Geometric Variational Problems

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Chapter 1

Introduction

These are notes of a course given in summer 2015 at the University of Freiburg. In 2007 Tobias Lamm lectured on this topic at FU Berlin, and I followed his notes. The subject of the course is the regularity theory for two-dimensional geometric variational problems, in particular compensation methods due to Henry Wente, Frédéric Hélein and Tristan Rivière. Along the lines we introduce certain Hardy and Lorentz spaces, and present the construction of a Coulomb gauge following Karen Uhlenbeck.

Chapter 2

Geometric variational problems

The purpose of this Chapter is to introduce the key examples of two-dimensional geometric variational problems. More background information on these examples can be found in the book of Jost [29].

2.1 The Dirichlet integral

Let $G \subset \mathbb{R}^2$ be a domain. For $u \in C^1(G, \mathbb{R}^n)$ the Dirichlet energy is defined by

$$\mathcal{E}(u) = \frac{1}{2} \int_G |Du|^2 \quad \text{where } |Du|^2 = \operatorname{tr}(Du^{\mathrm{T}} Du). \tag{2.1}$$

This definition clearly applies in any dimension, but the following interesting feature is specific to the case of dimension two.

Theorem 2.1.1 (Conformal invariance of the Dirichlet energy)**.** *Let $f : G \to G'$ be a conformal diffeomorphism of domains $G, G' \subset \mathbb{R}^2$. Then*

$$\mathcal{E}(v \circ f) = \mathcal{E}(v) \quad \textit{for all } v \in C^1(G', \mathbb{R}^n).$$

Proof. In dimension two, the fact that f is conformal means that

$$Df^{\mathrm{T}} Df = Df\, Df^{\mathrm{T}} = |\det Df|\, \mathrm{E}_2.$$

Now $D(v \circ f) = Dv \circ f\, Df$, thus

$$\begin{aligned} |D(v \circ f)|^2 &= \operatorname{tr}\big(Df^{\mathrm{T}} Dv^{\mathrm{T}} \circ f Dv \circ f Df\big) \\ &= \operatorname{tr}\Big(Df Df^{\mathrm{T}} Dv^{\mathrm{T}} \circ f Dv \circ f\Big) \\ &= |\det Df|\, |Dv|^2 \circ f. \end{aligned}$$

Thus by the transformation formula

$$\mathcal{E}(v \circ f) = \frac{1}{2} \int_G |Dv|^2 \circ f |\det Df| = \frac{1}{2} \int_{G'} |Dv|^2 = \mathcal{E}(v).$$

□

For $u \in C^1(G, \mathbb{R}^n)$ with $\mathcal{E}(u) < \infty$ and $\varphi \in C^1_c(G, \mathbb{R}^n)$ we have the first variation formula

$$\frac{d}{dt}\mathcal{E}(u + t\varphi)|_{t=0} = \int_G \langle Du, D\varphi \rangle. \tag{2.2}$$

Assuming that $u \in C^2(G, \mathbb{R}^n)$ we can integrate by parts to obtain

$$\frac{d}{dt}\mathcal{E}(u + t\varphi)|_{t=0} = -\int_G \langle \Delta u, \varphi \rangle.$$

In other words the Euler-Lagrange operator for the Dirichlet energy is $-\Delta$. We say that u is a critical point of the Dirichlet energy if the first variation in (2.2) always vanishes. For $u \in C^2(G, \mathbb{R}^n)$ we can then conclude that u is harmonic:

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{on } G. \tag{2.3}$$

The conformal invariance implies an equivariance property for the Laplace operator, more precisely we compute for a conformal diffeomorphism $f : G \to G'$

$$\begin{aligned} -\int_G \langle \Delta(v \circ f), \varphi \rangle &= \frac{d}{dt}\mathcal{E}(v \circ f + t\varphi)|_{t=0} \\ &= \frac{d}{dt}\mathcal{E}(v + t\varphi \circ f^{-1})|_{t=0} \\ &= -\int_{G'} \langle \Delta v, \varphi \circ f^{-1} \rangle \\ &= -\int_G \langle \Delta v \circ f, \varphi \rangle \, |\det Df|. \end{aligned}$$

We conclude that

$$\Delta v \circ f = \frac{1}{|\det Df|}\Delta(v \circ f) \quad \text{for a conformal diffeomorphism } f : G \to G'. \tag{2.4}$$

How comes geometry into play? For an immersion $u : G \to \mathbb{R}^n$ the area functional is [1]

$$\mathcal{A}(u) = \int_G \sqrt{\det g} \quad \text{where } g_{\alpha\beta} = \langle \partial_\alpha u, \partial_\beta u \rangle. \tag{2.5}$$

Note that $\sqrt{\det g}$ is the Jacobian Ju, in fact by definition

$$\sqrt{\det g} = \sqrt{|\partial_1 u|^2 |\partial_2 u|^2 - \langle \partial_1 u, \partial_2 u \rangle^2} = |\partial_1 u \wedge \partial_2 u| = Ju.$$

The first variation of the area in direction $\varphi \in C^1_c(G, \mathbb{R}^n)$ is computed as follows. First

$$\frac{\partial}{\partial t}\langle \partial_\alpha(f + t\varphi), \partial_\beta(f + t\varphi) \rangle|_{t=0} = \langle \partial_\alpha f, \partial_\beta \varphi \rangle + \langle \partial_\beta f, \partial_\alpha \varphi \rangle.$$

Using this one gets

$$\frac{d}{dt}\mathcal{A}(u + t\varphi)|_{t=0} = \int_G g^{\alpha\beta} \langle \partial_\beta f, \partial_\alpha \varphi \rangle \sqrt{\det g}.$$

[1] greek indices $\alpha, \beta, \ldots$ in $1, 2$, latin indices $i, j, \ldots$ in $1, \ldots, n$.

Here $(g^{\alpha\beta})$ denotes the (symmetric) inverse matrix to $(g_{\alpha\beta})$. Integrating by parts yields

$$\frac{d}{dt}\mathcal{A}(u+t\varphi)|_{t=0} = -\int_G \langle \vec{H}, \varphi\rangle \sqrt{\det g} \quad \text{where } \vec{H} = \frac{1}{\sqrt{\det g}}\partial_\alpha\big(\sqrt{\det g}\, g^{\alpha\beta}\partial_\beta u\big). \tag{2.6}$$

We now compare the area functional to the Dirchlet energy. First we have

$$Ju = |\partial_1 u \wedge \partial_2 u| \leq |\partial_1 u|\,|\partial_2 u| \leq \frac{1}{2}(|\partial_1 u|^2 + |\partial_2 u|^2) = \frac{1}{2}|Du|^2.$$

Thus $\mathcal{A}(u) \leq \mathcal{E}(u)$ with equality if and only if u is conformally parametrized, that is

$$\langle \partial_1 u, \partial_2 u\rangle = 0 \quad \text{and} \quad |\partial_1 u|^2 = |\partial_2 u|^2. \tag{2.7}$$

Assuming that u is conformally parametrized, with induced metric $g_{\alpha\beta} = e^{2\lambda}\delta_{\alpha\beta}$ and Jacobian $Ju = e^{2\lambda}$, the mean curvature vector becomes

$$\vec{H} = e^{-2\lambda}\Delta u. \tag{2.8}$$

It follows that a conformally parametrized immersion $u : G \to \mathbb{R}^n$ is a minimal surface if and only if u is Euclidean harmonic. These facts provide a close relation between the Dirichlet energy and the area functional in two dimensions.

The question whether any immersed surface admits a reparametrization which is conformal is of fundamental importance; it has local and global aspects. In 1825 Gauß proved that any real-analytic surface admits locally a conformal reparametrization; this was extended to surfaces of class $C^{1,1}$ by Lichtenstein in 1911. For oriented surfaces the parameter changes are then holomorphic, so that the immersion induces a global complex structure on the parameter domain, which becomes naturally a Riemann surface.

The following result goes in a different direction: it shows that certain critical points of geometric variational problems satisfy automatically the conformality relations. The result plays a crucial rôle in proving that the classical approach to the Plateau problem actually produces a minimal surface. Moreover, this generalizes to other problems for surfaces with precribed mean curvature. Historically, the result was also relevant in the regularity theory, because earlier regularity proofs needed to assume conformality [21]. In the literature one often finds the term *stationary* for a map which is critical with respect to variations of the independent variables.

Theorem 2.1.2. *Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Assume that $u \in C^1(D, \mathbb{R}^n)$ has finite energy and satisfies*

$$\frac{d}{dt}\mathcal{E}(u \circ \phi_t)|_{t=0} = 0,$$

for any smooth family $\phi : \overline{D} \times (-\varepsilon, \varepsilon) \to \overline{D}$ of diffeomorphisms $\phi_t = \phi(\cdot, t)$ with $\phi_0 = \mathrm{id}_{\overline{D}}$. Then u satisfies the conformally relations (2.7).

It is only asserted that u is *weakly* conformal, leaving open whether u is immersed or not. The conformality may however be useful in analyzing the behavior of u at points with vanishing Jacobian, showing that they are isolated and have the character of branchpoints, see [11].

Proof. We assume that $u : \mathbb{H} \to \mathbb{R}^n$ where $\mathbb{H}$ is the upper half plane. There is an explicit conformal equivalence between D and $\mathbb{H}$, so that the result transfers to D via conformal invariance. Consider a vector field $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $X \circ \tau = \tau X$, where $\tau(x,y) = (x,-y)$, in particular $X_2(x,0) = 0$ for all $x \in \mathbb{R}$. The associated flow $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is smooth, and $\phi_t = \phi(\cdot, t)$ is diffeomorphic with inverse ϕ_{-t}. Uniqueness for the initial value problem implies

$$\phi(\tau(z), t) = \tau(\phi(z,t)) \quad \text{for all } z \in \mathbb{R}^2,\ t \in \mathbb{R}.$$

In particular $\phi_t(\mathbb{R}) = \mathbb{R}$ and $\phi_t(\mathbb{H}) = \mathbb{H}$. Substituting $\zeta = (\xi, \eta) = \phi_{-t}(z)$ we have

$$\mathcal{E}(u \circ \phi_t) = \frac{1}{2}\int_{\mathbb{H}} |Du(\phi_t(\zeta))D\phi_t(\zeta)|^2\, d\xi d\eta = \frac{1}{2}\int_{\mathbb{H}} |Du(z)D\phi_{-t}(z)^{-1}|^2 \det D\phi_{-t}(z)\, dxdy.$$

We want to differentiate under the integral sign at $t = 0$. As $\phi(z,t) = z$ for $z \notin \operatorname{spt} X$, the integrand and its derivative for $t \in (-\varepsilon, \varepsilon)$ are bounded by $C\,|Du|^2$ which is integrable. We compute

$$\begin{aligned}
\frac{\partial}{\partial t} D\phi_{-t}(z)\cdot v|_{t=0} &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\phi_{-t}(z+sv)|_{s=0,t=0} \\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\phi_{-t}(z+sv)|_{t=0,s=0} \\
&= -\frac{\partial}{\partial s} X(z+sv)|_{s=0} \\
&= -DX(z)\cdot v.
\end{aligned}$$

This implies

$$\frac{\partial}{\partial t} D\phi_{-t}(z)^{-1}|_{t=0} = DX(z), \qquad \frac{\partial}{\partial t}\det D\phi_{-t}(z) = -\operatorname{div} X(z).$$

Putting $X = (a,b)$ we find

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u\circ\phi_t)|_{t=0} &= \int_{\mathbb{H}} \big(\langle Du(z), Du(z)DX(z)\rangle - \frac{1}{2}|Du(z)|^2 \operatorname{div} X(z)\big)\, dxdy \\
&= \int_{\mathbb{H}} \big(\langle \partial_i u, \partial_j u\rangle \partial_i X^j - \frac{1}{2}|Du|^2 \partial_i X^i\big)\, dxdy \\
&= \int_{\mathbb{H}} \big(\frac{1}{2}(|u_x|^2 - |u_y|^2) a_x + \langle u_x, u_y\rangle a_y\big)\, dxdy \\
&\quad + \int_{\mathbb{H}} \big(\langle u_x, u_y\rangle b_x - \frac{1}{2}(|u_x|^2 - |u_y|^2) b_y\big)\, dxdy.
\end{aligned}$$

Taking $X|_{\mathbb{H}} \in C_c^\infty(\mathbb{H}, \mathbb{R}^2)$ arbitrary, we see that the integrable function

$$h(z) = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle$$

is a weak solution to the Cauchy-Riemann equations, and hence holomorphic. Next we take $X = (\varphi_y, \varphi_x)$ where $\varphi \in C_c^\infty(\mathbb{R}^2)$ is odd, that is $\varphi(x,-y) = -\varphi(x,y)$. Then X is admissible whence

$$\int_{\mathbb{H}} \langle u_x, u_y\rangle \Delta\varphi\, dxdy = 0.$$

Using odd reflection the function $\langle u_x, u_y\rangle$ extends to a weakly harmonic function which is integrable on $\mathbb{R}^2$. By the mean value formula, we conclude that $\langle u_x, u_y\rangle$ is identically zero. The Cauchy-Riemann equations now yield that $h(z)$ is constant and in fact vanishes, again by integrability. □

2.2 Surfaces of prescribed mean curvature in $\mathbb{R}^3$

Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. For $u \in C^1(\overline{D}, \mathbb{R}^3)$ we consider the functional

$$\mathcal{F}(u) = \frac{1}{2} \int_D |Du|^2 + \int_D u^* \omega, \tag{2.9}$$

where $\omega \in C^1(\mathbb{R}^3, \Lambda_2(\mathbb{R}^3))$ is a given two-form. To interpretate the second term geometrically, let us assume for simplicity that $u \in C^2(\overline{D}, \mathbb{R}^3)$. Consider the cone over u defined by

$$F : D \times [0,1] \to \mathbb{R}^3,\ F(z,t) = tu(z).$$

Writing $d\omega = H\, dV_{\mathbb{R}^3}$ we get by Stokes' theorem

$$\begin{aligned}
\int_{D\times[0,1]} F^*(H dV_{\mathbb{R}^3}) &= \int_{D\times[0,1]} F^* d\omega \\
&= \int_{D\times[0,1]} dF^*\omega \\
&= \int_D F(\cdot,1)^*\omega - \int_D F(\cdot,0)^*\omega + \int_{\partial D\times[0,1]} F^*\omega \\
&= \int_D u^*\omega + \int_{\partial D\times[0,1]} F^*\omega.
\end{aligned}$$

The C^2 assumption was used when interchanging F^* and d. Introducing the multiplicity function $\theta_F(X) = \sum_{F(z,t)=X} \operatorname{sign} \det DF(z,t)$, we get by the transformation formula

$$\int_{F(D\times[0,1])} H\theta_F\, d\mathcal{L}^3 = \int_D u^*\omega + \int_{\partial D\times[0,1]} F^*\omega.$$

The second integral on the right depends only on $u|_{\partial D}$, thus it reduces to a constant when restricting to a class of maps with prescribed boundary values. Up to that constant, the integral $\int_D u^*\omega$ then corresponds to the volume of the cone, weighted with the function H and counted with multiplicities. A special case is obtained by chosing

$$\omega = \frac{1}{3} X \llcorner dV_{\mathbb{R}^3}.$$

Then $d\omega = dV_{\mathbb{R}^3}$ and $F^*\omega = 0$ on $\partial D \times [0,1]$, no matter what boundary condition. Hence

$$\frac{1}{3} \int_D \langle u, u_x \wedge u_y \rangle\, dxdy = \int_D u^*\omega = \int_{F(D\times[0,1])} \theta_F\, d\mathcal{L}^3. \tag{2.10}$$

Next we calculate the first variation of the functional.

Lemma 2.2.1. *Let $\mathcal{F}(u)$ be the functional in (2.9), and put $d\omega = H\, dV_{\mathbb{R}^3}$. Then for any $u \in C^2(\overline{D}, \mathbb{R}^3)$ and $\varphi \in C^2_c(D, \mathbb{R}^3)$ we have*

$$\frac{d}{d\varepsilon} \mathcal{F}(u + \varepsilon\varphi)|_{\varepsilon=0} = \int_D \langle Du, D\varphi \rangle + \int_D H \circ u\, \langle u_x \wedge u_y, \varphi \rangle. \tag{2.11}$$

Proof. We consider the affine homotopy

$$F : D \times [0,\varepsilon] \to \mathbb{R}^3,\ F(z,t) = u(z) + t\varphi(z).$$

Applying Stokes' formula on $D \times [0,\varepsilon]$ we get

$$\int_{D\times[0,\varepsilon]} F^*(H\, dV_{\mathbb{R}^3}) = \int_D (u+\varepsilon\varphi)^*\omega - \int_D u^*\omega + \int_{\partial D\times[0,\varepsilon]} F^*\omega.$$

The last integral on the right vanishes since $\partial_t F(z,t) = \varphi(z) = 0$ on $\partial D \times [0,\varepsilon]$. Taking the derivative at $\varepsilon = 0$ yields

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}\int_D (u+\varepsilon\varphi)^*\omega|_{\varepsilon=0} &= \frac{\partial}{\partial\varepsilon}\int_0^\varepsilon\int_D H(F(x,y,t))\det DF(x,y,t)\, dxdydt|_{\varepsilon=0}\\
&= \int_D H(u(x,y))\det DF(x,y,0)\, dxdy\\
&= \int_D H(u(x,y))\det(u_x,u_y,\varphi)\, dxdy\\
&= \int_D H(u(x,y))\langle u_x\wedge u_y,\varphi\rangle\, dxdy.
\end{aligned}$$

The claim follows by combining with the first variation of the Dirichlet energy. □

We see that regular critical points of $\mathcal{F}$ are solutions of the elliptic system

$$\Delta u = (H\circ u)\ u_x\wedge u_y \quad \text{in } D. \tag{2.12}$$

If u is in addition a conformal immersion, with induced metric $g_{ij} = e^{2\lambda}\delta_{ij}$, then we can rewrite the equation in the form

$$e^{-2\lambda}\Delta u = (H\circ u)\ \nu \quad \text{where } \nu = \frac{u_x\wedge u_y}{|u_x\wedge u_y|}.$$

We know from (2.8) that the left hand side is the mean curvature vector, hence the surface u has prescribed mean curvature $H\circ u$. A special case is when H is constant, then u is called a constant mean curvature or CMC surface. One may then take as differential form

$$\omega = \frac{H}{3} X \llcorner dV_{\mathbb{R}^3} \quad (H \text{ constant}).$$

The partial differential equation (2.12) is called the prescribed mean curvature equation or constant mean curvature equation, respectively. The use of this terminology does not require solutions to be conformally parametrized. The geometric interpretation, however, is only available for u conformally parametrized. To obtain such solutions, one may potentially use Theorem 2.1.2. Namely for smooth diffeomorphisms $\phi_t : \overline{D} \to \overline{D}$, $t \in (-\varepsilon,\varepsilon)$, one has

$$\int_D (u\circ\phi_t)^*\omega = \int_D \phi_t^* u^*\omega = \int_D u^*\omega.$$

If u is a critical point of $\mathcal{F}$ with respect to variations $u\circ\phi_t$, then one concludes

$$\frac{d}{dt}\mathcal{E}(u\circ\phi_t)|_{t=0} = \frac{d}{dt}\mathcal{F}(u\circ\phi_t)|_{t=0} = 0.$$

The conformality relations now follow from Theorem 2.1.2. This applies, for example, to minimizers of $\mathcal{F}$ under Plateau boundary conditions.

To derive the prescribed mean curvature equation from the vanishing of the first variation we have imposed strong regularity assumptions on the function u. In contrast, the existence theory will only give us functions $u \in W^{1,2} \cap L^\infty(D, \mathbb{R}^3)$, say. It is a key issue to show that these weak solutions are regular. The special case of constant mean curvature was solved by Wente in 1969 [53], whereas the general case of variable mean curvature (for which $H \circ u$ is a priori only bounded and measurable) was proved by Riviére much later in 2007 [39].

2.3 Harmonic maps

As a second example we now introduce harmonic maps. Let $M \subset \mathbb{R}^n$ be a m-dimensional smooth compact submanifold where $1 \leq m \leq n-1$. By definition, a harmonic map $u : D \to M$ is a critical point of the Dirichlet energy under the constraint that $u(D) \subset M$; that is only variations staying in M are allowed. To derive the Euler-Lagrange equation, we need some basic geometric facts.

Lemma 2.3.1. *Let $U_\varrho(M) = \{X \in \mathbb{R}^n : \operatorname{dist}(X, M) < \varrho\}$. For small $\varrho > 0$, the nearest point projection $\pi^M : U_\varrho(M) \to M$ is well-defined and smooth. One has for all $X \in M$ the equations*

$$\begin{aligned} D\pi^M(X)\tau &= \tau && \text{for all } \tau \in T_X M.\\ D\pi^M(X)\nu &= 0 && \text{for all } \nu \in T_X M^\perp.\\ D^2\pi^M(X)(\tau,\tau) &= A(X)(\tau,\tau) && \text{for all } \tau \in T_X M. \end{aligned}$$

Here A is the (normal-valued) second fundamental form of $M \subset \mathbb{R}^n$.

