chi, A\rangle\, dx = 0 \quad \text{for all smooth } \chi : \overline{B} \to \mathfrak{so}_m.$$

To show claim (3) we employ linear Hodge theory. By Lemma 3.3.1 in Chapter 3 there exists a form $\xi \in W^{1,2}(B, \Lambda^2)$ such that

$$A = d^*\xi \quad \text{where} \int_{\partial B} *(\nu \llcorner \xi) = 0 \text{ and } \|\xi\|_{W^{1,2}(B)} \leq \|A\|_{L^2(B)}.$$

Our proof in Chapter 3 was only in two dimensions, but its generalization is straightforward. For any smooth $\chi : \overline{B} \to \mathfrak{so}_m$ we compute by partial integration

$$\begin{aligned}
\int_B \langle d\chi, A\rangle\, dx &= \int_B d\chi \wedge *d^*\xi \\
&= (-1)^n \int_B d\chi \wedge d*\xi \\
&= (-1)^n \int_{\partial B} \chi\, i^*_{\partial B}(d*\xi).
\end{aligned}$$

The weak version of the Hodge gauge condition (1) implies $i^*_{\partial B}(d*\xi) = 0$. □

In dimension $n = 2$ we get $\xi = 0$ on the boundary. Namely for $\xi = \xi_0\, dx^1 \wedge dx^2$ we have $d * \xi = d\xi_0$, and the normalization becomes $\int_{\partial B} \xi_0\, d\theta = 0$. The presented gauge theorem is weak in the sense that no additional regularity of P and A is asserted. It is possible that P has singularities which change the topological type of the bundle. In dimensions $n \leq 4$ Uhlenbeck proved a stronger version where P is estimated in $W^{2,2}$, and accordingly A in $W^{1,2}$. These estimates depend on a smallness assumption for the L^2 norm of the curvature $F = dA + A \wedge A$. For $n \leq 3$ the smallness threshold can always be achieved by scaling, whereas in the critical dimension $n = 4$ it is a necessary, nontrivial condition.

6.2 Equations of the form $\Delta u = \Omega\nabla u$

Let $B \subset \mathbb{R}^2$ be the unit ball in $\mathbb{R}^2$. In this section we study the regularity properties of solutions of elliptic systems of the form

$$-\Delta u = \Omega\nabla u, \tag{6.5}$$

where $u \in W^{1,2}(B, \mathbb{R}^m)$ and $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$. Before coming to the detailed study let us give some examples for systems of the type (6.5).

1) From (**??**) we see that harmonic maps into spheres satisfy an equation of the form (6.5) with $(\Omega^{ij}) = (u^i\nabla u^j - u^j\nabla u^i) \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$.

2) It is easy to see that surfaces with prescribed mean curvature $H \in L^\infty(\mathbb{R}^3)$ (i.e. solutions of (**??**)) solve a system of the form (6.5) with

$$\Omega = -2H(u)\begin{pmatrix} 0 & \nabla^\perp u^3 & -\nabla^\perp u^2 \\ -\nabla^\perp u^3 & 0 & \nabla^\perp u^1 \\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0 \end{pmatrix} \in L^2(B, so(3) \otimes \wedge^1\mathbb{R}^2).$$

3) Harmonic maps into general target manifolds.
Here we let $u \in W^{1,2}(B, N)$, where $N \hookrightarrow \mathbb{R}^m$ is a smooth and compact Riemannian manifold without boundary. Then we know from the discussions in chapter 2 that harmonic maps into N are critical points of the functional

$$E(u) = \frac{1}{2}\int_B |\nabla u|^2 dv_g.$$

To compute the critical points of E we let $\varphi \in C_c^1(B, \mathbb{R}^m)$ with $\varphi(x) \in T_{u(x)}N$ for all $x \in B$. Then we compute

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} E(u + t\varphi) \\ &= -\int_B \Delta u \varphi. \end{aligned}$$

Since this is true for all such φ we know that

$$\Delta u \perp T_u N.$$

Therefore if we let $\{\nu_{n+1}, \dots, \nu_m\}$ be a smooth local orthonormal frame for the normal bundle near $u(x)$ we can write

$$\Delta u(x) = \sum_{i=n+1}^{m} \lambda_i(x)\nu_i(u(x)),$$

where the λ_i are scalar functions. Using the fact that $\langle \nabla u, \nu_i(u)\rangle = 0$ for every $i \in \{n+1, \dots, m\}$ we get

$$\begin{aligned}\lambda_i =& \langle \Delta u, \nu_i(u)\rangle \\ =& \operatorname{div}\langle \nabla u, \nu_i(u)\rangle - \langle \nabla_j u, (d_k\nu_i)(u)\nabla_j u^k\rangle\end{aligned}$$

and hence

$$\begin{aligned}\Delta u =& \sum_i \lambda_i \nu_i(u) \\ =& -\sum_{i=n+1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{2}\langle \nabla_j u, (d_k\nu_i)(u)\nabla_j u^k\rangle \nu_i(u) \\ =& -A(u)(\nabla u, \nabla u).\end{aligned}$$