Proof. We only show the formulae. For a curve $\gamma(s)$ in M with $\gamma(0) = X$, $\gamma'(0) = \tau \in T_X M$ we have $\pi^M \circ \gamma = \gamma$. Differentiating yields

$$(D\pi^M \circ \gamma)\gamma' = \gamma', \text{ i.e. } D\pi^M(X)\tau = \tau \quad \text{ for } s = 0.$$

Mow one proves $\pi^M(X + t\nu) = X$ for $\nu \in T_X M^\perp$ and $|t|$ small. Differentiating at $t = 0$ gives the second formula. Finally differentiating twice along $\gamma(s)$ implies

$$(D^2\pi^M \circ \gamma)(\gamma', \gamma') + (D\pi^M \circ \gamma)\gamma'' = \gamma''.$$

At $s = 0$ one concludes $D^2\pi^M(X)(\tau,\tau) = (\gamma'')^\perp = A(X)(\tau,\tau)$. □

Now assume $u \in C^1(D, \mathbb{R}^n)$ has finite Dirichlet energy and maps into the submanifold M. For $\varphi \in C_c^\infty(D, \mathbb{R}^n)$ one computes

$$\frac{d}{dt}\mathcal{E}(\pi^M \circ (u + t\varphi))|_{t=0} = \int_D \langle Du, D\big(D\pi^M \circ u)\varphi\rangle.$$

Assuming $u \in C^2(D)$ we can integrate by parts to get

$$\frac{d}{dt}\mathcal{E}(\pi^M \circ (u + t\varphi))|_{t=0} = -\int_D \langle \Delta u, (D\pi^M \circ u)\varphi\rangle = -\int_D \langle (D\pi^M \circ u)\Delta u, \varphi\rangle.$$

For a regular critical point, under the constraint $u(D) \subset M$, it follows that

$$(\Delta u)^\top = (D\pi^M \circ u)\Delta u = 0. \tag{2.13}$$

To get an equation which is analytically better tractable, one computes

$$\begin{aligned}(D\pi^M \circ u)\Delta u &= \partial_\alpha\big((D\pi^M \circ u)\partial_\alpha u\big) - \partial_\alpha(D\pi^M \circ u)\,\partial_\alpha u \\ &= \partial_\alpha(\partial_\alpha u) - (D^2\pi^M \circ u)(\partial_\alpha u, \partial_\alpha u) \\ &= \Delta u - (A \circ u)(\partial_\alpha u, \partial_\alpha u).\end{aligned}$$

We arrive at the harmonic map system

$$\Delta u = (A \circ u)(\partial_\alpha u, \partial_\alpha u). \tag{2.14}$$

It should be noted that the concept of harmonic maps is naturally defined in the context of maps $u : (M, \gamma) \to (M, g)$ between Riemannian manifolds, without assuming M embedded. The energy is then locally defined by

$$\mathcal{E}(u) = \frac{1}{2}\int_M \gamma^{\alpha\beta}(g \circ u)(\partial_\alpha u, \partial_\beta u)\,\sqrt{\det \gamma}.$$

In a geometric context this definition is preferable. However, it is difficult to avoid the embedding in regularity questions. There one usually needs a local coordinate representation $u : U \to V$, $u(z) = (u_1(z), \ldots, u_m(z))$, to get started. But maps in the natural energy space $W^{1,2}$ are not necessarily continuous. Therefore a passage to local coordinates on M may be impossible, no matter how small U is chosen. A classical example for this is as follows.

Beispiel 2.3.2. *Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^3$ and $\mathbb{S}^2$ the unit 2-sphere. By the above, a map $u : \mathbb{B} \to \mathbb{S}^2$ is harmonic if and only if*

$$-\Delta u = |Du|^2 u. \tag{2.15}$$

Namely, we have $A(X)(\tau, \tau) = -|\tau|^2 X$, which yields $(A \circ u)(\partial_\alpha u, \partial_\alpha u) = -|\partial_\alpha u|^2 u = -|Du|^2 u$. One checks that the following map belongs to $W^{1,2}(B, \mathbb{R}^3)$ and is a weak solution:

$$u : \mathbb{B} \to \mathbb{S}^2,\ u(x) = \frac{x}{|x|}.$$

An arbitrarily small neighborhood of the origin is mapped to the full two-sphere. We will eventually show that weak solutions are without singularities in two dimensions, however this is not clear a priori.

2.4 Two-dimensional geometric variational problems

The goal of this section is to classify all two-dimensional variational integrals of first order which are conformally invariant. We will see that the example of the Dirichlet energy plus a pullback of a 2-form already constitutes the general case, if we allow a general Riemannian metric in the target.

Let $f : \mathbb{R}^n \times \mathbb{R}^{2\times n} \to \mathbb{R}$, $f = f(X, A)$, be an integrand. On any bounded domain $G \subset \mathbb{R}^2$, we have the associated functional

$$\mathcal{F}(u, G) = \int_G f(u(z), Du(z))\, dxdy \quad \text{for } u : G \to \mathbb{R}^n.$$

We say that $\mathcal{F}$ is conformally invariant if for any conformal diffeomorphism $\phi : G \to \phi(G))$ we have the property

$$\mathcal{F}(u \circ \phi^{-1}, \phi(G)) = \mathcal{F}(u, G) \quad \text{for all } u : G \to \mathbb{R}^n. \tag{2.16}$$

The following classification is due to Michael Grüter.

Theorem 2.4.1 ([22]). *Consider a functional $\mathcal{F}(u, G) = \int_G f(u, Du)$, such that f and $D_A^2 f$ are continuous on $\mathbb{R}^n \times \mathbb{R}^{n\times 2}$. Assume also that*

$$f(X, A) > 0 \quad \text{whenever } A \neq 0.$$

If $\mathcal{F}(u, G) = \int_G f(u, Du)$ is conformally invariant, then it has the representation

$$\mathcal{F}(u) = \frac{1}{2} \int_G g(u)(Du, Du) + \int_G u^*\omega, \tag{2.17}$$

where g is a Riemannian metric and ω is a 2-form, both continuous on $\mathbb{R}^n$.

Proof. We compute using the transformation formula

$$\begin{aligned} \mathcal{F}(u \circ \phi^{-1}, \phi(G)) &= \int_{\phi(G)} f\big(u(\phi^{-1}(w)), Du(\phi^{-1}(w)) D\phi^{-1}(w)\big)\, dudv \\ &= \int_G f\big(u(z), Du(z) D\phi(z)^{-1}\big) |\det D\phi(z)|\, dxdy. \end{aligned}$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a conformal diffeomorphism. Taking $G = D_\varepsilon(0)$ and $u(z) = X + Az$ for given $X \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times 2}$, we obtain after dividing by $|D_\varepsilon|$ and letting $\varepsilon \searrow 0$

$$f(X, A) = f(X, AD\phi(0)^{-1}) |\det D\phi(0)|.$$

Chosing $\phi(z) = z/t$ for $t > 0$, and differentiating twice at $t = 0$ shows

$$f(X, A) = \frac{1}{2} D_A^2 f(X, 0)(A, A).$$

Chosing $\phi(z) = Sz$ for $S \in \mathbb{S}O(2)$ yields $f(X, A) = f(X, AS^{\mathrm{T}})$, hence

$$D_A^2 f(X, 0)(A, A) = D_A^2 f(X, 0)(AS^{\mathrm{T}}, AS^{\mathrm{T}}).$$

We now define a symmetric and an antisymmetric form, both continuous on $\mathbb{R}^n$, by

$$\begin{aligned} g(X)(V, W) &= D_A^2 f(X, 0)(e_1 \otimes V, e_1 \otimes W), \\ \omega(X)(V, W) &= D_A^2 f(X, 0)(e_1 \otimes V, e_2 \otimes W). \end{aligned}$$

The symmetry of $g(X)$ is clear. To show that $\omega(X)$ is antisymmetric, we first note

$$(e_\alpha \otimes V) S^{\mathrm{T}} z = \langle e_\alpha, S^{\mathrm{T}} z\rangle V = \langle S e_\alpha, z\rangle V = (S e_\alpha \otimes V) z.$$

Taking $Se_1 = e_2$, $Se_2 = -e_1$ we compute

$$\begin{aligned}
\omega(X)(V,W) &= D_A^2 f(X,0)(e_1 \otimes V, e_2 \otimes W) \\
&= -D_A^2 f(X,0)(Se_2 \otimes V, Se_1 \otimes W) \\
&= -D_A^2 f(X,0)((e_2 \otimes V)S^{\mathrm{T}}, (e_1 \otimes W)S^{\mathrm{T}}) \\
&= -D_A^2 f(X,0)(e_1 \otimes W, e_2 \otimes V) \\
&= -\omega(X)(W,V).
\end{aligned}$$

We conclude, using again the invariance with respect to S,

$$\begin{aligned}
f(u,Du) &= \frac{1}{2} D_A^2 f(u,0)(e_\alpha \otimes \partial_\alpha u, e_\beta \otimes \partial_\beta u) \\
&= \frac{1}{2}\big(D_A^2 f(u,0)(e_1 \otimes \partial_1 u, e_1 \otimes \partial_1 u) + D_A^2 f(u,0)(e_2 \otimes \partial_2 u, e_2 \otimes \partial_2 u)\big) \\
&\quad + D_A^2 f(u,0)(e_1 \otimes \partial_1 u, e_2 \otimes \partial_2 u) \\
&= \frac{1}{2}\big(g(u)(\partial_1 u, \partial_1 u) + g(u)(\partial_2 u, \partial_2 u)\big) + \omega(u)(\partial_1 u, \partial_2 u).
\end{aligned}$$

Finally the assumed positivity implies that g is a Riemannian metric, namely for $V \neq 0$

$$g(X)(V,V) = D_A^2 f(X,0)(e_1 \otimes V, e_1 \otimes V) = 2f(X, e_1 \otimes V) > 0.$$

This finishes the proof of the theorem. □

The calculation of the Euler-Lagrange equation for the Riemannian Dirichlet energy is straightforward using local coordinates. We compute for $\varphi \in C_c^\infty(G, \mathbb{R}^n)$

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_g(u+t\varphi)|_{t=0} &= \int_G \Big(g_{jk}(u)\partial_\alpha \varphi^j \partial_\alpha u^k + \frac{1}{2}\partial_i g_{jk}(u)\varphi^i \partial_\alpha u^j \partial_\alpha u^k\Big) \\
&= -\int_G \varphi^j \Big(g_{jk}(u)\Delta u^k + \partial_i g_{jk}(u)\partial_\alpha u^i \partial_\alpha u^k - \frac{1}{2}\partial_j g_{ik}(u)\partial_\alpha u^i \partial_\alpha u^k\Big) \\
&= -\int_G g_{jk}(u)\varphi^j \Big(\Delta u^k + g^{kl}(u)\partial_i g_{lm}(u)\partial_\alpha u^i \partial_\alpha u^m - \frac{1}{2}g^{kl}(u)\partial_l g_{im}(u)\partial_\alpha u^i \partial_\alpha u^m\Big) \\
&= -\int_G g_{jk}(u)\varphi^j \Big(\Delta u^k + \Gamma_{im}^k(u)\partial_\alpha u^i \partial_\alpha u^m\Big).
\end{aligned}$$

Here the Γ_{ij}^k are the Christoffel symbols of the metric g; we used that the term $\partial_\alpha u^i \partial_\alpha u^m$ is symmetric in i and m. For the pullback integral, we proceed as in the case of codimension one. Introducing the 3-Form $\Omega = d\omega$, we have putting $F : D \times [0,\varepsilon] \to \mathbb{R}^n$, $F(z,t) = u(z) + t\varphi(z)$,

$$\begin{aligned}
\frac{d}{d\varepsilon}\int_G (u+\varepsilon\varphi)^*\omega\,|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_{G\times[0,\varepsilon]} F^*\Omega\,|_{\varepsilon=0} \\
&= \int_G (\Omega \circ u)(u_x, u_y, \varphi) \\
&= \int_G g_{jk}(u)\varphi^j g^{kl}(u)(\Omega \circ u)(u_x, u_y, e_l).
\end{aligned}$$

In summary, the Euler-Lagrange operator $L_f(u)$ of the functional (2.17) is given by

$$L_f(u)^k = -\Delta u^k - \Gamma_{ij}^k(u)\partial_\alpha u^i \partial_\alpha u^j + g^{kl}(u)(\Omega \circ u)(u_x, u_y, e_l), \tag{2.18}$$

where Γ_{ij}^k are the Christoffelsymbols of g and $\Omega = d\omega$. The nonlinear operator

$$(\Delta^N u)^k = \Delta u^k + \Gamma_{ij}^k(u)\partial_\alpha u^i \partial_\alpha u^j$$

is sometimes called the tension field or intrinsic Laplacian of u. The system (2.18) is semilinear with principal term given by the standard Laplacian, and a right hand side which is a quadratic form of the gradient, possibly depending nonlinearly on u. The key question is now whether a regularity theory is available for such systems. Here are the bad news.

Beispiel 2.4.2. *Consider the scalar equation*

$$-\Delta u = |Du|^2 \quad \text{on } G = D_{1/e}(0).$$

We claim that the function $u(z) = \log\log\frac{1}{r}$, $r = |z|$, *belongs to* $W^{1,2}(G)$ *and solves the equation in the weak sense. For this we compute*

$$\begin{aligned} u'(r) &= \frac{1}{r\log r}, \\ u''(r) &= -\frac{1}{r^2\log r} - \frac{1}{r^2\log^2 r}, \\ |Du|^2 &= u'(r)^2 = \frac{1}{r^2\log^2 r}, \\ \Delta u &= u''(r) + \frac{1}{r}u'(r) = -\frac{1}{r^2\log^2 r}. \end{aligned}$$

Away from the origin the equation holds in the classical sense. Substituting $r = e^{-t}$ *where* $t \in [1,\infty)$ *we have*

$$\int_0^{1/e} \frac{dr}{r\log^s\frac{1}{r}} = \int_1^\infty \frac{dt}{t^s} = \begin{cases} \infty & \text{for } s = 1 \\ \frac{1}{s-1} & \text{for } s > 1. \end{cases}$$

Thus $Du \in L^2(G,\mathbb{R}^2)$. *Using cutoff arguments one now proves that* Du *is the weak derivative, and that the equation holds weakly on the full domain* G. *We also see that solutions to the Dirichlet problem may be nonunique, since* $u = 0$ *on* ∂G. *Furthermore, we have a counterexample to an* L^1 *theory for the Laplacian:* Δu *is integrable while* D^2u *is not.*

The fact that $u(x) = \log\log\frac{1}{r}$ is unbounded is important in the previous example. Namely, if the weak solution was bounded then a regularity result by Ladyzhenskaya and Ural'tseva (1961) would imply that it is Hölder continuous; further regularity would then follow easily. In the case of harmonic maps the boundedness of the weak solution is for granted, by assuming the target manifold to be compact. Nevertheless the result of Ladyzhenskaya and Ural'tseva does not apply, because it is limited to scalar equations. This is seen from the following modification of our example, due to Hildebrandt and Widman.

Beispiel 2.4.3. *The map* $u(r) = \exp(i\log\log\frac{1}{r})$ *is a bounded weak solution to the system*

$$-\Delta u = |Du|^2\Lambda u \quad \text{where } \Lambda = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In summary we see that the regularity for the prescribed mean curvature equation and for harmonic maps is subtle, and that the particular structure of the nonlinearity needs to be exploited in some way.

Chapter 3

Harmonic maps into spheres

3.1 Formulation as a conservation law

Let $B = \{x \in \mathbb{R}^d : |x| < 1\}$ where $d \geq 2$ is arbitrary. For a smooth compact submanifold $M \subset \mathbb{R}^n$, we consider the class of Sobolev mappings

$$W^{1,2}(B, M) = \{u \in W^{1,2}(B, \mathbb{R}^n) : u(x) \in M \text{ almost everywhere}\}. \tag{3.1}$$

We say that $u \in W^{1,2}(B, M)$ is (weakly) harmonic if and only if

$$\int_B \langle Du, D\phi \rangle = \int_B \langle A(u)(\partial_\alpha u, \partial_\alpha u), \phi \rangle \quad \text{for all } \phi \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^n). \tag{3.2}$$

Here A is the second fundamental form of M, taking values in the normal bundle of M. In Chapter 1 we gave the same definition, except that we allowed only $\phi \in C_c^\infty(B, \mathbb{R}^n)$. But (3.2) is equivalent by the following approximation argument.

Given $\phi \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^n)$, we first chose $\psi_k \in C_c^\infty(B, \mathbb{R}^n)$ with $\psi_k \to \phi$ in $W^{1,2}(B, \mathbb{R}^n)$ and almost everywhere. For $R > \|\phi\|_{L^\infty(B)}$ we define the functions $\phi_k \in W_0^{1,2}(B, \mathbb{R}^n)$ by

$$\phi_k(x) = \begin{cases} \psi_k(x) & \text{for } |\psi_k(x)| < R, \\ R\, \pi_{\mathbb{S}^{n-1}}(\varphi_k(x)) & \text{for } |\varphi_k(x)| \geq R. \end{cases}$$

Here $\pi_{\mathbb{S}^{n-1}}(X) = X/|X|$ is the projection map. For a. e. $x \in B$ we have $|\psi_k(x)| \to |\phi(x)| < R$, which implies $\phi_k(x) = \psi_k(x) \to \phi(x)$ as $k \to \infty$. This shows $\phi_k \to \phi$ almost everywhere. Now $D\pi_{\mathbb{S}^{n-1}}$ has bounded norm. Thus $|\phi_k| \leq R$ and $|D\phi_k| \leq C|D\psi_k|$, which implies $\phi_k \to \phi$ weakly in $W^{1,2}(B, \mathbb{R}^n)$ up to a subsequence. It follows that the left hand side of (3.2) converges. The right hand side also passes to the limit by a. e. convergence, using the bound

$$|A(u)(\partial_\alpha u, \partial_\alpha u)\phi_k| \leq CR\, |Du|^2 \in L^1(B).$$

By the standard product rule the space $W^{1,2} \cap L^\infty(B)$ is an algebra; this is used several times below. Moreover if one of the functions belongs to $W_0^{1,2}(B, \mathbb{R}^n)$, then this is also true for the product.

In the present section we focus on $M = \mathbb{S}^{n-1}$. In this case there is a useful reformulation of the equation due to Yun Mei Chen and Jalal Shatah.

Theorem 3.1.1 ([8, 42])**.** *For $u \in W^{1,2}(B, \mathbb{S}^{n-1})$ the following are equivalent:*

(a) *u is weakly harmonic.*

(b) *For any $\Lambda \in \mathbb{R}^{n\times n}$ with $\Lambda^{\mathrm{T}} = -\Lambda$ we have $\operatorname{div}(Du^{\mathrm{T}}\Lambda u) = 0$, that is*

$$\int_B \langle Du \cdot \operatorname{grad}\varphi, \Lambda u\rangle = 0 \quad \textit{for all } \varphi \in C_c^\infty(B). \tag{3.3}$$

Proof. Let u be weakly harmonic. Along u the map $\varphi\Lambda u$ is tangential as $\langle \Lambda u, u\rangle = 0$, and belongs to $W_0^{1,2} \cap L^\infty(B, \mathbb{R}^n)$. Thus we get from the equation, using that $A(u)(\partial_\alpha u, \partial_\alpha u)$ is normal and $\langle Du, \Lambda Du\rangle = 0$,

$$0 = \int_B \langle Du, D(\varphi\Lambda u)\rangle = \int_B \partial_\alpha\varphi\langle Du \cdot e_\alpha, \Lambda u\rangle = \int_B \langle Du \cdot \operatorname{grad}\varphi, \Lambda u\rangle.$$

For the reverse direction, we consider for $1 \le i < j \le n$ the tangent vectorfields

$$\Lambda_{ij}\omega = (e_i \otimes e_j - e_j \otimes e_i)\omega = \omega^i e_j - \omega^j e_i \in T_\omega\mathbb{S}^{n-1}.$$

The $\Lambda_{ij}\omega$ span $T_\omega\mathbb{S}^{n-1}$, in fact we have for any $\xi \in T_\omega\mathbb{S}^{n-1}$ the representation

$$\xi = (\omega \otimes \xi - \xi \otimes \omega)\omega = \sum_{1\le i<j\le n} (\omega^i\xi^j - \omega^j\xi^i)\Lambda_{ij}\omega.$$

For a given variation $\phi \in C_c^\infty(B, \mathbb{R}^n)$ we obtain

$$\begin{aligned} \phi &= \langle \phi, u\rangle u + \sum_{1\le i<j\le n} \varphi_{ij}\Lambda_{ij}u, \quad \text{where} \\ \varphi_{ij} &= u^i(\phi^j - \langle\phi, u\rangle u^j) - u^j(\phi^i - \langle \phi, u\rangle u^i). \end{aligned} \tag{3.4}$$

Now $\varphi_{ij} \in W_0^{1,2} \cap L^\infty(B)$, and (b) holds for $W_0^{1,2}$-functions by approximation. Thus

$$\int_B \langle Du, D(\varphi_{ij}\Lambda_{ij}u)\rangle = \int_B \langle Du \cdot \operatorname{grad}\varphi_{ij}, \Lambda_{ij}u\rangle = 0.$$

Furthermore $\langle\phi, u\rangle u \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^n)$, and

$$\int_B \langle Du, D(\langle \phi, u\rangle u)\rangle = \int_B |Du|^2\langle u, \phi\rangle.$$

The claim follows using the representation (3.4). □

Equation (3.3) may also be derived by Noether's theorem. For any map u the Dirichlet energy is invariant under rotations $R(t)u = \exp(t\Lambda)u$. This implies for any $\Omega \subset B$

$$0 = \frac{d}{dt}\mathcal{E}(R(t)u, \Omega)|_{t=0} = \int_\Omega \langle Du, D(\Lambda u)\rangle = \int_\Omega \partial_\alpha\langle\partial_\alpha u, \Lambda u\rangle - \int_\Omega \langle \Delta u, \Lambda u\rangle.$$

If $u : B \to \mathbb{S}^{n-1}$ is harmonic, then $(\Delta u)^\top = 0$ and the second integral vanishes. Since Ω was arbitrary we get the conservation law

$$\operatorname{div}(Du^{\mathrm{T}}u) = \partial_\alpha\langle\partial_\alpha u, \Lambda u\rangle = 0 \quad \text{in } B.$$

The regularity of u was used in this argument: since Λu does not have compact support in Ω the weak equation does apply without further justification. On the other hand, the interpretation by Noether's theorem shows more clearly the key rôle of the Killing fields for the result, that is vector fields which generate isometries. In fact, the theorem was generalized to manifolds M with isometry group acting transitively by Frédéric Hélein in [24].

As an application of Theorem 3.1.1 we now show that the set of weakly harmonic maps is closed under weak convergence. This is not clear, because the weak convergence of $Du_k \to Du$ in $L^2(B, \mathbb{R}^{n\times d})$ does not obviously imply $A(u_k)(Du_k, Du_k) \to A(u)(Du, Du)$ in the sense of distributions. Still, the proof is trivial using the reformulation as a conservation law.