Moreover, using the definition of A, we see that (using that $\sum_k \nabla u^k \nu_i^k(u) = 0$ for every i)

$$\begin{aligned}\Delta u^s =& -\sum_{i,k}\langle \nabla u, (d_k\nu_i)(u)\nabla u^k\rangle \nu_i^s(u) \\ =& -\sum_{i,k,l}\nabla u^k(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l),\end{aligned}$$

and hence u solves an equation of the form (6.5) with

$$(\Omega_{sk}) = (\sum_{i,l}(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l)) \in L^2(B, so(m)\otimes \wedge^1\mathbb{R}^2).$$

4) Conformally invariant variational problems.
We consider the functional

$$E_\omega(u) = \frac{1}{2}\int_B (|\nabla u|^2 + \omega(u)(\partial_x u, \partial_y u))dx,$$

where ω is a C^1 two-form on $\mathbb{R}^m$ such that the L^∞-norm of $d\omega$ is bounded. By Theorem 2.4.1 we see that every conformally invariant energy in two-dimensions can be written in this way. The Euler-Lagrange equation of E_ω can easily be computed to be

$$\Delta u^i + A^i(u)(\nabla u, \nabla u) + \lambda^i_{jl}(u)\partial_x u^j \partial_y^l = 0,$$

where $\lambda^i_{jl}(u) = d\omega(u)(e_i, e_j, e_l)$ and where $\{e_i\}_{\{i=1,\dots,m\}}$ is the standard basis of $\mathbb{R}^m$. Using that $\lambda^i_{jl} = -\lambda^j_{il}$ we calculate

$$\lambda^i_{jl}(u)\partial_x u^j \partial_y^l = \frac{1}{4}(\lambda^i_{jl}(u) - \lambda^j_{il}(u))\nabla^\perp u^l \nabla u^j.$$

Combining this with the result of 3) we see that the Euler-Lagrange equation of every conformally invariant energy in two dimensions can be written in the form (6.5) with

$$\begin{aligned}\Omega_{sk} =& \sum_{i,l}(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l)\\ &- \sum_l \frac{1}{4}(\lambda_{kl}^s(u) - \lambda_{sl}^k(u))\nabla^\perp u^l\\ \in& L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2).\end{aligned}$$

After having collected all these examples of systems of the type (6.5) we now state the main Theorem of this chapter. This Theorem was only recently proved by Tristan Rivière [46] (see also [34], [47] and [60] for related results).

Theorem 6.2.1. *Let $u \in W^{1,2}(B,\mathbb{R}^m)$ be a solution of (6.5) with $\Omega \in L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2)$. Then u is continuous and therefore by Theorem **??** as smooth as the data permits.*

Proof. The Theorem will be proved in three steps.
Step 1:

Lemma 6.2.2. *Let $m\in\mathbb{N}$ and $\Omega \in L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2)$. Let $A \in L^\infty\cap W^{1,2}(B, M(m))$ and $B \in W^{1,2}(B, M(m))$ be solutions of*

$$\nabla A - A\Omega = \nabla^\perp B. \tag{6.6}$$

Then $u \in W^{1,2}(B,\mathbb{R}^m)$ is a solution of (6.5) with Ω iff

$$\operatorname{div}(A\nabla u + B\nabla^\perp u) = 0. \tag{6.7}$$

Proof. By a direct calculation (using that $\operatorname{div}\nabla^\perp = 0$ and $\nabla u\nabla^\perp v = -\nabla^\perp u\nabla v$) and using (6.6) we get

$$\begin{aligned}\operatorname{div}(A\nabla u + B\nabla^\perp u) =& (\nabla A - \nabla^\perp B)\nabla u + A\Delta u\\ =& A(\Delta u + \Omega\nabla u).\end{aligned}$$

This proves the Lemma. □

Step 2:

Lemma 6.2.3. *There exists $\varepsilon > 0$, $c > 0$ such that for every $\Omega \in L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2)$ with*

$$\int_B |\Omega|^2 dx < \varepsilon, \tag{6.8}$$

there exist $A \in L^\infty\cap W^{1,2}(B, Gl(m))$ and $B \in W^{1,2}(B, M(m))$ satisfying

$$\int_B (|\nabla A|^2 + |\nabla B|^2)dx + ||\operatorname{dist}(A, SO(n))||^2_{L^\infty} \le c\int_B |\Omega|^2 \quad \textit{and} \tag{6.9}$$

$$\nabla A - A\Omega - \nabla^\perp B = 0. \tag{6.10}$$

Proof. For $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$ with $\int_B |\Omega|^2 dx < \varepsilon$ we apply Theorem **??** to get the existence of $P \in W^{1,2}(B, SO(m))$ and $\xi \in W^{1,2}(B, so(m))$ such that $\xi = 0$ on ∂B,

$$\nabla^\perp \xi = P^{-1}\nabla P + P^{-1}\Omega P. \tag{6.11}$$

and

$$||\xi||_{W^{1,2}} + ||\nabla P||_{L^2} + ||\nabla P^{-1}||_{L^2} \le c||\Omega||_{L^2}. \tag{6.12}$$

Claim 1: There exist $\hat{A} \in W^{1,2} \cap L^\infty(B, M(m))$ and $B \in W^{1,2}(B, M(m))$ solving

$$\Delta \hat{A} = \nabla \hat{A} \nabla^\perp \xi + \nabla^\perp B \nabla P \quad \text{in} \quad B, \tag{6.13}$$

$$\Delta B = -\nabla^\perp \hat{A} \nabla P^{-1} - \operatorname{div}(\hat{A}\nabla\xi P^{-1} + \nabla \xi P^{-1}) \quad \text{in} \quad B, \tag{6.14}$$

$$\frac{\partial \hat{A}}{\partial \nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial B, \tag{6.15}$$

$$\int_B \hat{A} = 0. \tag{6.16}$$

To prove this claim we apply Theorem **??** (combined with remark **??**) and standard L^2-theory to get

$$||\hat{A}||_{W^{1,2}} + ||\hat{A}||_{L^\infty} \le c||\nabla\xi||_{L^2}||\nabla\hat{A}||_{L^2} + c||\nabla P||_{L^2}||\nabla B||_{L^2} \quad \text{and} \tag{6.17}$$

$$||B||_{W^{1,2}} \le c||\nabla P^{-1}||_{L^2}||\nabla \hat{A}||_{L^2} + c||\nabla\xi||_{L^2}||\hat{A}||_{L^\infty} + c||\nabla\xi||_{L^2}. \tag{6.18}$$

Using (6.12) and choosing ε small enough we combine (6.17) and (6.18) to get

$$||\hat{A}||_{W^{1,2}} + ||\hat{A}||_{L^\infty} + ||B||_{W^{1,2}} \le c||\Omega||_{L^2}. \tag{6.19}$$

The existence of the desired solution of (6.13)-(6.16) (and hence the proof of Claim 1) now follows from a standard fixed-point argument.
Next we define $\tilde{A} = \hat{A} + id$ and we see from (6.13)-(6.16) that $\tilde{A}$ and B solve

$$\Delta \tilde{A} = \nabla \tilde{A} \nabla^\perp \xi + \nabla^\perp B \nabla P \quad \text{in} \quad B, \tag{6.20}$$

$$\Delta B = -\nabla^\perp \tilde{A} \nabla P^{-1} - \operatorname{div}(\tilde{A}\nabla\xi P^{-1}) \quad \text{in} \quad B, \tag{6.21}$$

$$\frac{\partial \tilde{A}}{\partial \nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial B, \tag{6.22}$$

$$\int_B \tilde{A} = |B|. \tag{6.23}$$

Moreover we get from (6.19) that

$$||\nabla \tilde{A}||_{L^2} + ||\operatorname{dist}(\tilde{A}, SO(m))||_{L^\infty} + ||B||_{W^{1,2}} \le c||\Omega||_{L^2}. \tag{6.24}$$

Now it is easy to see that (6.20) can be rewritten as

$$\operatorname{div}(\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp B P) = 0 \tag{6.25}$$

and hence, by Lemma **??**, there exists $C \in W^{1,2}(B, M(m) \otimes \wedge^1\mathbb{R}^2)$ such that

$$\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp B P = \nabla^\perp C. \tag{6.26}$$