Corollary 3.1.2. *Let $u_k \in W^{1,2}(B, \mathbb{S}^{n-1})$ be weakly harmonic, and suppose $u_k \to u$ weakly in $W^{1,2}(B, S^{n-1})$. Then $u \in W^{1,2}(B, \mathbb{S}^{n-1})$ is also weakly harmonic.*

Proof. We have $Du_k \to Du$ weakly in $L^2(\mathbb{R}^{n\times d})$, and $u_k \to u$ strongly in $L^2(B, \mathbb{R}^n)$ by Rellich's theorem. Then $(\partial_\alpha \varphi)\Lambda u_k \to (\partial_\alpha\varphi)\Lambda u$ strongly in $L^2(B, \mathbb{R}^n)$ for $\varphi \in C_c^\infty(B)$, and

$$\int_B \langle Du \cdot \operatorname{grad}\varphi, \Lambda u\rangle = \lim_{k\to\infty} \int_B \langle Du_k \cdot \operatorname{grad}\varphi, \Lambda u_k\rangle.$$

The result now follows by Theorem 3.1.1. □

3.2 Wente's inequality

In this section we are back to dimension $d = 2$. Let us start with the classical Dirichlet problem for the Poisson equation, assuming that

$$-\Delta u = f \text{ weakly in } D \qquad \text{where } u \in W_0^{1,2}(D).$$

A very fundamental estimate, due to Calderon and Zygmund, asserts that if $f \in L^p(D)$ where $1 < p < \infty$, then $u \in W^{2,p}(D)$ and

$$\|u\|_{W^{2,p}(D)} \leq C(p)\, \|f\|_{L^p(D)}.$$

In the case of the harmonic map or prescribed mean curvature systems, this does not apply directly because the right hand side, being quadratic in the gradient, belongs a priori only to $L^1(D)$. The Calderon-Zygmund theory does not extend to the space L^1; the difficulty was already observed in Example 2.4.2. Henry Wente found that this drawback can be compensated if the right hand side has a special algebraic structure, namely when f is a Jacobi determinant. We refer to Hélein's book [26] for an in-depth discussion.

Theorem 3.2.1 ([53])**.** *For $a, b \in W^{1,2}(D)$ there is a unique function $u \in W_0^{1,2}(D)$ solving*

$$\int_D \langle du, d\varphi\rangle = \int_D \varphi\{a,b\} \text{ for all } \varphi \in W_0^{1,2} \cap L^\infty(D), \qquad \text{where } \{a,b\} = a_y b_y - a_y b_x. \tag{3.5}$$

The solution belongs to $C^0(\overline{D})$, and satisfies the a priori estimates

$$||u||_{C^0(\overline{D})} + ||du||_{L^2(D)} \leq C\, \|da||_{L^2(D)}|\, |db||_{L^2(D)}. \tag{3.6}$$

Proof. The uniqueness of the solution is trivial. For existence the key step is to prove the estimates, assuming that a, b and hence u are smooth on the closed disk. The rest will follow by a simple approximation argument. We also observe before that the statement of the theorem is conformally invariant: if $\phi : U \to D$ is a conformal diffeomorphism with pullback metric $e^{2\lambda}\delta_{\alpha\beta}$, then according to equation (2.4)

$$\Delta(u \circ \phi) = e^{2\lambda}\Delta u \circ \phi.$$

On the other hand we compute

$$\begin{aligned} \{a \circ \phi, b \circ \phi\} dx \wedge dy &= d(a \circ \phi) \wedge d(b \circ \phi) \\ &= d(\phi^* a) \wedge d(\phi^* b) \\ &= \phi^*(da) \wedge \phi^*(db) \\ &= \phi^*(da \wedge db) \\ &= e^{2\lambda}\{a, b\} \circ \phi \; dx \wedge dy. \end{aligned}$$

This shows $-\Delta(u \circ \phi) = \{a \circ \phi, b \circ \phi\}$, i.e. the conformal invariance of the equation. Moreover the L^2 norms of du, da and db and also the C^0 norm of u are invariant. Now for given $a \in D$ there is an automorphism ϕ of the disk with $\phi(0) = a$. The well-known formula is

$$\phi(z) = \frac{z + a}{1 + \overline{a}z} \qquad \text{for } z \in D.$$

To get the C^0 estimate it is then sufficient to bound u at the origin, because $u(a) = (u \circ \phi)(0)$. The value at the origin is given by Green's formula (writing $r = |z|$ and assuming a, b smooth)

$$\begin{aligned} u(0) &= \frac{1}{2\pi}\int_D (\log r)\, da \wedge db \\ &= \frac{1}{2\pi}\lim_{\varepsilon \searrow 0}\int_{D\setminus D_\varepsilon(0)} d\big((\log r) a\, db\big) - \frac{1}{2\pi}\lim_{\varepsilon \searrow 0}\int_{D\setminus D_\varepsilon(0)} \frac{dr}{r}\, a \wedge db \\ &= -\frac{1}{2\pi}\int_0^1 \frac{dr}{r}\int_0^{2\pi} a(r,\varphi)\frac{\partial b}{\partial \varphi}(r,\varphi)\, d\varphi. \end{aligned}$$

Let $\hat{a}(r)$ be the mean value of $a(r, \cdot)$ on $[0, 2\pi]$, and estimate

$$\begin{aligned} \Big|\int_0^{2\pi} a(r,\varphi)\frac{\partial b}{\partial \varphi}(r,\varphi)\, d\varphi\Big| &= \Big|\int_0^{2\pi} (a(r,\varphi) - \hat{a}(r))\frac{\partial b}{\partial \varphi}(r,\varphi)\, d\varphi\Big| \\ &\leq \big\|a - \hat{a}(r)\big\|_{L^2(0,2\pi)} \cdot \Big\|\frac{\partial b}{\partial \varphi}\Big\|_{L^2(0,2\pi)} \\ &\leq \Big\|\frac{\partial a}{\partial \varphi}\Big\|_{L^2(0,2\pi)} \cdot \Big\|\frac{\partial b}{\partial \varphi}\Big\|_{L^2(0,2\pi)}. \end{aligned}$$

In the last step we used the Poincaré inequality on $(0, 2\pi)$ for functions having zero mean value; this follows easily by Fourier expansion. We conclude

$$\begin{aligned} |u(0)| &\leq \frac{1}{2\pi}\int_0^1 \Big\|\frac{\partial a}{\partial \varphi}\Big\|_{L^2(0,2\pi)} \cdot \Big\|\frac{\partial b}{\partial \varphi}\Big\|_{L^2(0,2\pi)} \frac{dr}{r} \\ &\leq \frac{1}{2\pi}\Big(\int_0^1\int_0^{2\pi} \frac{1}{r^2}\Big|\frac{\partial a}{\partial \varphi}\Big|^2 r dr d\varphi\Big)^{1/2} \cdot \Big(\int_0^1\int_0^{2\pi} \frac{1}{r^2}\Big|\frac{\partial b}{\partial \varphi}\Big|^2 r dr d\varphi\Big)^{1/2} \\ &\leq \frac{1}{2\pi}\|da\|_{L^2(D)} \cdot \|db\|_{L^2(D)}. \end{aligned}$$

The L^2 bound for the gradient now follows simply by testing with u:

$$\int_D |du|^2 = \int_D u\{a,b\} \le \|u\|_{C^0(D)}\|da\|_{L^2(D)}\|db\|_{L^2(D)} \le \frac{1}{2\pi}\|da\|^2_{L^2(D)}\|db\|^2_{L^2(D)}.$$

In the smooth case the estimates are settled. Given $a, b \in W^{1,2}(D)$ we approximate $a_k \to a$, $b_k \to b$ in $W^{1,2}(D)$, where a_k, b_k are in $C^\infty(\overline{D})$. Let u_k be the solution of the Dirichlet problem

$$-\Delta u_k = \{a_k, b_k\} \text{ in } D, \quad u_k|_{\partial D} = 0.$$

Then $u_k - u_l$ is zero on ∂D and satisfies

$$-\Delta(u_k - u_l) = \{a_k, b_k\} - \{a_l, b_l\} = \{a_k - a_l, b_k\} + \{a_l, b_k - b_l\}.$$

We have $\|da_k\|_{L^2(D)} + \|db_k\|_{L^2(D)} \le C$ for all k. By uniqueness and the estimates, we obtain

$$\begin{aligned}\|u_k - u_l\|_{C^0(\overline{D})} + \|d(u_k - u_l)\|_{L^2(D)} &\le C\left(\|d(a_k - a_l)\|_{L^2(D)} + \|d(b_k - b_l)\|_{L^2(D)}\right)\\ &\to 0 \quad \text{as } k, l \to \infty.\end{aligned}$$

Thus $u_k \to u$ in both $W_0^{1,2}(D)$ and $C^0(\overline{D})$, and u satisfies the desired estimates. To get the weak equation for $\varphi \in W_0^{1,2} \cap L^\infty(D)$, we compute

$$\int_D \langle du, d\varphi\rangle = \lim_{k\to\infty}\int_D \langle du_k, d\varphi\rangle = \lim_{k\to\infty}\int_D \varphi\{a_k, b_k\} = \int_D \varphi\{a,b\}.$$

□

As a consequence of Wente's theorem, we get the following regularity result.

Corollary 3.2.2. *Let $a, b \in W^{1,2}(D)$ and assume that $v \in L^1(D)$ solves*

$$-\int_D v\Delta\varphi = \int_D \varphi\{a,b\} \quad \textit{for all } \varphi \in C^\infty(\overline{D}),\ \varphi|_{\partial D} = 0. \tag{3.7}$$

Then $v \in W^{1,2}(D) \cap C^0(\overline{D})$, and it satisfies the estimates (3.6).

Remark 3.2.3. *Equation 3.7 holds for any $v \in W_0^{1,p}(D)$, $p \in [1,2]$, solving*

$$\int_D \langle Dv, D\varphi\rangle = \int_D \varphi\{a,b\} \quad \textit{for all } \varphi \in C^\infty(\overline{D}),\ \varphi|_{\partial D} = 0.$$

In fact, by Sobolev trace theory one gets using $v = 0$ on ∂D

$$\int_D \langle Dv, D\varphi\rangle = \int_{\partial D} v\frac{\partial\varphi}{\partial r} - \int_D v\Delta\varphi = -\int_D v\Delta\varphi.$$

Proof. The remark applies to $u \in W_0^{1,2}(D)$ as in Theorem 3.2.1. Subtracting yields

$$\int_D (u - v)\Delta\varphi = 0 \quad \text{for all } \varphi \in C^\infty(\overline{D}), \varphi|_{\partial D} = 0.$$

Schwarz reflection extends $u - v$ to a (weakly, thus classically) harmonic function on $\mathbb{R}^2$, which proves $v \in W^{1,2}(D) \cap C^0(\overline{D})$. In fact, $v = u$ by the uniqueness part of Theorem 3.2.1. □

3.3 Regularity of harmonic maps $u \in W^{1,2}(D, \mathbb{S}^{n-1})$

The regularity of two-dimensional harmonic maps into submanifolds of $\mathbb{R}^n$ was proved by F. Hélein around 1990, in three papers with increasing generality. The first studied the case where the target manifold is a round sphere. As an auxiliary tool we will need a Hodge decomposition on the disk, which is discussed now.

For $k = 0, 1, 2$ let $W^{1,2}(D, \Lambda^k)$ be the $W^{1,2}$-space of alternating k-linear forms. The Hodge star, the exterior derivative d and its L^2 adjoint d^* are as in the following table[1]:

	$*$	d	d^*
$u \in W^{1,2}(D)$	$u\,dx \wedge dy$	$u_x\,dx + u_y\,dy$	0
$a\,dx + b\,dy \in W^{1,2}(D, \Lambda^1)$	$a\,dy - b\,dx$	$(b_x - a_y)\,dx \wedge dy$	$-(a_x + b_y)$
$u\,dx \wedge dy \in W^{1,2}(D, \Lambda^2)$	u	0	$u_y\,dx - u_x\,dy.$

From the table one easily checks that $d^* = - * d *$ and $dd^* + d^*d = -\Delta$ on forms of class $W^{2,2}$; in our notation $\Delta = \partial_x^2 + \partial_y^2$ acting on the coefficients. One further has the following rules of partial integration, where $i : \partial D \to \mathbb{R}^2$ is the inclusion map:

$$\begin{aligned} \int_D \langle du, \omega \rangle - \int_D u\, d^*\omega &= \int_{\partial D} u\, i^*(*\omega) \quad \text{for } u \in W^{1,2}(D),\ \omega \in W^{1,2}(D, \Lambda^1), \\ \int_D \langle d\omega, \theta \rangle - \int_D \langle \omega, d^*\theta \rangle &= \int_{\partial D} (*\theta)\, i^*\omega \quad \text{for } \omega \in W^{1,2}(D, \Lambda^1),\ \theta \in W^{1,2}(D, \Lambda^2). \end{aligned}$$

The following is a special case of the decomposition theorem of W. Hodge.

Lemma 3.3.1 ([28]). *Any $\eta \in L^2(D, \Lambda^1)$ has an orthogonal decomposition*

$$\eta = d\alpha + d^*\beta \quad \text{where } \alpha \in W_0^{1,2}(D),\ \beta \in W^{1,2}(D, \Lambda^2).$$

α, β are unique, assuming $\int_{\partial D} *(\nu \llcorner \beta) = 0$. Moreover if $d^*\eta = 0$ then $\alpha = 0$.

Proof. We start with uniqueness. For $\alpha \in C_c^\infty(D)$, $\beta \in W^{1,2}(D, \Lambda^2)$ we have

$$\int_D \langle d\alpha, d^*\beta \rangle = \int_D \langle d(d\alpha), \beta \rangle - \int_{\partial D} (*\beta) i^*(d\alpha) = 0.$$

This proves $dW_0^{1,2}(D) \perp d^*W^{1,2}(D, \Lambda^2)$. Next assume that $d^*\eta = 0$. Multiplying the decomposition $\eta = d\alpha + d^*\beta$ with $d\zeta$ and integrating yields

$$0 = \int_D \langle \eta, d\zeta \rangle = \int_D \langle d\alpha, d\zeta \rangle \quad \text{for all } \zeta \in W_0^{1,2}(D).$$

Putting $\zeta = \alpha$ shows $\alpha = 0$. Now if $\eta = 0$, then $d^*\beta = 0$ which implies β constant. The normalization then yields $\beta = 0$, which proves the uniqueness.

[1] Identifying co-/vectors one may write $d^*(u\,dx \wedge dy) = - * du = \nabla^\perp u$ where $\nabla^\perp u = (u_y, -u_x)$.

For existence we introduce the Sobolev space

$$W^{1,2}_{\top}(D, \Lambda^1) = \{\omega \in W^{1,2}(D, \Lambda^1) : i^*\omega = 0\},$$

and the bounded , symmetric operator $L : W^{1,2}_{\top}(D, \Lambda^1) \to W^{1,2}_{\top}(D, \Lambda^1)'$ given by

$$\langle L\omega, \varphi\rangle = \int_D \big(\langle d\omega, d\varphi\rangle + \langle d^*\omega, d^*\varphi\rangle\big).$$

We show that L is strongly coercive. Namely we have for $\omega = a\,dx + b\,dy \in W^{1,2}_{\top}(D, \Lambda^1)$

$$\begin{aligned}
\langle L\omega, \omega\rangle &= \int_D \big((b_x - a_y)^2 + (a_x + b_y)^2\big) \\
&= \int_D \big(|da|^2 + |db|^2\big) + 2\int_D da \wedge db \\
&= \int_D \big(|da|^2 + |db|^2\big) + \int_{\partial D} \big(ab_\theta - a_\theta b\big).
\end{aligned}$$

By assumption $(a, b) = \lambda(\cos\theta, \sin\theta)$ along ∂D, which yields

$$\begin{aligned}
a_\theta &= \lambda_\theta \cos\theta - \lambda \sin\theta, \\
b_\theta &= \lambda_\theta \sin\theta + \lambda \cos\theta.
\end{aligned}$$

Inserting yields $ab_\theta - a_\theta b = \lambda^2(\cos^2\theta + \sin^2\theta) = |\omega|^2$, that is

$$\langle L\omega, \omega\rangle = \int_D |D\omega|^2 + \int_{\partial D} |\omega|^2.$$

Now by a standard indirect argument using Rellich's lemma we get the estimate

$$\|\omega\|^2_{L^2(D)} \le C(\|D\omega\|^2_{L^2(D)} + \|\omega\|^2_{L^2(\partial D)}),$$

completing the proof of coercivity. The Riesz representation theorem applies to show that for any $\eta \in L^2(D, \Lambda^1)$ there is a unique $\omega \in W^{1,2}_{\top}(D, \Lambda^1)$ such that

$$\langle L\omega, \varphi\rangle = \langle \eta, \varphi\rangle_{L^2(D)} \quad \text{for all } \varphi \in W^{1,2}_{\top}(D, \Lambda^1).$$

Now $\omega \in W^{2,2}_{\top}(D, \Lambda^1)$ by regularity theory (see discussion below), thus

$$\begin{aligned}
\int_D \langle d\omega, d\varphi\rangle &= \int_D \langle d^*d\omega, \varphi\rangle + \int_{\partial D} (*d\omega)\underbrace{i^*\varphi}_{=0}, \\
\int_D \langle d^*\omega, d^*\varphi\rangle &= \int_D \langle dd^*\omega, \varphi\rangle - \int_{\partial D} (d^*\omega) i^*(*\varphi).
\end{aligned}$$

Taking $\varphi \in C^\infty_c(D, \Lambda^1)$ shows $(dd^* + d^*d)\omega = \eta$. Furthermore, any $\chi \in C^\infty(\partial D)$ extends to $\chi \in C^\infty(\overline{D})$ with $\chi = 0$ on $D_{1/2}(0)$. Taking $\varphi = \chi\,dr$ and using $*dr = d\theta$ on ∂D yields

$$0 = \int_{\partial D} (d^*\omega) i^*(*\chi\,dr) = \int_{\partial D} (d^*\omega)\chi,$$

hence $d^*\omega = 0$ on ∂D. Taking $\alpha = d^*\omega$ and $\beta = d\omega$ proves the desired decomposition. □

We include an ad hoc argument for $W^{2,2}$ regularity. Let H be the upper halfplane and $\phi : H \to D$ a conformal equivalence, with pullback metric $g_{\alpha\beta} = e^{2\lambda}\delta_{\alpha\beta}$. For 1-forms ω and 2-forms θ one has the following transformation rules:

$$\begin{aligned}
\phi^*(d^*\omega) &= d^*_g(\phi^*\omega) = e^{-2\lambda} d^*(\phi^*\omega),\\
\phi^*(d^*\theta) &= d^*_g(\phi^*\theta) = d^*(e^{-2\lambda}\phi^*\theta).
\end{aligned}$$

From this one computes putting $\gamma = \phi^*\omega$

$$\phi^*\big(-\Delta\omega\big) = e^{-2\lambda}\big(-\Delta\gamma - 2(d\lambda)\, d^*\gamma + 2(*d\lambda) * d\gamma\big).$$

The form ω in the proof satisfies $-\Delta\omega = \eta$, with boundary conditions $i^*\omega = 0$, $d^*\omega = 0$ on ∂D. Therefore $\gamma = \phi^*\omega$ solves the boundary value problem

$$\begin{aligned}
-\Delta\gamma - 2(d\lambda)\, d^*\gamma + 2(*d\lambda) * d\gamma &= e^{2\lambda}\phi^*\eta,\\
i^*_{\partial H}\gamma &= 0 \quad \text{on } \partial H,\\
d^*\gamma &= 0 \quad \text{on } \partial H.
\end{aligned}$$

Writing $\gamma = u\,dx + v\,dy$ the first equation on the boundary becomes $u = 0$ on ∂H, a Dirichlet condition. Using $d^*\gamma = -(u_x + v_y)$ the second equation is reduced to $v_y = 0$ on ∂H, a Neumann condition. Now $W^{2,2}$ regularity is reduced to estimates for the scalar Laplacian under these boundary conditions. The equations are coupled via the first order terms, however these can be put on the right hand side, since ω is already estimated in $W^{1,2}$ by coercivity. Recalling the definition of α and β we arrive at the estimate

$$\|\alpha\|_{W^{1,2}(D)} + \|\beta\|_{W^{1,2}(D)} \le C\,\|\eta\|_{L^2(D)}. \tag{3.8}$$

For $\eta \in L^p(D, \Lambda^1)$ where $p \in (1, \infty)$, one can also derive $W^{1,p}(D)$ estimates for α and β using standard Calderon-Zygmund theory.

Theorem 3.3.2 ([23]). *Any 2-dimensional harmonic map $u \in W^{1,2}(D, \mathbb{S}^{n-1})$ is continuous.*

Proof. The proof is divided into two steps: first one shows that the map is continuous, then one shows smoothness. The second step was known previously to the work of Hélein; it is not specific to harmonic maps but applies to general elliptic systems with quadratic nonlinearity in the gradient. Here we focus on the first step, and deal with the second afterwards.