Since by (6.22) and the definition of ξ we have

$$\begin{aligned}(\nabla\tilde{A} - \tilde{A}\nabla^{\perp}\xi - \nabla^{\perp}BP)\cdot\nu =& \frac{\partial\tilde{A}}{\partial\nu} - \tilde{A}\nabla^{\perp}\xi\cdot\nu - \nabla^{\perp}BP\cdot\nu \\ =&0\end{aligned}$$

on ∂B we can moreover assume that $C = 0$ on ∂B. Using a rotation by $\frac{\pi}{2}$ (one can also view $\nabla^{\perp}$ as $\star d$ and then the rotation by $\frac{\pi}{2}$ is just another application of $\star$) we see that (6.26) is equivalent to

$$-\nabla CP^{-1} = \nabla^{\perp}\tilde{A}P^{-1} + \tilde{A}\nabla\xi P^{-1} + \nabla B,$$

and hence, using (6.21), we calculate

$$\begin{aligned}-\operatorname{div}(\nabla CP^{-1}) &= \nabla^{\perp}\tilde{A}\nabla P^{-1} + \operatorname{div}(\tilde{A}\nabla\xi P^{-1}) + \Delta B \\ &= 0. \end{aligned} \tag{6.27}$$

Claim 2: Every solution C of (6.27) with $C = 0$ on ∂B vanishes identically.
To see this we apply again Lemma **??** and get the existence of $D \in W^{1,2}(B, M(m) \otimes \wedge^1\mathbb{R}^2)$ such that

$$\nabla^{\perp}D = \nabla CP^{-1}. \tag{6.28}$$

Since $C = 0$ on ∂B we easily see that $\frac{\partial D}{\partial\nu} = 0$ on ∂B and we can also assume that $\int_B D = 0$. Hence C and D solve

$$\Delta C = \nabla^{\perp}D\nabla P \quad \text{in} \quad B, \tag{6.29}$$

$$\Delta D = \nabla C\nabla^{\perp}P^{-1} \quad \text{in} \quad B, \tag{6.30}$$

$$C = 0 \quad \text{and} \quad \frac{\partial D}{\partial\nu} = 0 \quad \text{on} \quad \partial B, \tag{6.31}$$

$$\int_B D = 0. \tag{6.32}$$

From this we see that we can apply Theorem **??** for (6.29) and (6.30) (in this case combined with remark **??**) to get

$$||\nabla C||_{L^2} + ||\nabla D||_{L^2} \le c(||\nabla P||_{L^2}||\nabla D||_{L^2} + ||\nabla P^{-1}||_{L^2}||\nabla C||_{L^2}). \tag{6.33}$$

By choosing ε small enough we get from (6.12) that $C = D = 0$ and this shows the claim.
From (6.26) we now see that $\tilde{A}$ and B solve

$$\nabla\tilde{A} - \tilde{A}\nabla^{\perp}\xi - \nabla^{\perp}BP = 0. \tag{6.34}$$

Defining $A = \tilde{A}P^{-1}$ we see that

$$\begin{aligned}||\nabla A||_{L^2} + ||\operatorname{dist}(A, SO(m))||_{L^\infty} &\le c(||\nabla\tilde{A}||_{L^2} + ||\tilde{A}||_{L^\infty}||\nabla P^{-1}||_{L^2} + ||\operatorname{dist}(\tilde{A}, SO(m))||_{L^\infty}) \\ &\le c||\Omega||_{L^2}, \end{aligned} \tag{6.35}$$

where we used (6.12) and (6.24). Moreover we use (6.11) and (6.34) to calculate

$$\begin{aligned} 0 = \nabla\tilde{A} - \tilde{A}\nabla^{\perp}\xi - \nabla^{\perp}BP &= \nabla AP + A\nabla P - AP\nabla^{\perp}\xi - \nabla^{\perp}BP \\ &= A\nabla P - A\nabla P + (\nabla A - A\Omega - \nabla^{\perp}B)P \end{aligned}$$

and therefore

$$\nabla A - A\Omega - \nabla^{\perp}B = 0. \tag{6.36}$$

This finishes the proof of the Lemma. □

Step 3:
For every point $x \in B$ we choose a radius $r_x > 0$ such that $\int_{B_{r_x}(x)} |\Omega|^2 < \varepsilon$, where ε is the same as in Lemma 6.2.3. In the following we write $B_{r_x}(x) = B$. Then we can apply Lemma 6.2.3 to get the existence of A and B solving (6.6). Hence we can apply Lemma 6.2.2 to see that

$$\operatorname{div}(A\nabla u) = \nabla B\nabla^{\perp}u = -\nabla^{\perp}B\nabla u \quad \text{and} \tag{6.37}$$