Using the identity $u^j du^j = \frac{1}{2} d|u|^2 = 0$, we write the harmonic map equation in the form

$$-\Delta u^i = |Du|^2 u^i = \langle du^j, u^i du^j - u^j du^i \rangle.$$

From the conservation law in Theorem 3.1.1 we know that

$$d^*(u^i du^j - u^j du^i) = 0 \quad \text{weakly for } 1 \le i, j \le n.$$

Lemma 3.3.1, the Hodge lemma, yields forms $\beta^{ij} = b^{ij} dx \wedge dy \in W^{1,2}(D, \Lambda^2)$ with

$$u^i du^j - u^j du^i = d^*\beta^{ij} = - * db^{ij}.$$

Inserting yields, recalling the notation $\{a, b\} = a_x b_y - a_y b_x$,

$$-\Delta u^i = -\langle du^j, *db^{ij} \rangle = \{u^j, b^{ij}\}.$$

The right hand side has the desired Jacobi determinant structure. To arrange for zero boundary values, we let $h \in W^{1,2}(D, \mathbb{R}^n)$ be the harmonic extension of $u|_{\partial D}$. Then $v = u - h \in W^{1,2}(D, \mathbb{R}^n)$ solves the problem

$$-\Delta v^i = \{u^j, b^{ij}\} \text{ in } D, \quad v = 0 \text{ on } \partial D.$$

We have $v \in C^0(\overline{D}, \mathbb{R}^n)$ by Wente, see Theorem 3.2.1. But the harmonic function h is smooth in D, which proves the continuity of $u = v + h$. □

We now take up the problem of higher regularity for systems of harmonic map type in arbitrary dimensions. This goes back to S. Hildebrandt, K.-O. Widman, and M. Wiegner

Theorem 3.3.3 ([27, 54]). *Let $u \in W^{1,2} \cap L^\infty(B_2(0), \mathbb{R}^n)$ be a weak solution of the equation $-\Delta u = A(u)(Du, Du)$ on $B_2(0) \subset \mathbb{R}^m$. Assume that for constants $a, M < \infty$*

$$|A(z)(p,p)| \leq a\,|p|^2 \quad \text{for all } |z| \leq M,\, p \in \mathbb{R}^{n \times m}, \tag{3.9}$$

$$\|u\|_{L^\infty(B_2(0))} \leq M. \tag{3.10}$$

Then the following holds:

(1) *Let $\alpha \in (0,1)$. If $aM \leq \varepsilon_0 = \varepsilon_0(\alpha)$ then $u \in C^{0,\alpha}(B_1(0), \mathbb{R}^n)$.*

(2) *For $\alpha \in (\frac{2}{3}, 1)$ we get further $u \in C^{1,\mu}(B_1(0), \mathbb{R}^n)$, where $\mu = \frac{3}{2}\alpha - 1 \in (0, \frac{1}{2})$.*

Proof. For $x \in B_1(0)$ and $\varrho \in (0,1]$, let $v \in W^{1,2}(B_\varrho(x), \mathbb{R}^n)$ be harmonic with $v - u \in W_0^{1,2}(B_\varrho(x))$. We have the standard estimates

$$\begin{aligned} \sup_{B_\varrho(x)} |v| &\leq \|u\|_{L^\infty(B_\varrho(x))} \leq M, \\ \varrho \sup_{B_{\varrho/2}(x)} |Dv| + \varrho^2 \sup_{B_{\varrho/2}(x)} |D^2 v| &\leq \frac{C}{\varrho^{m/2-1}} \|Dv\|_{L^2(B_\varrho(x))}. \end{aligned}$$

For $w = u - v$ and $\phi \in W_0^{1,2} \cap L^\infty(B_\varrho(x), \mathbb{R}^n)$ we infer

$$\int_{B_\varrho(x)} \langle Dw, D\phi \rangle = \int_{B_\varrho(x)} \langle A(u)(Du, Du), \phi \rangle.$$

We take $\phi = w$. Using $|w| \leq |u| + |v| \leq 2M$ almost everywhere, we estimate

$$\int_{B_\varrho(x)} |Dw|^2 = \int_{B_\varrho(x)} \langle A(u)(Du, Du), w \rangle \leq 2aM \int_{B_\varrho(x)} |Du|^2.$$

Now let $\theta \in (0, \frac{1}{2}]$. Using $|Du|^2 \leq 2(|Dv|^2 + |Dw|^2)$ we get

$$\begin{aligned} (\theta\varrho)^{2-m} \int_{B_{\theta\varrho}(x)} |Du|^2 &\leq 2(\theta\varrho)^{2-m} \int_{B_{\theta\varrho}(x)} |Dv|^2 + 2(\theta\varrho)^{2-m} \int_{B_{\theta\varrho}(x)} |Dw|^2 \\ &\leq C\theta^m (\theta\varrho)^{2-m} \int_{B_\varrho(x)} |Dv|^2 + 4aM(\theta\varrho)^{2-m} \int_{B_\varrho(x)} |Du|^2 \\ &\leq C\theta^2 \big(1 + aM\theta^{-m}\big)\, \varrho^{2-m} \int_{B_\varrho(x)} |Du|^2. \end{aligned}$$

In the last step we used that $v(x)$ minimizes the Dirichlet energy with given boundary values. Assume for the moment that

$$aM \leq \theta^m, \tag{3.11}$$

so that for any $x \in B_1(0)$, $\varrho \in (0,1]$ we have the inequality

$$\phi(x,\theta\varrho) \leq C\theta^2\phi(x,\varrho) \quad \text{where } \phi(x,\varrho) = \varrho^{2-m}\int_{B_\varrho(x)} |Du|^2.$$

Given $\varrho \in (0,1]$ we choose $k \in \mathbb{N}_0$ with $\theta^{k+1} < \varrho \leq \theta^k$, and iterate

$$\begin{aligned}\phi(x,\varrho) &\leq \theta^{2-m}\phi(x,\theta^k)\\ &\leq \theta^{2-m}(C\theta^2)^k\phi(x,1)\\ &\leq \theta^{2-m-2\alpha}(C\theta^{2-2\alpha})^k\,\varrho^{2\alpha}\int_{B_2(0)}|Du|^2.\end{aligned}$$

For given $\alpha \in [0,1)$ we chose $\theta = \theta(\alpha) \in (0,\frac{1}{2}]$ with $C\theta^{2-2\alpha} \leq 1$, and take $\varepsilon_0 = \theta^m$ in assumption (1). Then (3.11) holds, and we conclude

$$\varrho^{2-m}\int_{B_\varrho(x)}|Du|^2 \leq C(\alpha)\varrho^{2\alpha}\int_{B_2(0)}|Du|^2.$$

By Morrey's Dirichlet growth theorem $u(x)$ is α-Hölder continuous on $B_1(0)$. To prove that Du is also Hölder continous, go back to the inequality

$$⨍_{B_\varrho(x)}|Dw|^2 = ⨍_{B_\varrho(x)}\langle A(u)(Du,Du),w\rangle \leq a\sup_{B_\varrho(x)}|w|⨍_{B_\varrho(x)}|Du|^2.$$

Now $w = u - v = (u - u_{x,\varrho}) - (v - u_{x,\varrho})$, where $u_{x,\varrho}$ is the mean value on $B_\varrho(x)$. We know already $|u - u_{x,\varrho}| \leq C\varrho^\alpha$, furthermore the maximum principle yields, as $v = u$ on $\partial B_\varrho(x)$,

$$\sup_{B_\varrho(x)}|v - u_{x,\varrho}| \leq \sup_{\partial B_\varrho(x)}|u - u_{x,\varrho}| \leq C\varrho^\alpha.$$

Thus we have the estimate

$$⨍_{B_\varrho(x)}|Dw|^2 \leq C\varrho^\alpha⨍_{B_\varrho(x)}|Du|^2 \leq C\,\varrho^\alpha\varrho^{2\alpha-2} = C\varrho^{3\alpha-2}.$$

For convenience we put $3\alpha - 2 =: 2\mu$. Clearly

$$\begin{aligned}⨍_{B_{\theta\varrho}(x)}|Dw|^2 &\leq \theta^{-m}⨍_{B_\varrho(x)}|Dw|^2 \leq C\theta^{-m}\varrho^{2\mu},\\ |Dw_{x,\theta\varrho}|^2 &\leq ⨍_{B_{\theta\varrho}(x)}|Dw|^2 \leq C\theta^{-m}\varrho^{2\mu}.\end{aligned}$$

Here we use the notation $f_{x,\varrho} = ⨍_{B_\varrho(x)} f$. We compute further, recalling $u = v + w$,

$$\begin{aligned}⨍_{B_{\theta\varrho}(x)}|Du - (Du)_{x,\theta\varrho}|^2 &\leq 2⨍_{B_{\theta\varrho}(x)}|Dv - (Dv)_{x,\theta\varrho}|^2 + 2⨍_{B_{\theta\varrho}(x)}|Dw|^2 + 2|(Dw)_{x,\theta\varrho}|^2\\ &\leq 2⨍_{B_{\theta\varrho}(x)}|Dv - (Dv)_{x,\theta\varrho}|^2 + C\theta^{-m}\varrho^{2\mu}.\end{aligned}$$

Using the Poincaré inequality and interior estimates for harmonic functions, we obtain

$$\begin{aligned}
\fint_{B_{\theta\varrho}(x)} |Dv - (Dv)_{x,\theta\varrho}|^2 &\leq C(\theta\varrho)^2 \fint_{B_{\theta\varrho}(x)} |D^2 v|^2 \\
&\leq C(\theta\varrho)^2 \varrho^{-2} \fint_{B_\varrho(x)} |Dv - (Du)_{x,\varrho}|^2 \\
&\leq C\theta^2 \fint_{B_\varrho(x)} |D(v-\ell)|^2 \quad \text{where } \ell(y) = (Du)_{x,\varrho}\, y \\
&\leq C\theta^2 \fint_{B_\varrho(x)} |Du - (Du)_{x,\varrho}|^2.
\end{aligned}$$

In the last step the minimizing property of $v - \ell$ was used. Combining we arrive at

$$\phi(x, \theta\varrho) \leq C\theta^2 \phi(x, \varrho) + C\theta^{-m} \varrho^{2\mu} \quad \text{where } \phi(x, \varrho) = \fint_{B_\varrho(x)} |Du - (Du)_{x,\varrho}|^2.$$

Using induction, we see that

$$\begin{aligned}
\phi(x, \theta^k) &\leq (C\theta^2)^k \phi(x, 1) + C\theta^{-m} \theta^{(k-1)2\mu} \sum_{j=0}^{k-1} (C\theta^{2-2\mu})^j \\
&\leq \theta^{2\mu k} \Big((C\theta^{2-2\mu})^k \int_{B_2(0)} |Du|^2 + C\theta^{-m-2\mu} \sum_{j=0}^{k-1} (C\theta^{2-2\mu})^j \Big).
\end{aligned}$$

Now fix $\alpha \in (\frac{2}{3}, 1)$ or equivalently $\mu \in (0, \frac{1}{2})$, and chose $\theta \in (0, \frac{1}{2}]$ with $C\theta^{2-2\mu} \leq \frac{1}{2}$. Then for $\theta^{k+1} < \varrho \leq \theta^k$ we infer

$$\phi(x, \varrho) \leq \theta^{-m} \phi(x, \theta^k) \leq C\varrho^{2\mu} \Big(\int_{B_2(0)} |Du|^2 + 1 \Big).$$

Campanato's lemma implies that Du is μ-Hölder continuous on $B_1(0)$. □

Corollary 3.3.4. *Let $u \in W^{1,2} \cap L^\infty(U, M)$ be a harmonic map on the open set $U \subset \mathbb{R}^m$ into the smooth submanifold $M \subset \mathbb{R}^n$. Assume that $u(x)$ is continuous at $x_0 \in U$, more precisely*

$$\lim_{\varrho \searrow 0} \|u - p\|_{L^\infty(B_\varrho(x_0))} = 0 \quad \textit{for some } p \in M.$$

Then $u(x)$ is smooth in a full neighborhood of x_0.

Proof. By translations we may assume $x_0 = 0$ and $p = 0$. The $u_\lambda : B \to M$, $u_\lambda(x) = u(\lambda x)$, are harmonic and satisfy

$$\|u_\lambda\|_{L^\infty(B_2(0))} = \|u\|_{L^\infty(B_{2\lambda}(0))} \to 0 \quad \text{as } \lambda \searrow 0.$$

For fixed $\alpha \in (0, 1)$, Theorem 3.3.3 yields $u_\lambda \in C^{1,\mu}(B_1(0), \mathbb{R}^n)$ for some $\mu > 0$. Thus $A(u)(Du, Du)$ is of class $C^{0,\mu}$ near the origin, which means that its Newtonian potential and hence $u(x)$ are locally $C^{2,\mu}$ on a neighborhood of the origin. Repeated application of the Schauder estimates shows that $u(x)$ is smooth on that neighborhood. □

In subsequent work Hildebrandt-Widman and Wiegner proved Hölder continuity assuming only $aM < 1$. By consequence, any harmonic map with image strictly contained in a hemisphere is smooth. This result is sharp in view of the example $u(x) = \frac{x}{|x|}$ into $\mathbb{S}^{n-1} \subset \mathbb{S}^n$.

The techniques of Hélein presented in this section were generalized to harmonic maps into the sphere for $m \geq 3$ by C. Evans [12]. Assuming that the map is also stationary with respect to domain variations, he obtained regularity away from a closed set of $(m-2)$-dimensional Hausdorff measure zero. Any hopes to prove regularity for weakly harmonic maps in dimension $m \geq 3$ without extra assumptions were dashed by T. Rivière [36]. He constructed a harmonic map from the 3-dimensional ball B into $\mathbb{S}^2$ which is discontinuous on any open subset of B.

Chapter 4

Hardy space

In this chapter we study the Hardy space $\mathcal{H}^1(R^n)$.

4.1 Higher integrability of Jacobi determinants

We start by collecting some basic results about the maximal function.

Lemma 4.1.1. *Let $f : \mathbb{R}^n \to [0, \infty)$ be measurable and $0 < p < \infty$. Then we have*

$$\int_{\mathbb{R}^n} f^p(x)\, dx = p \int_0^\infty \alpha^{p-1} |\{x : f(x) > \alpha\}|\, d\alpha. \tag{4.1}$$

Proof. Let χ_f be the characteristic function of the set $\{(x, \alpha) \in \mathbb{R}^n \times [0, \infty) : f(x) > \alpha\}$. Then χ_f is $\mathcal{L}^n \times \mathcal{L}^1$ measurable, and

$$f(x)^p = p \int_0^{f(x)} \alpha^{p-1}\, d\alpha = p \int_0^\infty \alpha^{p-1} \chi_f(x, \alpha)\, d\alpha.$$

Integrating we get by Fubini's theorem

$$\int_{\mathbb{R}^n} f(x)^p\, dx = p \int_0^\infty \alpha^{p-1} \int_{\mathbb{R}^n} \chi_f(x, \alpha)\, dx\, d\alpha = p \int_0^\infty \alpha^{p-1} |\{x : f(x) > \alpha\}|\, d\alpha.$$

□

Definition 4.1.2. *For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define its maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ by*

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x)|\, dx. \tag{4.2}$$

When dealing with the maximal function one needs two basic principles, namely the Vitali covering lemma and the Calderon Zygmund decomposition. For the proof of the first we refer to [13], while the proof of the second is included for convenience.

Theorem 4.1.3 (Vitali)**.** *Let B_j, $j \in J$, be a family of closed balls in $\mathbb{R}^n$ with strictly positive radius and $\sup_{j \in J} \operatorname{diam}(B_j) < \infty$. There is a disjoint subfamily $\{B_{j_k}\}_{k \in \mathbb{N}}$ such that*

$$\bigcup_{j \in J} B_j \subset \bigcup_{k=1}^\infty 5B_{j_k},$$

where $5B$ *is the concentric ball with* 5 *times the radius. In particular if* $E \subset \bigcup_{j\in J} B_j$ *then*

$$|E| \leq 5^n \sum_{k=1}^{\infty} |B_{j_k}|. \tag{4.3}$$

Theorem 4.1.4 (Calderon-Zygmund [6])**.** *Let* $f \in L^1(\mathbb{R}^n)$ *with* $f \geq 0$. *For any* $\alpha > 0$ *there exists a countable family* $\mathcal{G}$ *of closed cubes with pairwise disjoint interior, such that the following holds:*

(i) $\alpha < \fint_Q f(x)\,dx \leq 2^n\alpha$ *for any* $Q \in \mathcal{G}$.

(ii) $f(x) \leq \alpha$ *for almost all* $x \in \mathbb{R}^n \backslash G$, *where* $G = \bigcup_{Q\in\mathcal{G}} Q$.

(iii) $|G| \leq \dfrac{C}{\alpha}\|f\|_{L^1(\mathbb{R}^n)}$.

Proof. Chose a subdivision of $\mathbb{R}^n$ into congruent cubes P having volume $|P| \geq \frac{1}{\alpha}\|f\|_{L^1(\mathbb{R}^n)}$, and denote this family by $\mathcal{F}_0$. Clearly for $P \in \mathcal{F}_0$

$$\frac{1}{|P|}\int_P f(x)\,dx \leq \alpha. \tag{4.4}$$

Using induction we now define families $\mathcal{F}_k$, $\mathcal{G}_k$ of cubes for $k = 1, 2, \ldots$. For this divide each $P \in \mathcal{F}_{k-1}$ into 2^n congruent subcubes using edge bisection. Then denote by $\mathcal{F}_k$ the subfamily for which (4.4) holds, and by $\mathcal{G}_k$ the subfamily where (4.4) fails. The union of the cubes in $\mathcal{F}_k$, $\mathcal{G}_k$ is denoted by F_k,G_k; we note $\mathbb{R}^n = F_k \cup \bigcup_{j=1}^k G_j$. The process is iterated as long as $\mathcal{F}_k$ is nonempty. We prove the result for $\mathcal{G} = \bigcup \mathcal{G}_k$. Two cubes $Q \in \mathcal{G}_k$, $Q' \in \mathcal{G}_\ell$ with $k < \ell$ have disjoint interior, since Q' comes from some $P \in \mathcal{F}_k$. If Q belongs to $\mathcal{G}_k$ and comes from $P \in \mathcal{F}_{k-1}$, then

$$\alpha < \frac{1}{|Q|}\int_Q f(x)\,dx \leq \frac{2^n}{|P|}\int_P f(x)\,dx \leq 2^n\alpha.$$

This proves (i). For $x \notin G$, we have $x \in P_k$ for a sequence $P_k \in \mathcal{F}_k$, thus

$$\frac{1}{|P_k|}\int_{P_k} f(x)\,dx \leq \alpha \quad \text{where } |P_k| \to 0.$$

By the Lebesgue differentiation theorem, see Cor. 2, Sect. 1.7 of [13], the left hand side converges to $f(x)$ a.e. which proves (ii). Finally (iii) follows since

$$|G| = \sum_{Q\in\mathcal{G}} |Q| \leq \frac{1}{\alpha}\sum_{Q\in\mathcal{G}}\int_Q |f(x)|\,dx = \frac{1}{\alpha}\int_G |f(x)|\,dx.$$

□

Our aim is to compare the function and its maximal function, in terms of integrability. The following are the key inequalities.

Lemma 4.1.5. *Let* $f : \mathbb{R}^n \to [0, \infty]$ *be measurable and* $\alpha > 0$. *Then for* $C = C(n) < \infty$

$$|\{x : Mf(x) > \alpha\}| \quad \leq \quad \frac{C}{\alpha}\int_{\{x:f(x)>\frac{\alpha}{2}\}} f(x)\,dx. \tag{4.5}$$

Reversely, we have for constants $C = C(n) < \infty$ and $\lambda = \lambda(n) > 0$

$$\frac{1}{\alpha}\int_{\{x:f(x)>\alpha\}} f(x)\,dx \quad\leq\quad C\,|\{x : Mf(x) > \lambda\alpha\}|. \tag{4.6}$$

Proof. To prove (4.5) we chose for each $x \in \mathbb{R}^n$ with $Mf(x) > \alpha$ a radius $r_x > 0$ such that

$$\int_{B^x} f(y)\,dy > \alpha|B^x| \quad \text{where } B^x = \overline{B_{r_x}(x)}.$$

By Vitali, Theorem 4.1.3, there are disjoint B^{x_k}, $k \in \mathbb{N}$, such that the set $\{x : Mf(x) > \alpha\}$ is covered by the enlarged balls $5B^{x_k}$. Hence

$$|\{x : Mf(x) > \alpha\}| \leq 5^n \sum_{k=1}^{\infty} |B^{x_k}| \leq \frac{5^n}{\alpha}\sum_{k=1}^{\infty}\int_{B^{x_k}} |f(y)|\,dy \leq \frac{5^n}{\alpha}\int_{\mathbb{R}^n} |f(x)|\,dx. \tag{4.7}$$

The trick to obtain the improved inequality (4.5) is to consider

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) \geq \alpha/2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f(x) \leq f_1(x) + \frac{\alpha}{2}$ for all $x \in \mathbb{R}^n$, which implies $Mf(x) \leq Mf_1(x) + \frac{\alpha}{2}$ and thus

$$\{x : Mf(x) > \alpha\} \subset \{x : Mf_1(x) > \frac{\alpha}{2}\}.$$

Applying the previous estimate (4.7) to f_1 yields

$$|\{x : Mf(x) > \alpha\}| \leq |\{x : Mf_1(x) > \frac{\alpha}{2}\}| \leq \frac{2\cdot 5^n}{\alpha}\int_{\{x:f(x)>\alpha/2\}} f(x)\,dx.$$

We now turn to the proof of (4.6), applying Theorem 4.1.4 by Calderon-Zygmund. For any point $x \in Q$ with $Q \in \mathcal{G}$ we have, putting $d = \operatorname{diam} Q$,

$$\alpha < \frac{1}{|Q|}\int_Q f(y)\,dy \leq \frac{|B_d(x)|}{|Q|}\frac{1}{|B_d(x)|}\int_{B_d(x)} f(y)\,dy \leq 2^n n^{n/2} Mf(x).$$

Using (ii) and (i) from Theorem 4.1.4 we infer

$$\begin{aligned} \frac{1}{\alpha}\int_{\{x:f(x)>\alpha\}} f(x)\,dx &\leq \frac{1}{\alpha}\sum_{Q\in\mathcal{G}}\int_Q f(x)\,dx \\ &\leq 2^n \sum_{Q\in\mathcal{G}} |Q| \\ &\leq 2^n |\{x : Mf(x) > 2^{-n}n^{-n/2}\alpha\}|. \end{aligned}$$

This proves (4.6). □

As a first application, we see that there is no difference between f and Mf as regards L^p integrability for $1 < p < \infty$.

Theorem 4.1.6 (Hardy-Littlewood). *For $f \in L^p(\mathbb{R}^n)$ with $1 < p \le \infty$ we have*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \text{ where } C = C(n,p) < \infty. \tag{4.8}$$

Proof. The case $p = \infty$ is trivial with $C(n,\infty) = 1$. For $1 < p < \infty$ we estimate using (4.5)

$$\begin{aligned}
\int_{\mathbb{R}^n} |Mf(x)|^p \, dx &= p \int_0^\infty \alpha^{p-1} |\{x : |f(x)| > \alpha\}| \, d\alpha \\
&\le Cp \int_0^\infty \alpha^{p-2} \int_{\{x \in \mathbb{R}^n : |f(x)| > \alpha/2\}} |f(x)| \, dx \, d\alpha \\
&= Cp \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} \alpha^{p-2} \, d\alpha \, dx \\
&\le \frac{Cp}{p-1} \int_{\mathbb{R}^n} |f(x)|^p \, dx.
\end{aligned}$$

□

Remark 4.1.7. *The function Mf is never in $L^1(\mathbb{R}^n)$ unless $f \equiv 0$. In fact, for any $R < \infty$ we have $B_R(0) \subset B_{2|x|}(x)$ for $|x| \ge R$, yielding the lower bound*

$$Mf(x) \ge \frac{1}{|B_{2|x|}(x)|} \int_{B_{2|x|}(x)} |f(y)| \, dy \ge \frac{c}{|x|^n} \int_{B_R(0)} |f(y)| \, dy.$$

If $\|f\|_{L^1(B_R(0)} > 0$ then the right hand side is not integrable. To get an example where the maximal function is locally not integrable consider $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \frac{1}{|x|^n \log^2 |x|} \chi_{B_{1/e}(0)} \ge 0 \qquad \text{where } e = 2.718\ldots.$$

We compute substituting $r = e^{-t}$ for $0 < \varrho \le 1/e$

$$\int_{B_\varrho(0)} f(x) \, dx = \int_0^\varrho \frac{dr}{r \log^2 r} = \int_{-\log \varrho}^\infty \frac{dt}{t^2} = -\log \varrho.$$

In particular $f \in L^1(\mathbb{R}^n)$. On the other hand as $B_{2|x|}(x) \supset B_{|x|}(0)$, we estimate for $|x| \le 1/e$

$$Mf(x) \ge \frac{c(n)}{|x|^n} \int_{B_{2|x|}(x)} f(y) \, dy \ge \frac{c(n)}{|x|^n} \int_{B_{|x|}(0)} f(y) \, dy \ge -\frac{c(n)}{|x|^n \log |x|}.$$

The right hand side is not integrable near the origin (for the integrals see also example 2.4.2).

Contrary to the case $p > 1$, the L^1 integrability of the maximal function Mf implies an improved integrability of f. This was discovered by E. Stein.

Theorem 4.1.8 (Stein [44]). *Let $f : \mathbb{R}^n \to [0,\infty)$ be measurable having support contained in a ball B. Then one has an estimate*

$$\int_{\mathbb{R}^n} f(x) \log^+ f(x) \, dx \le C\big(B, \|Mf\|_{L^1(B)}\big). \tag{4.9}$$

Proof. By (4.6) we can estimate, recalling $\lambda = \lambda(n) > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\log^+ f(x)\,dx &= \int_{\mathbb{R}^n} f(x)\int_1^\infty \chi_f(x,\alpha)\,\frac{d\alpha}{\alpha}\,dx\\ &= \int_1^\infty \frac{1}{\alpha}\int_{\{x:f(x)>\alpha\}} dx\,d\alpha\\ &\le C\int_1^\infty |\{x: Mf(x)>\lambda\alpha\}|\,d\alpha\\ &= C\int_\lambda^\infty |\{x: Mf(x)>\beta\}|\,d\beta\\ &= C\int_{\{x:Mf(x)>\lambda\}} Mf(x)\,dx.\end{aligned}$$

Let us assume for simplicity that $B = B_1$; the general case follows by scaling. For $|x| > 1$ we have, using that f is supported in B,

$$Mf(x) \le \frac{C}{(|x|-1)^n}\|f\|_{L^1},$$

in particular $Mf(x) \le \lambda$ for $(|x|-1)^n \ge c\,\|f\|_{L^1}$ where $c = c(n) > 0$. Thus we have for a radius $R = \big(C\|f\|_{L^1}\big)^{1/n} + 1$

$$\int_{\{x:|x|>\frac{3}{2},\,Mf(x)>\lambda\}} Mf(x)\,dx \le C\|f\|_{L^1}R^n \le C\big(1+\|f\|_{L^1}^2\big) \le C\big(1+\|Mf\|_{L^1(B)}^2\big).$$

In the remaining annulus, we consider the reflection

$$\phi: B_1\backslash B_{\frac{1}{2}} \to B_{\frac{3}{2}}\backslash B_1,\ \phi(x) = (2-|x|)\frac{x}{|x|}.$$

As B_1 is contained in the halfplane $\{y\in\mathbb{R}^n : \langle y, \frac{x}{|x|}\rangle \le 1\}$, Pythagoras implies that $|y-x| \le |y-\phi(x)|$ for any $y \in B$, and hence $B_r(\phi(x))\cap B \subset B_r(x)\cap B$ for any $r > 0$. We conclude using the transformation formula

$$\int_{1<|x|<\frac{3}{2}} Mf(y)\,dy \le C\int_{\frac{1}{2}<|x|<1} Mf(\phi(x))\,dx \le C\int_{\frac{1}{2}<|x|<1} Mf(x)\,dx.$$

The theorem follows by combining the estimates. □

Remark 4.1.9. *The set of functions for which $|f|\log^+|f|$ is integrable is called the $L\log L$ class. As noted by Stein the above theorem is sharp, in the sense that the $L\log L$ property implies the local integrability of Mf. To see this we write for any set $E \subset \mathbb{R}^n$*

$$\begin{aligned}\int_E Mf\,dx &= 2\int_0^\infty |\{x\in E: Mf(x)>2\alpha\}|\,d\alpha\\ &\le 2\,|E| + 2\int_1^\infty |\{x: Mf(x)>2\alpha|\,d\alpha.\end{aligned}$$

Now (4.5) yields, with $\chi_f(x,\alpha)$ the characteristic function of $\{|f(x)|>\alpha\}$,

$$\begin{aligned}\int_1^\infty |\{x : Mf(x) > 2\alpha\}|\,d\alpha &\le \int_1^\infty \Big(\frac{C}{\alpha}\int_{\mathbb{R}^n} \chi_{\{x:|f(x)|>\alpha\}}|f(x)|\,dx\Big)\,d\alpha \\ &= C\int_{\mathbb{R}^n}|f(x)|\int_1^\infty \chi_f(x,\alpha)\frac{d\alpha}{\alpha}\,dx \\ &= C\int_{\mathbb{R}^n}|f(x)|\log^+|f(x)|\,dx.\end{aligned}$$

Thus for an arbitrary set E we obtain

$$\int_E Mf(x)\,dx \le C\big(|E| + \int_{\mathbb{R}^n}|f(x)|\log^+|f(x)|\,dx\big). \tag{4.10}$$

Next we review some facts about degree theory and Jacobi determinants. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain of class C^1. For a map $u\in C^2(\overline{\Omega},\mathbb{R}^n)$ the oriented multiplicity function $i_u:\mathbb{R}^n\to\mathbb{Z}$ is given by

$$i_u(y) = \begin{cases}\sum_{u(x)=y}\operatorname{sign}\det Du(x) & \text{if } y\notin u(\partial\Omega) \text{ is a regular value,}\\ 0 & \text{else.}\end{cases} \tag{4.11}$$

Here $y\notin u(\partial\Omega)$ is a regular value if and only if $\det Du(x)\neq 0$ for all $x\in u^{-1}\{y\}$. By the inverse function theorem and compactness, each regular value has only finitely many preimages so that the sum is defined. Our main tool in the following is the transformation formula: for any $g\in L^1(\Omega)$, the function $y\mapsto\sum_{u(x)=y} g(x)$ is integrable on $\mathbb{R}^n$ and

$$\int_\Omega g(x)|\det Du(x)|\,d\mathcal{L}^n(x) = \int_{\mathbb{R}^n}\Big(\sum_{u(x)=y} g(x)\Big)\,d\mathcal{L}^n(y).$$

In particular, the set of points in $\mathbb{R}^n\backslash u(\partial\Omega)$ which are not regular has Lebesgue measure zero. This is actually a step in the proof of the transformation formula, see [13, Section 3.3]. As $u(\partial\Omega)$ is also a null set, the first alternative in the definition of i_u applies almost everywhere.

Now let $h\in C^0_c(\mathbb{R}^n)$. By the transformation formula, we calculate

$$\begin{aligned}\int_{\mathbb{R}^n} h(y)\,i_u(y)\,d\mathcal{L}^n(y) &= \int_\Omega h(u(x))\operatorname{sign}\det Du(x)|\det Du(x)|\,d\mathcal{L}^n(x)\\ &= \int_\Omega h(u(x))\det Du(x)\,d\mathcal{L}^n(x)\\ &= \int_\Omega u^*(h(y)\,dy),\end{aligned}$$

where $dy = dy^1\wedge\ldots\wedge dy^n$. Inserting $h=\operatorname{div}\phi$ where $\phi\in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$ we infer

$$\int_{\mathbb{R}^n}(\operatorname{div}\phi)\,i_u\,d\mathcal{L}^n = \int_\Omega u^*(\operatorname{div}\phi\,dy) = \int_\Omega u^*d(\phi\llcorner dy) = \int_\Omega du^*(\phi\llcorner dy) = \int_{\partial\Omega}u^*(\phi\llcorner dy).$$

Taking $\operatorname{spt}\phi\subset\mathbb{R}^n\backslash u(\partial B)$ yields $Di_u=0$ on $\mathbb{R}^n\backslash u(\partial B)$, hence i_u is constant on the components of that set. For general ϕ we compute further

$$\begin{aligned}u^*(\phi\llcorner dy)(e_1,\ldots,\widehat{e_j},\ldots,e_n) &= \det(\phi\circ u,\partial_1 u,\ldots,\widehat{\partial_j u},\ldots,\partial_n u)\\ &= (\phi^i\circ u)\,\det(e_i,\partial_1 u,\ldots,\widehat{\partial_j u},\ldots,\partial_n u)\\ &= (-1)^{j-1}(\phi^i\circ u)\operatorname{cof}(Du)_{ij}.\end{aligned}$$

Here $\mathrm{cof}_{ij}(Du)$ equals $(-1)^{i+j}$ times the ij-minor, i.e. the subdeterminant when the i-th row and j-th column of Du is deleted. Now assume $|\phi| \le 1$, so that by Cauchy-Schwarz

$$\Big| \int_{\partial\Omega} u^*(\phi \llcorner dy) \Big| \le \int_{\partial\Omega} |u^*(\phi \llcorner dy)| \, d\mathcal{H}^{n-1} \le \int_{\partial\Omega} |\mathrm{cof}(Du)| \, d\mathcal{H}^{n-1}.$$

Recalling the definition of the variation measure $|Di_u|$ we arrive at

$$|Di_u|(\mathbb{R}^n) \le \int_{\partial\Omega} |\mathrm{cof}(Du)| \, d\mathcal{H}^{n-1}. \tag{4.12}$$

The following is Lemma 1.3 in [33], see also Theorem 2.10 in [43] for the case $n = 3$.

Lemma 4.1.10. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. For any $x \in \Omega$ and almost all $r \in (0, \mathrm{dist}(x, \partial\Omega))$, we have for a constant $C = C(n) < \infty$*

$$\Big| \int_{B_r(x)} \det Du \, d\mathcal{L}^n \Big|^{\frac{n-1}{n}} \le C \int_{\partial B_r(x)} |\mathrm{cof}(Du)| \, d\mathcal{H}^{n-1}. \tag{4.13}$$

Proof. We first assume that $u \in C^2(\Omega, \mathbb{R}^n)$. Using once more the transformation formula, and the fact that $i_{u,B_r(x)}$ is integer-valued, we have

$$\int_{B_r(x)} \det Du \, d\mathcal{L}^n = \int_{\mathbb{R}^n} i_{u,B_r(x)} \, d\mathcal{L}^n \le \int_{\mathbb{R}^n} |i_{u,B_r(x)}|^{\frac{n}{n-1}} \, d\mathcal{L}^n.$$

By the Sobolev embedding theorem, see [13, Sec. 5.6], and by (4.12) we can continue

$$\Big(\int_{\mathbb{R}^n} |i_{u,B_r(x)}|^{\frac{n}{n-1}} \, d\mathcal{L}^n \Big)^{\frac{n-1}{n}} \le C \, |Di_{u,B_r(x)}|(\mathbb{R}^n) \le C \int_{\partial B_r(x)} |\mathrm{cof}(Du)| \, d\mathcal{H}^{n-1}.$$

This proves the lemma for maps $u \in C^2(\Omega, \mathbb{R}^n)$. Now let $u \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, and chose $u_k \in C^2(\Omega, \mathbb{R}^n)$ with $u_k \to u$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. By Fatou we have for $R < \mathrm{dist}(x, \partial\Omega)$

$$\int_0^R \liminf_{k\to\infty} \int_{\partial B_r(x)} |\mathrm{cof}(Du_k) - \mathrm{cof}(Du)| \, d\mathcal{H}^{n-1} dr \le \lim_{k\to\infty} \int_{B_R(x)} |\mathrm{cof}(Du_k) - \mathrm{cof}(Du)| \, d\mathcal{L}^n = 0.$$

Thus for almost all $r \in (0, \mathrm{dist}(x, \partial\Omega))$ inequality (4.13) follows by approximation. □

We have now collected all ingredients to prove Müller's higher integrability theorem. The main idea is to estimate the maximal function of the Jacobi determinant by the maximal function of the $(n-1) \times (n-1)$ minors using (4.13). The advantage is that the minors come with a power $\frac{n}{n-1}$, so that Theorem 4.1.6 by Hardy-Littlewood can be applied.

Theorem 4.1.11 ([33]). *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be open and bounded. If $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ has $\det Du \ge 0$ almost everywhere, then for any compact set $K \subset \Omega$*

$$\int_K \det Du \, \log^+(\det Du) \le C(K, \|u\|_{W^{1,n}(\Omega)}). \tag{4.14}$$

Proof. Let $g = \chi_K \det Du$. The result follows from Theorem 4.1.8, once we have the estimate

$$||Mg||_{L^1(B)} \le C(B, ||u||_{W^{1,n}(\Omega)}) \quad \text{for some ball } B \supset K. \tag{4.15}$$

We show an improved version where the right hand side depends only on the $L^{\frac{n}{n-1}}$-norm of $\mathrm{cof}(Du)$. Let $d = \mathrm{dist}(K, \partial\Omega)$. For $r \ge d/4$ and all $x \in \mathbb{R}^n$ we have the trivial inequality

$$\fint_{B_r(x)} |g| \, d\mathcal{L}^n \le \frac{C}{d^n} \int_\Omega |\det Du| \, d\mathcal{L}^n \le \frac{C}{d^n} \int_\Omega |\mathrm{cof}(Du)|^{\frac{n}{n-1}} \, d\mathcal{L}^n. \tag{4.16}$$

For the last step we use $Du \cdot \mathrm{cof}(Du)^{\mathrm{T}} = (\det Du)\,\mathrm{Id}$, which implies

$$|\det Du|^n = |\det Du|\,|\det \mathrm{cof}(Du)| \le |\det Du|\,|\mathrm{cof}(Du)|^n,$$

thus $|\det Du| \le |\mathrm{cof}(Du)|^{n/(n-1)}$. Now consider the case $r \le d/4$, in particular we may assume $\mathrm{dist}(x, \partial\Omega) > d/2$. As $\det Du \ge 0$, Lemma 4.1.10 implies for almost all $\varrho \in (r, 2r)$

$$\Big(\int_{B_r(x)} |g| \, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \le \Big(\int_{B_\varrho(x)} \det Du \, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \le C \int_{\partial B_\varrho(x)} |\mathrm{cof}(Du)| \, d\mathcal{H}^{n-1}.$$

Integrating on $(r, 2r)$ and dividing by r^n gives, putting $\mathrm{cof}(Du) = 0$ on $\mathbb{R}^n \backslash \Omega$,

$$\Big(\fint_{B_r(x)} |g| \, d\mathcal{L}^n\Big)^{\frac{n-1}{n}} \le C \fint_{B_{2r}(x)} |\mathrm{cof}(Du)| \, d\mathcal{L}^n \le CM(\mathrm{cof}(Du))(x). \tag{4.17}$$

Combining (4.16) and (4.17) yields

$$Mg(x) \le CM(\mathrm{cof}(Du))(x) + \frac{C}{d^n} \int_\Omega |\mathrm{cof}(Du)|^{\frac{n}{n-1}} \, d\mathcal{L}^n.$$

As $\frac{n}{n-1} > 1$ we can apply Theorem 4.1.6 by Hardy-Littlewood to get

$$\|M(\mathrm{cof}(Du))\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C\|\mathrm{cof}(Du)\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C\|\mathrm{cof}(Du)\|_{L^{\frac{n}{n-1}}(\Omega)}.$$

We finally conclude, chosing any ball $B \supset K$,

$$\|Mg\|_{L^1(B)} \le C\left(B, \|\mathrm{cof}(Du)\|_{L^{\frac{n}{n-1}}(\Omega)}\right).$$

□

In [33] the estimate is stated for $\det Du \log(2 + \det Du)$. This follows easily from the version above, using $\log(2 + s) \le \log s + 1$ for $s \ge 2$. In Section 7 of [33] a counterexample is given, showing that the condition $\det Du \ge 0$ cannot be dropped.

4.2 The Hardy space

As is well-known a bounded sequence f_k in $L^1(\mathbb{R}^n)$ may have a weak limit which is not representable by an $L^1(\mathbb{R}^n)$ function, but only by a signed Radon measure. The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is continuously embedded into $L^1(\mathbb{R}^n)$ and has a norm which scales like the $L^1(\mathbb{R}^n)$ norm. However, as opposed to $L^1(\mathbb{R}^n)$ the unit ball in $\mathcal{H}^1(\mathbb{R}^n)$ is weakly sequentially compact.

To start we consider on $C^0_c(\mathbb{R}^n)$ the usual convergence $\phi_k \to \phi$ defined by

- $\bigcup_{k\in\mathbb{N}} \operatorname{spt}\phi_k$ is a bounded subset of $\mathbb{R}^n$,
- $\phi_k \to \phi$ uniformly on $\mathbb{R}^n$.

By the Riesz representation theorem any $\Lambda \in C^0_c(\mathbb{R}^n)'$ has the form

$$\Lambda(\phi) = \int_{\mathbb{R}^n} \phi\, \sigma d\mu,$$

where μ is a Radon measure and $\sigma : \mathbb{R}^n \to \{\pm 1\}$ is μ measurable. The convolution of $\Lambda \in C^0_c(\mathbb{R}^n)'$ with $\phi \in C^0_c(\mathbb{R}^n)$ is the function

$$\phi * \Lambda : \mathbb{R}^n \to \mathbb{R},\ (\phi * \Lambda)(x) = \Lambda(\phi^x) \quad \text{where } \phi^x(y) = \phi(x-y).$$

For example $(\phi * \delta_0)(x) = \phi(x)$. We note that $\phi * \Lambda \in C^0(\mathbb{R}^n)$ since the map $\mathbb{R}^n \to C^0_c(\mathbb{R}^n)$, $x \mapsto \phi^x$, is continuous, as is $\Lambda : C^0_c(\mathbb{R}^n) \to \mathbb{R}$. Any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ defines canonically a functional $\Lambda_f \in C^0_c(\mathbb{R}^n)'$ by

$$\Lambda_f(\phi) = \int_{\mathbb{R}^n} \phi(y) f(y)\, dy,$$

and the notions of convolutions are consistent in the sense that

$$(\phi * \Lambda_f)(x) = \Lambda_f(\phi^x) = \int_{\mathbb{R}^n} \phi(x-y) f(y)\, dy = (\phi * f)(x).$$

Now we introduce the following class of test functions.

$$\mathcal{T} = \{\phi \in C^\infty_c(\mathbb{R}^n) : \operatorname{spt}\phi \subset B_1(0) \text{ and } \|D\phi\|_{L^\infty} \le 1\}. \tag{4.18}$$

Clearly $\|\phi\|_{L^\infty(\mathbb{R}^n)} \le 1$ for $\phi \in \mathcal{T}$. For the rescalings we use the notation

$$\phi_t(x) = t^{-n}\phi\big(\frac{x}{t}\big) \quad \text{for } t > 0, \tag{4.19}$$

thus

$$\operatorname{spt}\phi_t \subset B_t(0) \quad \text{and} \quad \|D\phi_t\|_{L^\infty(\mathbb{R}^n)} \le C\, t^{-(n+1)}.$$

In the following definition the term *grand* means that one maximizes over all kernels in $\mathcal{T}$, rather than working with a specific one.

Definition 4.2.1. *The grand maximal function of $\Lambda \in C^0_c(\mathbb{R}^n)'$ is defined by*

$$\Lambda^*(x) = \sup_{\phi\in\mathcal{T}} \sup_{t>0} |\phi_t * \Lambda(x)|. \tag{4.20}$$

Let $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\operatorname{spt}\phi \subset B_R(0)$ and $\|D\phi\|_{L^\infty(\mathbb{R}^n)} = \alpha > 0$. Then the function $\psi(x) = \frac{1}{R\alpha}\phi(Rx)$ belongs to $\mathcal{T}$, and we calculate

$$(\phi * \Lambda)(x) = \Lambda\big(y \mapsto \phi(x-y)\big) = \Lambda\big(y \mapsto R^{n+1}\alpha\psi_R(x-y)\big) = R^{n+1}\alpha(\psi_R * \Lambda)(x).$$

Thus for any $\phi \in C^\infty_c(\mathbb{R}^n)$ we have the inequality

$$|(\phi * \Lambda)(x)| \le R^{n+1}\|D\phi\|_{L^\infty(\mathbb{R}^n)}\Lambda^*(x) \quad \text{if } \operatorname{spt}\phi \subset B_R(0). \tag{4.21}$$

The Hardy space can be defined in several ways, whose equivalence is by no means obvious, see [14] or [46]. A nice introduction is due to Semmes [41].

Definition 4.2.2. *$\mathcal{H}^1(\mathbb{R}^n)$ is the set of all $\Lambda \in C^0_c(\mathbb{R}^n)'$ for which $\Lambda^* \in L^1(\mathbb{R}^n)$. We put*

$$\|\Lambda\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|\Lambda^*\|_{L^1(\mathbb{R}^n)}. \tag{4.22}$$

As $(\Lambda_1 + \Lambda_2)^* \leq \Lambda_1^* + \Lambda_2^*$ and $(\alpha\Lambda)^* = |\alpha|\Lambda^*$, the Hardy space is a normed vector space. The following lemma will allow us to consider its elements as L^1 functions.

Lemma 4.2.3. *The space $\mathcal{H}^1(\mathbb{R}^n)$ is continuously embedded into $L^1(\mathbb{R}^n)$.*

Proof. We use approximation by smoothing. Chose a fixed kernel $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\operatorname{spt}\phi \subset B_1(0)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. Putting $\check{\phi}(x) = \phi(-x)$ we have using that Λ is linear and continuous

$$\begin{aligned}
\int_{\mathbb{R}^n} (\phi_t * \Lambda)(x)\eta(x)\,dx &= \int_{\mathbb{R}^n} \Lambda(\phi_t^x)\,\eta(x)\,dx \\
&= \Lambda\Big(y \mapsto \int_{\mathbb{R}^n} \phi_t^x(y)\,\eta(x)\,dx\Big) \\
&= \Lambda\big(\check{\phi}_t * \eta\big) \to \Lambda(\eta) \quad \text{as } t \searrow 0.
\end{aligned}$$

Using (4.21) we have the bound

$$|\phi_t * \Lambda(x)| \leq t^{n+1}\|D\phi_t\|_{L^\infty(\mathbb{R}^n)}\Lambda^*(x) = \|D\phi\|_{L^\infty(\mathbb{R}^n)}\Lambda^*(x) \in L^1(\mathbb{R}^n).$$

As the $\phi_t * \Lambda$ are equiintegrable they converge weakly subsequentially in $L^1(\mathbb{R}^n)$ to some $f \in L^1(\mathbb{R}^n)$. Then $\Lambda_f = \Lambda$ by the above, and the sublimit improves to a limit. Finally

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \liminf_{t \searrow 0} \|\phi_t * \Lambda\|_{L^1(\mathbb{R}^n)} \leq \|D\phi\|_{L^\infty(\mathbb{R}^n)}\|\Lambda^*\|_{L^1(\mathbb{R}^n)}.$$

□

From now on the elements of $\mathcal{H}^1(\mathbb{R}^n)$ are regarded as $L^1(\mathbb{R}^n)$ functions. As pointed out at the beginning, the following weak compactness theorem distinguishes the space $\mathcal{H}^1(\mathbb{R}^n)$ from $L^1(\mathbb{R}^n)$.