$$\nabla^{\perp}(A\nabla u) = \nabla^{\perp}A\nabla u. \tag{6.38}$$

Now we apply Lemma **??** to get the existence of $\alpha \in W^{1,2}(B, \mathbb{R}^m)$ and $\beta \in W^{1,2}(B, \mathbb{R}^m \otimes \wedge^1\mathbb{R}^2)$ such that

$$A\nabla u = \nabla\alpha + \nabla^{\perp}\beta. \tag{6.39}$$

Using (6.37) we see that α solves

$$\Delta\alpha = \operatorname{div}(A\nabla u) = -\nabla^{\perp}B\nabla u. \tag{6.40}$$

Now we denote by $\overline{u}$ the mean value of u on $B_{\frac{1}{2}}$ and let $\tilde{u} \in W_0^{1,2}(\mathbb{R}^2, \mathbb{R}^m)$ be the extension with compact support of $u - \overline{u}$. Then we have that $\nabla\tilde{u} = \nabla u$ on $B_{\frac{1}{2}}$. Moreover we use Poincare's inequality to get

$$||\nabla\tilde{u}||_{L^2(\mathbb{R}^2)} \le c||u - \overline{u}||_{W^{1,2}(B_{\frac{1}{2}})} \le c||\nabla u||_{L^2(B)}.$$

We extend B in the same way and denote the resulting map by $\tilde{B} \in W_0^{1,2}(\mathbb{R}^2, M(m))$. Then we let $\tilde{\alpha}$ be the solution of

$$\Delta\tilde{\alpha} = -\nabla^{\perp}\tilde{B}\nabla\tilde{u} \tag{6.41}$$

on $\mathbb{R}^2$. Since by Corollary **??** $-\nabla^{\perp}\tilde{B}\nabla\tilde{u} \in \mathcal{H}^1(\mathbb{R}^2)$ with

$$||-\nabla^{\perp}\tilde{B}\nabla\tilde{u}||_{\mathcal{H}^1(\mathbb{R}^2)} \le c||\nabla B||_{L^2(B)}||\nabla u||_{L^2(B)}$$

we can apply Theorem 4.3.9 to get that $\tilde{\alpha} \in W^{2,1}(\mathbb{R}^2)$. Since

$$\Delta(\alpha - \tilde{\alpha}) = 0$$

on $B_{\frac{1}{2}}$ we get that $\alpha \in W^{2,1}(B_{\frac{1}{4}})$ (harmonic functions are smooth). Next we observe that β solves

$$\Delta\beta = \nabla^{\perp}A\nabla u \tag{6.42}$$

and hence we can argue as before to get that $\beta \in W^{2,1}(B_{\frac{1}{4}})$. Therefore we see from (6.39) that

$$A\nabla u \in W^{1,1}(B_{\frac{1}{4}}) \tag{6.43}$$

and therefore (using (6.9))

$$\nabla u \in W^{1,1}(B_{\frac{1}{4}}) \quad \text{or} \quad u \in W^{2,1}(B_{\frac{1}{4}}). \tag{6.44}$$

Combining this with Corollary **??** we finish the proof of the Theorem. □

With the following counterexamples of Frehse [17] we show that one can not drop the condition that Ω has to be antisymmetric.

Remark 6.2.4. *Let $u = (u_1, u_2) \in W^{1,2}(B, S^1 \subset \mathbb{R}^2)$ be defined by*

$$u_1(x) = \sin \ln \ln \frac{2}{|x|},$$
$$u_2(x) = \cos \ln \ln \frac{2}{|x|}$$

then it is easy to check that u solves the elliptic system $-\Delta u = \Omega\nabla u$ with

$$\Omega = \begin{pmatrix} (u_1+u_2)\nabla u_1 & (u_1+u_2)\nabla u_2 \\ (u_2-u_1)\nabla u_1 & (u_2-u_1)\nabla u_2 \end{pmatrix}.$$

So in this case Ω is not antisymmetric and u is bounded but not continuous.
For $u \in W^{1,2}(B, \mathbb{R}^2)$ given by

$$u_1(x) = \ln \ln \frac{2}{|x|},$$
$$u_2(x) = \ln \ln \frac{2}{|x|}$$

we have that u solves the elliptic system $-\Delta u = \Omega\nabla u$ with

$$\Omega = \begin{pmatrix} \nabla u_1 & 0 \\ 0 & \nabla u_2 \end{pmatrix}.$$

Here we even don't have that u is bounded.

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