Theorem 4.2.4 (weak compactness in $\mathcal{H}^1(\mathbb{R}^n)$)**.** *Let f_k be a bounded sequence in $\mathcal{H}^1(\mathbb{R}^n)$. Then there exists an $f \in \mathcal{H}^1(\mathbb{R}^n)$, such that for a subsequence $f_k \to f$ in $C^0_c(\mathbb{R}^n)'$, and*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq \operatorname{liminf}_{k\to\infty} \|f_k\|_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.23}$$

Proof. By Lemma 4.2.3 we have $\|f_k\|_{L^1(\mathbb{R}^n)} \leq C$, so that $f_k \to \Lambda$ in $C^0_c(\mathbb{R}^n)'$ after passing to a subsequence. Now for $\phi \in \mathcal{T}$ and $t > 0$ we have

$$(\phi_t * f_k)(x) = \int_{\mathbb{R}^n} \phi_t^x(y) f_k(y)\,dy \stackrel{k\to\infty}{\longrightarrow} \Lambda(\phi_t^x) = (\phi_t * \Lambda)(x),$$

which implies

$$(\phi_t * \Lambda)(x) = \lim_{k\to\infty} (\phi_t * f_k)(x) \leq \liminf_{k\to\infty} f_k^*(x).$$

Take the supremum with respect to $\phi \in \mathcal{T}$ and $t > 0$. Then by Fatou's lemma

$$\|\Lambda\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|\Lambda^*\|_{L^1(\mathbb{R}^n)} \leq \liminf_{k\to\infty} \|f_k^*\|_{L^1(\mathbb{R}^n)} = \liminf_{k\to\infty} \|f_k\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

Finally $\Lambda = \Lambda_f$ where $f \in L^1(\mathbb{R}^n)$ by Lemma 4.2.3, which finishes the proof. □

Corollary 4.2.5 (Cancellation in $\mathcal{H}^1(\mathbb{R}^n)$)**.** *For every $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} f(y)\,dy = 0. \tag{4.24}$$

Proof. We first check the scaling of the Hardy norm. For $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $t > 0$ let $f_t(y) = t^{-n} f(\frac{y}{t})$. Compute for $\phi \in \mathcal{T}$, $t > 0$,

$$\begin{aligned}(\phi_s * f_t)(x) &= \int_{\mathbb{R}^n} \phi_s(x-y) t^{-n} f\big(\frac{y}{t}\big)\,dy \\ &= \int_{\mathbb{R}^n} \phi_s\big(t\,(\frac{x}{t} - z)\big)\, f(z)\,dz \\ &= t^{-n}\big(\phi_{\frac{s}{t}} * f\big)\big(\frac{x}{t}\big).\end{aligned}$$

We estimate on the right with $f^*(\frac{x}{t})$, and then take the supremum over $\phi \in \mathcal{T}$, $s > 0$, to get

$$f_t^*(x) = t^{-n} f^*\big(\frac{x}{t}\big) \qquad \text{for } x \in \mathbb{R}^n,\ t > 0.$$

We first get the inequality, for equality we use $(f_t)_{\frac{1}{t}} = f$. In particular we obtain

$$\|f_t\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \qquad \text{for all } t > 0.$$

Now for $f \in L^1(\mathbb{R}^n)$ we know that f_t converges to a multiple of the Dirac measure as $t \searrow 0$, in fact dominated convergence yields

$$\int_{\mathbb{R}^n} \phi(x) f_t(x)\,dx = \int_{\mathbb{R}^n} \phi(ty) f(y)\,dy \to \phi(0) \int_{\mathbb{R}^n} f(y)\,dy.$$

On the other hand we must have $f_t \to \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n)$ for a subsequence by Theorem 4.2.4. As the Dirac measure is not in $\mathcal{H}^1(\mathbb{R}^n)$ we conclude that $\int_{\mathbb{R}^n} f(y)\,dy = 0$. □

Alternative to the given argument, one can estimate more directly to show

$$\Big|\int_{\mathbb{R}^n} f(y)\,dy\Big| \leq C \liminf_{|x|\to\infty} \frac{f^*(x)}{|x|^n}.$$

For $f \in \mathcal{H}^1(\mathbb{R}^n)$ the right hand side must be zero. In the next lemma the grand maximal function is compared to the maximal function of Hardy-Littlewood.

Lemma 4.2.6. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The for any $x \in \mathbb{R}^n$ we have*

(1) $f^*(x) \leq C\, Mf(x)$,

(2) $Mf(x) \leq C\, f^*(x)$, *if* $f \geq 0$.

Proof. To prove the first statement we calculate for $\phi \in \mathcal{T}$, $t > 0$,

$$\begin{aligned}|(\phi_t * f)(x)| &= t^{-n}\Big|\int_{\mathbb{R}^n} \phi(\frac{x-y}{t}) f(y)\,dy\Big| \\ &\leq \|\phi\|_{L^\infty(\mathbb{R}^n)} t^{-n} \int_{B_t(x)} |f(y)|\,dy \\ &\leq C\, Mf(x).\end{aligned}$$

Taking the supremum over $\phi \in \mathcal{T}$, $t > 0$ shows claim (1). Next we chose $\phi \in \mathcal{T}$ such that $\phi \geq \frac{1}{4}\chi_{B_{1/2}(0)}$. For $f \geq 0$ we can then estimate

$$\begin{aligned}(\phi_t * f)(x) &= t^{-n}\int_{\mathbb{R}^n}\phi(\frac{x-y}{t}))f(y)\,dy\\ &\geq t^{-n}\frac{1}{4}\int_{B_{t/2}(x)} f(y)\,dy\\ &\geq c \fint_{B_{t/2}(x)} f(y)\,dy.\end{aligned}$$

Recalling the definiion of f^*, this implies

$$f^*(x) \geq \sup_{t>0}(\phi_t * f)(x) \geq c\sup_{t>0}\fint_{B_{t/2}(x)} f(y)\,dy = Mf(x).$$

□

Theorem 4.2.7. *Let $f \in L^1(\mathbb{R}^n)$ such that* $\operatorname{spt} f \subset B_R(0)$ *and* $\int_{\mathbb{R}^n} f(x)dx = 0$. *Then*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\big(\|Mf\|_{L^1(B_{2R}(0))} + C\|f\|_{L^1(\mathbb{R}^n)}\big).$$

Proof. We estimate the L^1 integral of f^* by splitting into the regions $B_{2R}(0)$ and $\mathbb{R}^n\backslash B_{2R}(0)$. For $|x| \leq 2R$ the inequality $f^*(x) \leq C\,Mf(x)$ from Lemma 4.2.6 yields

$$\int_{B_{2R}(0)} f^*(x)\,dx \leq C\,\|Mf\|_{L^1(B_{2R}(0))}. \tag{4.25}$$

For $|x| \geq 2R$ we have $\operatorname{dist}(x, B_R(0)) = |x| - R \geq \frac{1}{2}|x|$, hence

$$\phi_t * f(x) = \int_{B_1(0)} \phi(z)f(x-tz)\,dz = 0 \quad \text{when } 0 < t < \frac{1}{2}|x|.$$

For $t \geq \frac{1}{2}|x|$ we estimate using $\int_{\mathbb{R}^n} f(y)\,dy = 0$

$$\begin{aligned}|\phi_t * f(x)| &= \Big|\int_{B_R(0)}(\phi_t(x-y)-\phi_t(x))f(y)\,dy\Big|\\ &\leq \|D\phi_t\|_{L^\infty(\mathbb{R}^n)}R\int_{\mathbb{R}^n}|f(y)|\,dy\\ &\leq \frac{CR}{|x|^{n+1}}\int_{\mathbb{R}^n}|f(y)|\,dy.\end{aligned}$$

Integrating shows

$$\int_{\mathbb{R}^n\backslash B_{2R}(0)} f^*(x)\,dx \leq C\int_{\mathbb{R}^n}|f(y)|\,dy. \tag{4.26}$$

The theorem follows by combining (4.25) and (4.26). □

Corollary 4.2.8. *Let $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, with* $\operatorname{spt} f \subset B_R(0)$ *and* $\int_{\mathbb{R}^n} f(x)dx = 0$. *Then $f \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C(p)\,R^{n-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The Hardy-Littlewood inequality, see Theorem 4.1.6, implies

$$\|Mf\|_{L^1(B_{2R}(0))} \le C\,R^{n-\frac{n}{p}}\|Mf\|_{L^p(\mathbb{R}^n)} \le C(p)\,R^{n-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)}.$$

Since also $\|f\|_{L^1(\mathbb{R}^n)} \le C\,R^{n-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)}$, the claim follows from Theorem 4.2.7. □

Remark 4.2.9. *We give an example of a function $f \in \mathcal{H}^1(\mathbb{R})$ with compact support, whose maximal function is not locally integrable. Consider*

$$f(x) = \sum_{n=2}^{\infty} \frac{a_n(x)}{n(\log n)^2} \qquad \textit{where } a_n = -n\chi_{[-\frac{1}{n},0)} + n\chi_{(0,\frac{1}{n}]}.$$

The function $a_1(x)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$ by Corollary 4.2.8. As $a_n(x) = na_1(nx) = (a_1)_{\frac{1}{n}}(x)$, we have $\|a_n\|_{\mathcal{H}^1(\mathbb{R})} = \|a_1\|_{\mathcal{H}^1(\mathbb{R})}$ for all n, and

$$\|f\|_{\mathcal{H}^1(\mathbb{R})} \le C\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} < \infty.$$

On the other hand, for $\frac{1}{2(n+1)} < x \le \frac{1}{2n}$ with $n \ge 1$ we can estimate

$$f(x) \ge \sum_{k=n+1}^{2n} \frac{1}{(\log k)^2} \ge \frac{n}{(\log 2n)^2} \ge \frac{1}{4x(\log 1/x)^2}.$$

For $0 < x \le \frac{1}{2}$ we therefore have a lower bound

$$f(x)\log f(x) \ge \frac{1}{4x\log 1/x} + \frac{1}{4x(\log 1/x)^2}\log\frac{1}{4(\log 1/x)^2}.$$

The second term on the right is integrable, but the first is not. Hence $f(x)$ does not belong to the $L\log L$ class. By Stein's theorem 4.1.8 we conclude that Mf is not locally integrable, and that the integrability of f^* is due to a cancellation effect.

In 1993 Coifman, Lions, Meyer & Semmes [9] discovered that the famous div-curl lemma by Murat and Tartar is connected to Hardy space theory. We use differential forms to state the result in general form. We will need the following Hodge lemma.

Lemma 4.2.10. *Let $\beta \in L^q(\mathbb{R}^n, \Lambda^k)$ where $1 < q < \infty$. If $d\beta = 0$ in the sense of distributions, then there exists a form $\gamma \in W^{1,q}_{\text{loc}}(\mathbb{R}^n, \Lambda^{k-1})$ such that $d\gamma = \beta$, and*

$$\|D\gamma\|_{L^q(\mathbb{R}^n)} \le C\,\|\beta\|_{L^q(\mathbb{R}^n)} \qquad \textit{where } C = C(n,q). \tag{4.27}$$

Proof. We first show that for any $\beta \in L^q(\mathbb{R}^n, \Lambda^k)$ there exists a $\phi \in W^{2,q}_{\text{loc}}(\mathbb{R}^n, \Lambda^k)$ such that

$$\Delta\phi = \beta \quad \text{ and } \quad \|D^2\phi\|_{L^q(\mathbb{R}^n)} \le C\|\beta\|_{L^q(\mathbb{R}^n)} \quad \text{ where } C = C(n,q) < \infty.$$

For $\beta \in C^\infty_c(\mathbb{R}^n)$ we have the Newtonian potential

$$\phi(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\beta(y)\,dy \quad \text{ where } \Gamma(z) = \begin{cases} \frac{1}{(2-n)\omega_n}|z|^{2-n} & \text{for } n \ge 3, \\ \frac{1}{2\pi}\log|z| & \text{for } n = 2. \end{cases}$$

The L^q estimate of $D^2\phi$ is the classical Calderon-Zygmund inequality, see for instance [2]. For general $\beta \in L^q(\mathbb{R}^n)$ we approximate by $\beta_k \in C_c^\infty(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$. Then $\beta_k = \Delta\phi_k$ where

$$\limsup_{k\to\infty} \|D^2\phi_k\|_{L^q(\mathbb{R}^n)} \le C\|\beta\|_{L^q(\mathbb{R}^n)}.$$

Subtracting a linear function, we can arrange that

$$\int_{B_1(0)} \phi_k\, dx = 0 \quad \text{and} \quad \int_{B_1(0)} D\phi_k\, dx = 0.$$

Then a standard contradiction argument using Rellich's theorem implies

$$\|\phi_k\|_{W^{1,q}(B_R(0))} \le C(R,q)\|D^2\phi_k\|_{L^q(\mathbb{R}^n)}.$$

After passing to a subsequence we have $\phi_k \to \phi$ in $W^{1,q}_{\text{loc}}(\mathbb{R}^n)$, and $\|D^2\phi\|_{L^q(\mathbb{R}^n)} \le C\|\beta\|_{L^q(\mathbb{R}^n)}$. Thus ϕ is the desired solution of $\Delta\phi = \beta$. We next show that $\psi = d^*d\phi$ solves $\Delta\psi = d^*d\beta$ in the sense of distributions. Namely for $\zeta \in C_c^\infty(\mathbb{R}^n, \Lambda^k)$ we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \langle \psi, \Delta\zeta\rangle\, dx &= -\int_{\mathbb{R}^n} \langle d^*d\phi, (dd^* + d^*d)\zeta\rangle\, dx \\
&= -\int_{\mathbb{R}^n} \langle d\phi, dd^*d\zeta\rangle\, dx - \int_{\mathbb{R}^n} \langle d^*\phi, d^*d^*d\zeta\rangle\, dx \\
&= \int_{\mathbb{R}^n} \langle \beta, d^*d\zeta\rangle\, dx.
\end{aligned}$$

Now if $d\beta = 0$ then $d^*d\phi \in L^q(\mathbb{R}^n, \Lambda^k)$ is harmonic. By Weyl's lemma and the mean value inequality we conclude that $d^*d\phi$ is identically zero. This implies $-dd^*\phi = \beta$, and the lemma is proved by taking $\gamma = -d^*\phi$. □

The following is the key observation of Coifmann, Lions, Meyer and Semmes.

Theorem 4.2.11 ([9]). *Let $\alpha \in L^p(\mathbb{R}^n, \Lambda^k)$, $\beta \in L^q(\mathbb{R}^n, \Lambda^{n-k})$ where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $d\alpha = 0$ and $d\beta = 0$ weakly, then $\alpha \wedge \beta \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|\alpha \wedge \beta\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\, \|\alpha\|_{L^p(\mathbb{R}^n)}\|\beta\|_{L^q(\mathbb{R}^n)}. \tag{4.28}$$

Proof. The equation $d\alpha = 0$ has the alternative weak form

$$\int_{\mathbb{R}^n} \alpha \wedge d\zeta = 0 \quad \text{for all } \zeta \in C_c^\infty(\mathbb{R}^n, \Lambda^{n-k-1}). \tag{4.29}$$

To see this we calculate using standard rules for the Hodge star

$$\begin{aligned}
*(\alpha \wedge d\zeta) &= (-1)^{k(n-k)} * (\alpha \wedge *(*d\zeta)) \\
&= (-1)^{k(n-k)}\langle \alpha, *d\zeta\rangle \\
&= (-1)^{k(n-k)}\langle \alpha, *d*\omega\rangle \quad \text{where } \zeta = *\omega \\
&= (-1)^{n-k-1}\langle \alpha, d^*\omega\rangle.
\end{aligned}$$

By Lemma 4.2.10 there exists $\gamma \in W^{1,q}_{\text{loc}}(\mathbb{R}^n, \Lambda^{n-k-1})$ such that

$$d\gamma = \beta \quad \text{and} \quad \|D\gamma\|_{L^q(\mathbb{R}^n)} \le C\|\beta\|_{L^q(\mathbb{R}^n)}. \tag{4.30}$$

For any $\eta \in C_c^\infty(\mathbb{R}^n)$ and any constant $\gamma_0 \in \Lambda^{n-k-1}$, the form $\zeta = \eta(\gamma - \gamma_0) \in W^{1,q}(\mathbb{R}^n, \Lambda^{n-k-1})$ is admissible in (4.29) by approximation, thus we get

$$\int_{\mathbb{R}^n} \eta\, \alpha \wedge \beta = -\int_{\mathbb{R}^n} \alpha \wedge d\eta \wedge (\gamma - \gamma_0). \tag{4.31}$$

For given $\phi \in \mathcal{T}$ and $t > 0$, we take $\eta(y) = \phi_t^x(y) = \phi_t(x - y)$. Then $|d\eta| \le t^{-n-1}$ and

$$|\phi_t * (\alpha \wedge \beta)(x)| = \int_{\mathbb{R}^n} \phi_t^x \alpha \wedge \beta \le \frac{C}{t} \fint_{B_t(x)} |\alpha|\, |\gamma - \gamma_0|\, dy.$$

We now use Hölder's inequality with exponent $r \in (1, p]$. The second factor, which gets the power $s = \frac{r}{r-1} \in [q, \infty)$, is estimated by the Sobolev-Poincaré inequality. More precisely

$$\begin{aligned}
|\phi_t * (\alpha \wedge \beta)(x)| &\le \frac{C}{t}\Big(\fint_{B_t(x)} |\alpha|^r\, dy\Big)^{\frac{1}{r}} \Big(\fint_{B_t(x)} |\gamma - \gamma_0|^s\, dy\Big)^{\frac{1}{s}} \\
&\le C\Big(\fint_{B_t(x)} |\alpha|^r\, dy\Big)^{\frac{1}{r}} \Big(\fint_{B_t(x)} |D\gamma|^\lambda\, dy\Big)^{\frac{1}{\lambda}} \\
&\le CM(|\alpha|^r)(x)^{\frac{1}{r}}\, M(|D\gamma|^\lambda)(x)^{\frac{1}{\lambda}}.
\end{aligned}$$

Here we need $\frac{1}{\lambda} \le 1 + \frac{1}{n} - \frac{1}{r}$. Take the supremum over $t > 0$ and integrate, then use Hölder with exponents p, q to get

$$\begin{aligned}
\int_{\mathbb{R}^n} (\alpha \wedge \beta)^*\, dx &= C \int_{\mathbb{R}^n} M(|\alpha|^r)^{\frac{1}{r}}\, M(|D\gamma|^\lambda)^{\frac{1}{\lambda}}\, dx \\
&\le C\, \|M(|\alpha|^r)\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}^{\frac{1}{r}}\, \|M(|D\gamma|^\lambda)\|_{L^{\frac{q}{\lambda}}(\mathbb{R}^n)}^{\frac{1}{\lambda}} \\
&\le C\, \||\alpha|^r\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}^{\frac{1}{r}}\, \||D\gamma|^\lambda\|_{L^{\frac{q}{\lambda}}(\mathbb{R}^n)}^{\frac{1}{\lambda}} \\
&= C\, \|\alpha\|_{L^p(\mathbb{R}^n)} \|D\gamma\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

To apply the Hardy-Littlewood theorem 4.1.6 we needed that $r < p$ and $\lambda < q$. We eventually fix the parameters: first chose $r \in (1, p)$ such that

$$\frac{1}{r} < \frac{1}{p} + \frac{1}{n}. \tag{4.32}$$

This is possible since $\frac{1}{p} < \min(\frac{1}{p} + \frac{1}{n}, 1)$. Then take $\lambda \ge 1$ with

$$\frac{1}{q} < \frac{1}{\lambda} \le 1 + \frac{1}{n} - \frac{1}{r}. \tag{4.33}$$

This is also possible because $\frac{1}{q} < \min(1 + \frac{1}{n} - \frac{1}{r}, 1)$ by (4.32). Recalling the L^q estimate from Lemma 4.2.10, we arrive at the desired estimate

$$\|\alpha \wedge \beta\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C \|\alpha\|_{L^p(\mathbb{R}^n)} \|\beta\|_{L^q(\mathbb{R}^n)}.$$

□

The paper [9] states the theorem for vector fields $E \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $B \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{curl} E = 0$ and $\operatorname{div} B = 0$, claiming that

$$\|\langle E, B\rangle\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\,\|E\|_{L^p(\mathbb{R}^n)}\|B\|_{L^q(\mathbb{R}^n)}.$$

This result is also included in our formulation by considering the forms

$$\alpha = \sum_{i=1}^n E_i\,dx^i \quad \text{and} \quad \beta = B \llcorner dx^1 \wedge \ldots \wedge dx^n = \sum_{i=1}^n (-1)^i B_i\,dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n.$$

We then have $*(\alpha \wedge \beta) = \langle E, B\rangle$. One possible application is to Jacobi determinants of maps $u : \mathbb{R}^n \to \mathbb{R}^n$ with $Du \in L^n(\mathbb{R}^n, \mathbb{R}^{n\times n})$. The theorem then implies

$$\det Du = *(du^1 \wedge \ldots \wedge du^n) \in \mathcal{H}^1(\mathbb{R}^n).$$

In the case $\det Du \geq 0$ this yields another proof of Müller's theorem 4.1.11: one combines Stein's theorem 4.1.8 with Lemma 4.2.6 to obtain, for any ball $B \subset \mathbb{R}^n$,

$$\begin{aligned}
\|\det Du\,\log^+ \det Du\|_{L^1(B)} &\leq C\big(B, \|M(\det Du)\|_{L^1(B)}\big)\\
&\leq C(B, \|(\det Du)^*\|_{L^1(B)})\\
&= C\big(B, \|\det Du\|_{\mathcal{H}^1(\mathbb{R}^n)}\big)\\
&\leq C(B, \|Du\|_{L^n(B)}).
\end{aligned}$$

Furthermore, one obtains a generalization of Wente's theorem 3.2.1 to higher dimensions. This depends on the following regularity result by Fefferman and Stein [14] (see also next section).

Theorem 4.2.12 ([14]). *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$, and assume that $u : \mathbb{R}^n \to \mathbb{R}$ is a solution of*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

Then $u = u_0 + h$ where $h : \mathbb{R}^n \to \mathbb{R}$ is harmonic and $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\|D^2 u_0\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

For $n = 2$ the space $W^{2,1}$ embeds into $C^0 \cap W^{1,2}$, so that Wente's estimates are reproduced.

4.3 Atomic decomposition

In this section we prove that every element in $\mathcal{H}^1$ can be decomposed into so called atoms.

Definition 4.3.1. *A function $a \in L^1(\mathbb{R}^n)$ with the properties*

$$\operatorname{spt} a \subset B, \tag{4.34}$$

$$||a||_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{|B|}, \tag{4.35}$$

$$\int_{\mathbb{R}^n} a(x)dx = 0, \tag{4.36}$$

where $B \subset \mathbb{R}^n$ is a ball, is called $\mathcal{H}^1$-atom.

An example of $\mathcal{H}^1$-atoms is given by the functions in (**??**).

Lemma 4.3.2. *Let $a \in L^1(\mathbb{R}^n)$ be an $\mathcal{H}^1$-atom. Then we have that*

$$||a||_{\mathcal{H}^1(\mathbb{R}^n)} \leq c, \tag{4.37}$$

where the constant c is independent of the atom.

Proof. From the definition of an $\mathcal{H}^1$-atom we can assume that $\operatorname{spt} a \subset B = B_R$. Since a is additionally in $L^\infty(\mathbb{R}^n)$ with vanishing mean value we can apply Lemma **??** (more precisely estimate (**??**)) to get

$$||a||_{\mathcal{H}^1(\mathbb{R}^n)} \leq cR^n ||a||_{L^\infty(\mathbb{R}^n)} \leq c. \tag{4.38}$$

This proves the Lemma. □

In the next Lemma we show that a sum of $\mathcal{H}^1$-atoms also belongs to $\mathcal{H}^1$.

Lemma 4.3.3. *Let $a_k \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$, be a sequence of $\mathcal{H}^1$-atoms and let λ_k, $k \in \mathbb{N}$, be sequence in $\mathbb{R}$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$. Then we have that $f = \sum_{k=1}^\infty \lambda_k a_k \in \mathcal{H}^1(\mathbb{R}^n)$ with*

$$||f||_{\mathcal{H}^1(\mathbb{R}^n)} \leq c \sum_{k=1}^\infty |\lambda_k|. \tag{4.39}$$

Proof. For $N \in \mathbb{N}$ we consider the bounded function

$$f_N(x) = \sum_{k=1}^N \lambda_k a_k(x).$$

Using Lemma 4.3.2 and the assumption for $\sum |\lambda_k|$ we conclude that $f_N \in \mathcal{H}^1(\mathbb{R}^n)$ and moreover with the help of estimate (4.37) we get

$$\begin{aligned} ||f_N - f_M||_{\mathcal{H}^1} &\leq \sum_{k=M}^N |\lambda_k| ||a_k||_{\mathcal{H}^1} \\ &\leq c \sum_{k=M}^N |\lambda_k|, \end{aligned} \tag{4.40}$$

for every $N \geq M \in \mathbb{N}$. If we now define

$$f = f_1 + \sum_{N=1}^\infty (f_{N+1} - f_N) \in L^1(\mathbb{R}^n),$$

we see that (using (4.40))

$$\begin{aligned} ||f||_{\mathcal{H}^1} \leq& c|\lambda_1| + \sum_{N=1}^\infty ||f_{N+1} - f_N||_{\mathcal{H}^1} \\ \leq& c|\lambda_1| + \sum_{N=1}^\infty |\lambda_N| \\ \leq& c \sum_{N=1}^\infty |\lambda_N|, \end{aligned} \tag{4.41}$$

which shows that $f \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover we have (again using (4.40))

$$\begin{aligned} ||f - f_N||_{\mathcal{H}^1} \leq &\sum_{k=N}^{\infty} ||f_{k+1} - f_k||_{\mathcal{H}^1} \\ \leq &c \sum_{k=N}^{\infty} |\lambda_k|. \end{aligned} \tag{4.42}$$

This shows that $f_N \to f$ in $\mathcal{H}^1(\mathbb{R}^n)$ and proves the Lemma. □

The goal of the rest of this section is to prove a statement converse to the one in Lemma 4.3.3, namely that every function $f \in \mathcal{H}^1(\mathbb{R}^n)$ can be decomposed into a sum of $\mathcal{H}^1$-atoms. In order to do this we need a version of the Whitney decomposition which can be found for example in [45].

Lemma 4.3.4. *Let $F \subset \mathbb{R}^n$ be non-empty and closed and let $\Omega = \mathbb{R}^n \backslash F$. Then there exists a collection of cubes $\mathcal{F} = \{Q_1, \dots, Q_k, \dots\}$ so that*

(i) $\cup_k Q_k = \Omega$,

(ii) the Q_k are mutually disjoint and

(iii) $\operatorname{diam} Q_k \leq d(Q_k, F) \leq 4 \operatorname{diam} Q_k$ *for every $k \in \mathbb{N}$.*

Moreover there exists $1 < b$, such that if $Q_k^\star$ denotes the cube which has the same center as Q_k but whose side length is $(1+b)$ times the side length of Q_k, then these cubes satisfy $\cup_k Q_k = \cup_k Q_k^\star$ and the cubes $\{Q_k^\star\}$ have the bounded intersection property.

Proof. We let $\mathcal{Q}_0$ be the set of all cubes with side length 1 and whose corners are in $\mathbb{Z}^n$. Then we let $\mathcal{Q}_l = 2^{-l}\mathcal{Q}_0$ be the cubes with side length 2^{-l} and whose cornes are in $(2^{-l}\mathbb{Z})^n$. The diameter of the cubes in $\mathcal{Q}_l$ is $\sqrt{n}2^{-l}$. Now we let $\Omega = \cup_{l=-\infty}^{\infty} \Omega_l$ where $\Omega_l = \{x \in \Omega | \sqrt{n}2^{-l+1} < d(x,F) \leq \sqrt{n}2^{-l+2}\}$ and we define

$$\mathcal{F}_0 = \cup_{l \in \mathbb{Z}} \{Q \in \mathcal{Q}_l | Q \cap \Omega_l \neq \emptyset\}.$$

For $Q \in \mathcal{F}_0$, e.g. $Q \in \mathcal{Q}_l$ and $x \in Q \cap \Omega_l$, we have

$$d(Q,F) \leq d(x,F) \leq \sqrt{n}2^{-l+2} = 4 \operatorname{diam} Q \tag{4.43}$$

and

$$d(Q,F) \geq d(x,F) - \operatorname{diam} Q \geq \sqrt{n}2^{-l+1} - \sqrt{n}2^{-l} \geq \operatorname{diam} Q. \tag{4.44}$$

This proves (iii) of the Lemma for all cubes in $Q \in \mathcal{F}_0$ and shows in particular that $Q \subset \Omega$. From the definition we therefore get

$$\Omega = \cup_{Q \in \mathcal{F}_0} Q. \tag{4.45}$$

Next we let $\mathcal{F}$ be the family of the maximal cubes of $\mathcal{F}_0$, i.e. all cubes $Q \in \mathcal{F}_0$ such that if $Q' \in \mathcal{F}_0$, $Q \subset Q' \Rightarrow Q = Q'$. Since $F \neq \emptyset$ we see that every $x \in \Omega$ lies in at least one cube with maximal side length. This implies that $\mathcal{F}$ still covers Ω. The fact that the cubes in $\mathcal{F}$ are mutually disjoint follows directly from the definition.
It remains to prove the last statement of the Lemma. We prove this result in three steps:

1) If two cubes $Q_1, Q_2 \in \mathcal{F}$ touch each other, then we have

$$\frac{1}{4}\operatorname{diam} Q_2 \leq \operatorname{diam} Q_1 \leq 4 \operatorname{diam} Q_2. \tag{4.46}$$

To see this we note that since $Q_1 \in \mathcal{F}$ we have $d(Q_1, F) \leq 4\operatorname{diam} Q_1$. This implies $d(Q_2, F) \leq 4\operatorname{diam} Q_1 + \operatorname{diam} Q_1 = 5\operatorname{diam} Q_1$, since Q_1 and Q_2 touch. Since $Q_2 \in \mathcal{F}$ we conclude $\operatorname{diam} Q_2 \leq d(Q_2, F) \leq 5\operatorname{diam} Q_1$. By construction we have $\operatorname{diam} Q_2 = 2^k \operatorname{diam} Q_1$, for some $k \in \mathbb{Z}$, and therefore $\operatorname{diam} Q_2 \leq 4\operatorname{diam} Q_1$. The other inequality follows from symmetry considerations.

2) Let $N = (12)^n$. If $Q \in \mathcal{F}$ then there are at most N cubes in $\mathcal{F}$ which touch Q. If $Q \in \mathcal{Q}_l$ it is easy to see that there are 3^n cubes (including Q) which belong to $\mathcal{Q}_l$ and touch Q. Next, each cube in $\mathcal{Q}_l$ can contain at most 4^n cubes in $\mathcal{F}$ which have diameter larger than or equal to $\frac{1}{4}\operatorname{diam} Q$. Combining this with (4.46) finishes the proof of statement 2).

3) Let $0 < b < \frac{1}{4}$. Then each point $x \in \Omega$ is contained in at most N of the cubes $Q_k^\star$, $Q_k \in \mathcal{F}$.
Let $Q, Q_k \in \mathcal{F}$. We claim that $Q_k^\star$ intersects Q only if Q_k touches Q. In fact, if we consider the union of Q_k with all cubes in $\mathcal{F}$ which touch Q_k it is clear that this union contains $Q_k^\star$ (since the diameter of the cubes touching Q_k are all larger than or equal to $\frac{1}{4}\operatorname{diam} Q_k$). Therefore Q intersects $Q_k^\star$ only if Q touches Q_k. Since any point $x \in \Omega$ is contained in a cube Q we see from step 2) that there are at most N cubes $Q_k^\star$ which contain x.

Since we also have that if $Q_k \in \mathcal{F}$ then $Q_k^\star \subset \Omega$ we get $\Omega = \cup_{Q\in\mathcal{F}} Q^\star$ and this finishes the proof of the Lemma. □

The key step in the atomic decomposition is the following variant of the Calderon-Zygmund decomposition.

Theorem 4.3.5. *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$ and let $\alpha > 0$. Then there exists a decomposition $f = g + b$, with $b = \sum_{k=1}^\infty b_k$, and a countable family of cubes C_k, $k \in \mathbb{N}$, such that*

(i) For a.e. $x \in \mathbb{R}^n$ we have

$$|g(x)| \leq c\alpha. \tag{4.47}$$

(ii) Every function b_k is supported in C_k, satisfies $\int_{C_k} b_k(x)dx = 0$ and

$$||b_k||_{\mathcal{H}^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} b_k^\star(x)dx \leq c\int_{C_k} f^\star(x)dx. \tag{4.48}$$

(iii) The countable family of cubes C_k has the bounded intersection property and if we set $\Omega = \cup_{k=1}^\infty C_k$ we have

$$\Omega = \{x \in \mathbb{R}^n | f^\star(x) > \alpha\}. \tag{4.49}$$

Proof. For $\alpha > 0$ and $f \in \mathcal{H}^1(\mathbb{R}^n)$ we set

$$\Omega = \{x \in \mathbb{R}^n | f^\star(x) > \alpha\}$$

and note that Ω is open. (This follows from the fact that $f^\star$ is obtained by taking the sup over smooth functions and is therefore lower-semicontinuous. Hence the set $\mathbb{R}^n \backslash \Omega = \{x \in \mathbb{R}^n | f^\star(x) \leq \alpha\}$ is compact.) Next we apply the Whitney decomposition of Lemma 4.3.4 to Ω and $F = \mathbb{R}^n \backslash \Omega$ and we denote the cubes which we obtain from this decomposition by C_k'. Moreover we can choose two constants $1 < a < b \in \mathbb{R}$ such that, if $\tilde{C}_k$ and C_k denote the cubes having the same center as C_k' but scaled with the factors a and b, then $\Omega = \cup_{k=1}^{\infty} C_k$ and the family C_k has the bounded intersection property.
Next we consider a positive function ξ satisfying

$$\xi \in C_c^\infty([-\frac{a}{2}, \frac{a}{2}]^n), \quad \xi = 1 \quad \text{on} \quad [-\frac{1}{2}, \frac{1}{2}]^n.$$

Then we set

$$\xi_k = \xi(\frac{x - c_k}{l_k}), \tag{4.50}$$

where c_k denotes the center of the cube C_k' and l_k the length of its edges. Note that $\xi_k \in C_c^\infty(\tilde{C}_k)$ and $\xi_k \equiv 1$ on C_k'. If we define

$$\eta_k(x) = \frac{\xi_k(x)}{\sum_{j=1}^{\infty} \xi_j(x)} \in C_c^\infty(\tilde{C}_k) \tag{4.51}$$

it is not difficult to see that $\{\eta_k\}$ forms a partition of unity subordinated to $\tilde{C}_k$ (here we use the finite intersection property of the $\tilde{C}_k$'s in order to ensure that the sum in the denominator is finite). Moreover we observe that

$$l_k^n = |C_k'| \leq \int_{C_k} \eta_k(x) dx \leq |C_k| = b^n l_k^n \tag{4.52}$$

and

$$||\nabla \eta_k||_{L^\infty} \leq \frac{c}{l_k}, \tag{4.53}$$

for every $k \in \mathbb{N}$, where c does not depend on k.

Now for f and η_k as above we define

$$b_k(x) = (f(x) - a_k)\eta_k(x), \tag{4.54}$$

where

$$a_k = \frac{\int_{C_k} \eta_k(x) f(x) dx}{\int_{C_k} \eta_k(x) dx}.$$

Therefore b_k is supported in C_k and we see that

$$\int_{C_k} b_k(x) dx = 0. \tag{4.55}$$

Moreover we define

$$g(x) = \chi_{\mathbb{R}^n\setminus\Omega}(x)f(x) + \chi_\Omega(x)\sum_{k=1}^{\infty} a_k\eta_k(x). \tag{4.56}$$

Next we claim that

$$|a_k| \le c. \tag{4.57}$$

In order to see this we note that

$$a_k = \int_{C_k} f(x)\phi_k(x)dx = (f\star\phi)(0),$$

where $\phi = \frac{\eta_k}{\int_{C_k}\eta_k(x)dx}$ and therefore

$$||\nabla\phi||_{L^\infty} \le \frac{||\nabla\eta_k||_{L^\infty}}{\int_{C_k}\eta_k(x)dx} \le cl_k^{-n-1},$$

where we used (4.52) and (4.53) in the last step. Therefore we can argue as in the proof of Lemma **??** (more precisely as in the proof of (**??**) with $z \in B_{Cl_k}(c_k)\cap\Omega^c$, the constant C can be chosen in such a way because we consider a Whitney decomposition) to get

$$|a_k| = |(f\star\phi_k)(0)| \le cf^\star(z) \le c\alpha. \tag{4.58}$$

Moreover we see that

$$|a_k| \le cf^\star(x), \tag{4.59}$$

for every $x \in C_k$. Now we want to show that the statement (i) of the Theorem holds. In the case $x \notin \Omega$ we have

$$|g(x)| = |f(x)| \le f^\star(x) \le \alpha, \tag{4.60}$$

where we used the definition of Ω. If $x\in\Omega$ we have, using (4.58)

$$|g(x)| \le \sum_{k=1}^{\infty}|a_k|\eta_k \le c\alpha. \tag{4.61}$$

Now it remains to prove statement (ii) of the Theorem.
We claim that if we have

$$b_k^\star(x) \le cf^\star(x), \quad \text{for} \quad x\in C_k \tag{4.62}$$

$$b_k^\star(x) \le \frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}} \quad \text{for} \quad x\notin C_k, \tag{4.63}$$

then we are done. To see this we estimate

$$\begin{aligned}\int_{\mathbb{R}^n} b_k^\star(x)dx &= \int_{C_k} b_k^\star(x)dx + \int_{\mathbb{R}^n\setminus C_k} b_k^\star(x)dx \\ &\le c\int_{C_k} f^\star(x)dx + \int_{\mathbb{R}^n\setminus C_k}\frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}}dx.\end{aligned}$$

The last integral can be estimated by (remember $\alpha < f^\star(x)$ for $x \in C_k$)

$$\begin{aligned}\int_{\mathbb{R}^n \setminus C_k} \frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}}dx &\le c\alpha l_k^{n+1} \int_{\mathbb{R}^n \setminus B_{\frac{l_k}{2}}(c_k)} \frac{1}{|x-c_k|^{n+1}}dx \\ &\le c\alpha l_k^n \\ &\le c\alpha |C_k| \\ &\le c\int_{C_k} f^\star(x)dx.\end{aligned}$$

Altogether this shows that

$$\int_{\mathbb{R}^n} b_k^\star(x)dx \le c\int_{C_k} f^\star(x)dx. \tag{4.64}$$

We are left with proving the estimates (4.62) and (4.63). First we write for $\phi \in T$ and $x \in C_k$

$$\int_{\mathbb{R}^n} \phi_t(x-y)f(y)\eta_k(y)dy = \int_{\mathbb{R}^n} \varphi_{t,\phi}(y)f(y)dy,$$

where $\varphi_{t,\phi}(y) = \eta_k(y)\phi_t(x-y)$. We estimate, using (4.53)

$$||\nabla\varphi_{t,\phi}||_{L^\infty} \le \frac{c||\nabla\eta_k||_{L^\infty}}{t^n} + c||\nabla\phi_t||_{L^\infty} \tag{4.65}$$

$$\le c(\frac{1}{l_k t^n} + \frac{1}{t^{n+1}}). \tag{4.66}$$

Now we have two cases. The first one is that $t \le l_k$. In this situation we have that $||\nabla\varphi_{t,\phi}||_{L^\infty} \le \frac{c}{t^{n+1}}$ and we can again argue as in the proof of (**??**) ($\varphi_{t,\phi} \in C_c^\infty(B_t(x))$) to get

$$\sup_{\phi\in T}\sup_{0<t\le l_k} |\int_{\mathbb{R}^n} \varphi_{t,\phi}(y)f(y)dy| \le cf^\star(x). \tag{4.67}$$

In the case $t > l_k$ we have that $||\nabla\varphi_{t,\phi}||_{L^\infty} \le \frac{c}{l_k^{n+1}}$ and $\varphi_{t,\phi}$ is supported in $B_{cl_k(x)}$ (remember $\eta_k \in C_c^\infty(C_k)$), where c is chosen such that $C_k \subset B_{cl_k}(x)$. Therefore we can argue as above to get

$$\sup_{\phi\in T}\sup_{t>l_k} |\int_{\mathbb{R}^n} \varphi_{t,\phi}(y)f(y)dy| \le cf^\star(x). \tag{4.68}$$

Combining (4.67) and (4.68) we arrive at

$$(f\eta_k)^\star(x) = \sup_{\phi\in T}\sup_{0<t} |\int_{\mathbb{R}^n} \phi_t(x-y)f(y)\eta_k(y)dy| \tag{4.69}$$

$$\le cf^\star(x). \tag{4.70}$$

This implies that for $x \in C_k$ we have

$$b_k^\star(x) \le (f\eta_k)^\star(x) + |a_k|\eta_k^\star(x) \le cf^\star(x), \tag{4.71}$$

where we used (4.59) to get that $|a_k|\eta_k^\star(x) \leq |a_k| \leq cf^\star(x)$. This shows (4.62).
We use (4.55) to get

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_t(x-y)b_k(y)dy =& \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-c_k))b_k(y)dy \\ =& I_1 - I_2, \end{aligned} \tag{4.72}$$

where

$$\begin{aligned} I_1(x) =& \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-c_k))f(y)\eta_k(y)dy \\ I_2(x) =& \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-c_k))a_k(y)\eta_k(y)dy. \end{aligned}$$

This time we define $\psi_{t,\phi}(y) = (\phi_t(x-y) - \phi_t(x-c_k))\eta_k(y)$. Now we note that $\int_{\mathbb{R}^n} \phi_t(x-y)b_k(y)dy \neq 0$ only if the supports of b_k and $\phi_t(x-\cdot)$ intersect. This is the case if $t \geq c|x-c_k|$. From this we get by the mean value theorem

$$\begin{aligned} |\psi_{t,\phi}(y)| \leq& |\eta_k(y)|||\nabla\phi||_{L^\infty}|y-c_k| \\ \leq& \frac{cl_k}{|x-c_k|^{n+1}}. \end{aligned} \tag{4.73}$$

Another application of the mean value theorem and (4.53) yields

$$\begin{aligned} |\nabla\psi_{t,\phi}(y)| \leq& |\nabla\phi_t(x-y)| + |\nabla\eta_k(y)||\phi_t(x-y) - \phi_t(x-c_k)| \\ \leq& \frac{c}{t^{n+1}} + \frac{c}{l_k}\frac{cl_k}{t^{n+1}} \\ \leq& \frac{c}{t^{n+1}}. \end{aligned} \tag{4.74}$$

Next we argue again as in the proof of (**??**) (this time applied to the function $\psi_{t,\phi} \in C_c^\infty(B_{cl_k}(c_k))$, where c is chosen such that there exists $z \in B_{cl_k}(c_k) \cap \Omega^c$) to get

$$|\psi_{t,\phi} \star f| = |I_1(x)| \leq \frac{cl_k^{n+1}f^\star(z)}{|x-c_k|^{n+1}} \leq \frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}}, \tag{4.75}$$

where we also used that by (4.74) we have $|\nabla\psi_{t,\phi}(y)| \leq \frac{A}{l_k^{n+1}}$ with $A = \frac{cl_k^{n+1}}{|x-c_k|^{n+1}}$. On the other hand I_2 can be estimated by

$$|I_2(x)| \leq c|a_k||C_k|||\psi_{t,\phi}||_{L^\infty} \leq c\frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}}, \tag{4.76}$$

where we used (4.73) and (4.58). Combining (4.72), (4.75) and (4.76) we prove (4.63) and therefore the Theorem. □

In the next Theorem we finally prove the atomic decomposition.

Theorem 4.3.6. *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$. Then there exists a sequence a_k, $k \in \mathbb{N}$, of $\mathcal{H}^1$-atoms and a sequence $\lambda_k \in \mathbb{R}$, such that*

$$f = \sum_{k=1}^\infty \lambda_k a_k, \tag{4.77}$$

where the convergence is in the $\mathcal{H}^1$-norm, and moreover

$$\sum_{k=1}^{\infty} |\lambda_k| \leq c||f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.78}$$

Proof. For every $j \in \mathbb{Z}$ we apply Theorem 4.3.5 to $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $\alpha = 2^j > 0$. From this we get the decomposition

$$f(x) = g^j(x) + b^j(x) = g^j(x) + \sum_{k=1}^{\infty} b_k^j, \tag{4.79}$$

and the countable family of cubes $\{C_k^j\}_{k\in\mathbb{N}}$. We estimate

$$\begin{aligned} ||f - g^j||_{\mathcal{H}^1} = ||b^j||_{\mathcal{H}^1} &\leq \sum_k ||b_k^j||_{\mathcal{H}^1} \\ &\leq c\sum_k \int_{C_k^j} f^\star(x)dx \\ &\leq c\int_{\Omega^j} f^\star(x)dx \\ &= c\int_{\{x\in\mathbb{R}^n | f^\star(x) > 2^j\}} f^\star(x)dx, \end{aligned} \tag{4.80}$$

where we used (4.48) in the second line, the bounded intersection property of $\Omega^j = \cup_k C_k^j$ in the third line and (4.49) in the last step. Since $f^\star \in L^1(\mathbb{R}^n)$ we therefore conclude that

$$||f - g^j||_{\mathcal{H}^1} \to 0, \tag{4.81}$$

as $j \to \infty$. Since by (4.47) we also know that $|g^j(x)| \leq c2^j$ a.e. we conclude that

$$g^j \to 0, \tag{4.82}$$

in the sense of distributions as $j \to -\infty$. Combining (4.81) and (4.82) we get

$$\begin{aligned} f &= \lim_{N\to\infty} g^N + \lim_{N\to-\infty} g^N \\ &= \lim_{N\to\infty} \sum_{j=1}^{N}(g^{j+1} - g^j) + \lim_{N\to\infty}(\sum_{j=0}^{-N}(g^{j+1} - g^j) \\ &= \sum_{j=-\infty}^{\infty} (g^{j+1} - g^j), \end{aligned} \tag{4.83}$$

where the convergence is in the sense of distributions. From (4.79) we have that

$$g^{j+1}(x) - g^j(x) = (f(x) - b^{j+1}(x)) - (f(x) - b^j(x)) = b^j(x) - b^{j+1}(x). \tag{4.84}$$

Therefore $g^{j+1} - g^j$ is supported in $\Omega^j \supset \Omega^{j+1}$. Hence we can write

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty}(g^{j+1} - g^j)\eta_k^j, \tag{4.85}$$

where $\eta_k^j \in C_c^\infty(C_k^j)$ is the partition of unity subordinated to $\{C_k^j\}$ which was defined in the proof of Theorem 4.3.5. From (4.47) we also get for a.e. $x \in \mathbb{R}^n$

$$|g^{j+1}(x) - g^j(x)| \leq C2^j. \tag{4.86}$$

Now we define real numbers

$$\lambda_{j,k} = C2^j |B_k^j|, \tag{4.87}$$

where C is the constant from (4.86) and B_k^j is the smallest ball containing C_k^j, and functions

$$a_{j,k}(x) = \frac{1}{\lambda_{j,k}} A_{j,k}, \tag{4.88}$$

where $A_{j,k}$ is supported in C_k^j and is defined by

$$A_{j,k} = (g^{j+1}(x) - g^j(x))\eta_k^j(x). \tag{4.89}$$

This implies in particular that $a_{j,k}$ is supported in B_k^j and because of (4.86) and (4.87) we have

$$||a_{j,k}||_{L^\infty} \leq \frac{1}{\lambda_{j,k}} ||g^{j+1} - g^j||_{L^\infty} \leq \frac{1}{|B_k^j|}. \tag{4.90}$$

Moreover we have that

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} |\lambda_{j,k}| &\leq C \sum_{j=-\infty}^{\infty} 2^j |\Omega^j| \\
&= c \sum_{j=-\infty}^{\infty} 2^j |\{x \in \mathbb{R}^n | f^\star(x) > 2^j\}| \\
&= c \sum_{j=-\infty}^{\infty} (\sum_{k=-\infty}^{j} 2^k) |\{x \in \mathbb{R}^n | 2^{j+1} \geq f^\star(x) > 2^j\}| \\
&\leq c \sum_{j=-\infty}^{\infty} 2^j |\{x \in \mathbb{R}^n | 2^{j+1} \geq f^\star(x) > 2^j\}| \\
&\leq c \int_{\mathbb{R}^n} f^\star(x) dx \\
&= c||f||_{\mathcal{H}^1}.
\end{aligned} \tag{4.91}$$

Because of (4.85), (4.88) and (4.89) we also get that

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \lambda_{j,k} a_{j,k}. \tag{4.92}$$

This would yield the desired atomic decomposition of f if we would additionally have that $\int_{\mathbb{R}^n} a_{j,k}(x)dx = 0$. Since this is not true we have to modify the decomposition.
First we note that because of (4.84) and (4.89) we can write

$$A_{j,k}(x) = (b^j(x) - b^{j+1}(x))\eta_k^j(x) = b_k^j(x) - \sum_{l=1}^{\infty} b_l^{j+1}(x)\eta_k^j(x). \tag{4.93}$$

Then we define new functions

$$B_{j,k}(x) = A_{j,k}(x) + \sum_{l=1}^{\infty} b_{l,k}\eta_l^{j+1}(x), \tag{4.94}$$

where

$$b_{l,k} = \frac{\int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx}. \tag{4.95}$$

This implies that

$$\begin{aligned}\int_{\mathbb{R}^n} B_{j,k}(x)dx &= \int_{\mathbb{R}^n} A_{j,k}(x)dx + \sum_{l=1}^{\infty}\int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx \\ &= \int_{\mathbb{R}^n} b_k^j(x)dx - \sum_{l=1}^{\infty}\int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx + \sum_{l=1}^{\infty}\int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx \\ &=0.\end{aligned} \tag{4.96}$$

Moreover we have that

$$\sum_{k=1}^{\infty} b_{l,k} = \frac{\sum_{k=1}^{\infty}\int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^l(x)dx}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx} = \frac{\int_{\mathbb{R}^n} b_l^{j+1}(x)dx}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx} = 0. \tag{4.97}$$

and therefore

$$\sum_{k=1}^{\infty} B_{j,k} = \sum_{k=1}^{\infty} A_{j,k}. \tag{4.98}$$

This shows that (using (4.85) and (4.89))

$$f = \sum_{j=-\infty}^{\infty}\sum_{k=1}^{\infty} B_{j,k}. \tag{4.99}$$

Now we have to study the numbers $b_{l,k}$ in more detail. Since $\operatorname{spt} b_l^{j+1} \subset C_l^{j+1}$ we see that $b_{l,k} \neq 0$ only if $C_k^j \cap C_l^{j+1} \neq \emptyset$. This shows that $\operatorname{spt} B_{j,k} \subset B_k^j$. Next we want to show that $|B_{j,k}(x)| \leq c2^j$. To show this we need to estimate $|b_{l,k}|$. First of all we note that if $C_k^j \cap C_l^{j+1} \neq \emptyset$ we have

$$c_1 \operatorname{diam} C_k^j \leq \operatorname{diam} C_l^{j+1} \leq c_2 \operatorname{diam} C_k^j.$$

To see this we use the properties of the Whitney decomposition (Lemma 4.3.4) to conclude

$$\operatorname{diam} C_l^{j+1} \leq d(C_l^{j+1}, (\Omega^{j+1})^c) \leq d(C_l^{j+1}, (\Omega^j)^c) \leq \operatorname{diam} C_k^j + d(C_k^j, (\Omega^j)^c) \leq c \operatorname{diam} C_k^j.$$

The other inequality follows from symmetry considerations. Hence we also have that

$$cl_k^j \leq l_l^{j+1} \leq cl_k^j,$$

where l_k^j denotes the side length of the cube C_k^j. Therefore if we define

$$\phi(x) = \frac{\eta_k^j(x)}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx}$$

we have that $\operatorname{spt}\phi \subset B_k^j \subset B_{C_1 l_l^{j+1}}(c_l^{j+1})$, where C_1 is chosen such that $B_{C_1 l_l^{j+1}}(c_l^{j+1}) \cap (\Omega^{j+1})^c \neq \emptyset$, and $(\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx \geq (l_l^{j+1})^n)$

$$||\nabla\phi||_{L^\infty} \leq \frac{c}{l_k^j (l_l^{j+1})^n} \leq \frac{c}{(l_l^{j+1})^{n+1}}.$$

Therefore we can argue as in the proof of the Calderon-Zygmund decomposition (Lemma 4.3.5) to get for $z \in B_{C_1 l_l^{j+1}}(c_l^{j+1}) \cap (\Omega^{j+1})^c$

$$|b_{l,k}| = |\int_{\mathbb{R}^n} b_l^{j+1}(x)\phi(x)dx| \leq c(b_l^{j+1})^\star(z) \leq c2^j, \tag{4.100}$$

where we used (4.63) (note that $|z - c_l^{j+1}| \geq c l_l^{j+1}$ by our choices) in the last inequality. Altogether this shows that

$$|B_{j,k}(x)| \leq c2^j + \sum_{l=1}^{\infty} |b_{l,k}|\eta_l^{j+1}(x) \leq c2^j. \tag{4.101}$$

Therefore if we define

$$\tilde{b}_{j,k}(x) = \frac{1}{\lambda_{j,k}} B_{j,k}(x), \tag{4.102}$$

with $\lambda_{j,k}$ as in (4.87), we see that $\operatorname{spt}\tilde{b}_{j,k} \subset B_k^j$, $\int_{\mathbb{R}^n} \tilde{b}_{j,k}(x)dx = 0$ (here we use (4.96)) and

$$||\tilde{b}_{j,k}||_{L^\infty} \leq \frac{C2^j}{\lambda_{j,k}} \leq \frac{1}{|B_k^j|}. \tag{4.103}$$

This shows that the $\tilde{b}_{j,k}$'s are $\mathcal{H}^1$-atoms and from (4.99) we get

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \lambda_{j,k}\tilde{b}_{j,k}. \tag{4.104}$$

Together with (4.91) this proves the Theorem. □

In the rest of this section we want to show how one can use the atomic decomposition to extend the L^p-theory to the situation where the right hand side is only in $\mathcal{H}^1$. For sake of simplicity we restrict ourselves to the case of the Laplace operator (for the general case see [46]).

Lemma 4.3.7. *Let* $\Gamma(x) = \frac{1}{n(2-n)\omega_n}|x|^{2-n}$*, resp.* $\Gamma(x) = -\frac{1}{2\pi}\ln(\frac{1}{|x|})$*, be the fundamental solution of* Δ *for* $n \geq 3$*, resp.* $n = 2$*. Defining* $K_{ij}(x) = \partial_i\partial_j\Gamma(x)$*, for every* $i, j \in \{1, \ldots, n\}$*, we have that* *(i)* $||K_{ij} \star f||_{L^2} \le c||f||_{L^2}$, *for every* $f \in L^2(\mathbb{R}^n)$ *and*

(ii) $\int_{2|y|\le|x|} |K_{ij}(x-y) - K_{ij}(x)|dx \le c$.

Proof. (i) is the classical L^2 estimate for the fundamental solution of Δ and can be found in [19]. For (ii) we note that $|\partial_l K_{ij}| \le \frac{c}{|x|^{n+1}}$ for every $i,j,l \in \{1,\dots,n\}$. Therefore we get for $2|y| \le |x|$ with the help of the mean value theorem that

$$|K_{ij}(x-y) - K_{ij}(x)| \le \frac{c|y|}{|x|^{n+1}}. \tag{4.105}$$

Using this we estimate

$$\int_{2|y|\le|x|} |K_{ij}(x-y) - K_{ij}(x)|dx \le c|y| \int_{2|y|}^{\infty} r^{-2} dr \le c. \tag{4.106}$$

□

Now we can prove the regularity result for equations of the form $\Delta u = f \in \mathcal{H}^1$.

Theorem 4.3.8. *Let* $f \in \mathcal{H}^1(\mathbb{R}^n)$ *and let* $u \in W^{2,1}(\mathbb{R}^n)$ *be a solution of* $\Delta u = f$*, then we have*

$$||\nabla^2 u||_{L^1(\mathbb{R}^n)} \le c||f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.107}$$

Proof. From the above considerations it is easy to see that the result follows if we show that

$$||K_{ij} \star f||_{L^1(\mathbb{R}^n)} \le c||f||_{\mathcal{H}^1(\mathbb{R}^n)}, \tag{4.108}$$

for every $i,j \in \{1,\dots,n\}$.
To see this we first consider an $\mathcal{H}^1$-atom a which is supported in a ball B_r. From (i) of Lemma 4.3.7 we get that

$$\begin{aligned} ||K_{ij} \star a||^2_{L^2} &\le c||a||^2_{L^2(B_r)} \\ &\le c|B_r|||a||^2_{L^\infty} \\ &\le \frac{c}{|B_r|}, \end{aligned} \tag{4.109}$$

where we used the properties of an atom. From Hölder‘s inequality we then get

$$\begin{aligned} ||K_{ij} \star a||_{L^1(B_{2r})} &\le ||K_{ij} \star a||_{L^2} \sqrt{|B_{2r}|} \\ &\le c. \end{aligned} \tag{4.110}$$

Now we note that

$$K_{ij} \star a(x) = \int_{B_r} (K_{ij}(x-y) - K_{ij}(x)) a(y) dy, \tag{4.111}$$

where we used the cancellation property of an atom. From this it follows that

$$\begin{aligned} \int_{B^c_{2r}} |K_{ij} \star a|(x)dx &\le \int_{B^c_{2r}} \int_{B_r} |K_{ij}(x-y) - K_{ij}(x)||a(y)|dydx \\ &= \int_{B_r} |a(y)|(\int_{B^c_{2r}} |K_{ij}(x-y) - K_{ij}(x)|dx)dy. \end{aligned} \tag{4.112}$$

Since $|x| \geq 2r$ and $|y| \leq r$ we can apply (ii) from Lemma 4.3.7 to get

$$\int_{B_{2r}^c} |K_{ij} \star a|(x)dx \leq c \int_{B_r} |a(y)|dy \leq c. \tag{4.113}$$

Combining (4.110) amd (4.113) this shows that

$$\int_{\mathbb{R}^n} |K_{ij} \star a|(x)dx \leq c. \tag{4.114}$$

Now we use Theorem 4.3.6 to get $f = \sum_k \lambda_k a_k$ with $\sum_k |\lambda_k| \leq c||f||_{\mathcal{H}^1}$. Then we have

$$\begin{aligned} ||K_{ij} \star f||_{L^1} &\leq c \sum_k |\lambda_k| \int_{\mathbb{R}^n} |K_{ij} \star a_k|(x)dx \\ &\leq c \sum_k |\lambda_k| \\ &\leq c||f||_{\mathcal{H}^1}. \end{aligned} \tag{4.115}$$

This proves (4.108) and therefore the Theorem. □

Remark 4.3.9. *One can actually show that* $\nabla^2 u \in \mathcal{H}^1(\mathbb{R}^n)$ *(see [46]).*

Chapter 5

Regularity of geometric variational problems

This chapter addresses the regularity of critical points of two-dimensional, conformally invariant variational integrals. The case of harmonic maps was settled by Hélein [23], whereas the general result including surfaces of prescribed, variable mean curvature is due to T. Rivière [37]. It is his proof that is presented here.

5.1 Gauge transformation

In this section we prove an existence result for Coulomb gauges due to Uhlenbeck [50]. The issue is to construct a preferred gauge for a connection on a vector bundle over a Riemannian manifold. More precisely the theorem deals with a local situation $B \times \mathbb{R}^m$, where B is the unit ball in $\mathbb{R}^n$ and the $\mathbb{R}^m$ factor represents the coordinates with respect to a given frame. A connection is then given by a matrix-valued one-form $A = A_i(x)\,dx^i$ with $A_i(x) \in \mathbb{R}^{m \times m}$. It induces a notion of parallel vector fields along curves $\gamma : [a,b] \to B$ along curves $\gamma : [a,b] \to B$ by the linear ordinary differential equation

$$\frac{\nabla_A v}{dt} = v' + A(\gamma')\,v = 0 \quad \text{ where } v : [a,b] \to \mathbb{R}^m. \tag{5.1}$$

We should really write $(A \circ \gamma)(\gamma')$, however it is customary to omit the basepoint. The connection is often denoted by its local form $\nabla_A = d + A$. For simplicity we restrict to SO_m bundles; this means that the bundle is oriented and carries a Riemannian metric. The $\mathbb{R}^m$ factor represents the coordinates with respect to some oriented orthonormal frame, thus the bundle metric becomes the standard scalar product $\langle \cdot, \cdot \rangle$. The connections are required to be compatible in the sense of the product rule, for any vector fields ϕ, ψ along γ,

$$\frac{d}{dt}\langle v, w\rangle = \langle \frac{\nabla_A v}{dt}, \psi\rangle + \langle v, \frac{\nabla_A w}{dt}\rangle \quad \Leftrightarrow \quad \big\langle A(\gamma')\,e_i, e_j\big\rangle + \big\langle e_i, A(\gamma')\,e_j\big\rangle = 0.$$

Thus A is an $\mathfrak{so}_m$-valued one-form. Any oriented orthonormal frame $\mathcal{F} = \{v_1, \ldots, v_m\}$ over B induces new coordinates $v_{\mathcal{F}}$, such that $v = Pv_{\mathcal{F}}$ for some $P : B \to \mathrm{SO}_m$. It follows that

$$\begin{aligned}\Big(\frac{\nabla_A v}{dt}\Big)_{\mathcal{F}} &= P^{-1}\frac{\nabla_A}{dt}(Pv_{\mathcal{F}}) \\ &= P^{-1}(Pv_{\mathcal{F}})' + P^{-1}A(\gamma')Pv_{\mathcal{F}} \\ &= v_{\mathcal{F}}' + \big(P^{-1}dP(\gamma') + P^{-1}A(\gamma')P\big)v_{\mathcal{F}}.\end{aligned}$$

The map $P : B \to \mathrm{SO}_m$ is called a gauge transformation, and the one-form $P^{-1}dP + P^{-1}AP$ is the transformed connection. The group of gauge transformations acts isometrically on the space of connections with respect to the L^2 distance

$$\mathrm{dist}(\nabla_A, \nabla_B)^2 = \int_B |\nabla_A - \nabla_B|^2\,dx = \int_B |A - B|^2\,dx.$$

In fact we have

$$\int_B |P^{-1}\nabla_A P - P^{-1}\nabla_B P|^2\,dx = \int_B |P^{-1}(\nabla_A - \nabla_B)P|^2\,dx = \int_B |\nabla_A - \nabla_B|^2\,dx.$$

It is therefore natural to ask whether any gauge orbit contains an element which minimizes the distance to the trivial connection $d = \nabla_0$. If ∇_A is the desired minimizer, then we get by chosing $P = \exp(t\chi)$ with $\chi : B \to \mathfrak{so}_m$

$$0 = \frac{1}{2}\frac{d}{dt}\int_B |e^{-t\chi}d(e^{t\chi}) + e^{-t\chi}Ae^{t\chi}|\,dx|_{t=0} = \int_B \langle A, d\chi\rangle.$$

This means that the minimizer is a weak solution to the equations

$$d * A = 0 \text{ in } B, \qquad \nu \llcorner A = 0 \text{ on } \partial B.$$

These are called the Coulomb or Hodge gauge conditions. The following result is due to Uhlenbeck [50].

Theorem 5.1.1. *Let $A_0 \in L^2(B, \Lambda^1 \otimes \mathfrak{so}_m)$ be a connection on $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Then there exists a gauge transformation $P \in W^{1,2}(B, \mathrm{SO}_m)$, such that $A = P^{-1}dP + P^{-1}A_0P \in L^2(B, \Lambda^1 \otimes \mathfrak{so}_m)$ has the following properties:*

(1) *A solves the system $d^*A = 0$ in B, $\nu \llcorner A = 0$ on ∂B.*

(2) $\|dP\|_{L^2(B)} + \|A\|_{L^2(B)} \leq C\|A_0\|_{L^2(B)}$.

(3) *There is a $\xi \in W^{1,2}(B, \Lambda^2 \otimes \mathfrak{so}_m)$ with $i^*_{\partial B}(d * \xi) = 0$ on ∂B, such that*

$$d^*\xi = A \quad \text{and} \quad \|\xi\|_{W^{1,2}(B)} \leq C\|A\|_{L^2(B)}.$$

Proof. As outlined we consider a minimizing sequence $P_k \in W^{1,2}(B, \mathrm{SO}_m)$ for the functional

$$E(P) = \int_B |P^{-1}dP + P^{-1}A_0P|^2\,dx = \int_B |dP + A_0P|^2\,dx. \tag{5.2}$$

We have the inequality

$$E(P) \geq (1 - \varepsilon)\int_B |dP|^2\,dx - C_\varepsilon \int_B |A_0|^2. \tag{5.3}$$

Moreover $|P_k| = n$, thus we can assume $P_k \to P \in W^{1,2}(B, \mathrm{SO}_m)$ weakly in $W^{1,2}$, strongly in L^2 and pointwise almost everywhere. As $|A_0 P_k| = |A_0|$ we get $A_0 P_k \to A_0 P$ in $L^2(B)$ by Vitali's convergence theorem. Thus the infimum is attained by P, and

$$E(P) \leq \liminf_{k\to\infty} E(P_k) \leq E(\mathrm{Id}) = \int_B |A_0|^2.$$

Put $A = P^{-1}dP + P^{-1}A_0 P$. Then $\|A\|_{L^2(B)} \leq \|A_0\|_{L^2(B)}$ by the minimizing property, and from $dP = PA - A_0 P$ we see that

$$\|dP\|_{L^2(B)} \leq \|A\|_{L^2(B)} + \|A_0\|_{L^2(B)} \leq 2\|A_0\|_{L^2(B)}.$$

Moeoever, as explained above, A satisfies the weak Hodge gauge conditions

$$\int_B \langle d\chi, A\rangle\, dx = 0 \quad \text{for all smooth } \chi : \overline{B} \to \mathfrak{so}_m.$$

To show claim (3) we employ linear Hodge theory. By Lemma 3.3.1 in Chapter 3 there exists a form $\xi \in W^{1,2}(B, \Lambda^2)$ such that

$$A = d^*\xi \quad \text{where} \int_{\partial B} *(\nu \llcorner \xi) = 0 \text{ and } \|\xi\|_{W^{1,2}(B)} \leq \|A\|_{L^2(B)}.$$

Our proof in Chapter 3 was only in two dimensions, but its generalization is straightforward. For any smooth $\chi : \overline{B} \to \mathfrak{so}_m$ we compute by partial integration

$$\begin{aligned} \int_B \langle d\chi, A\rangle\, dx &= \int_B d\chi \wedge *d^*\xi \\ &= (-1)^n \int_B d\chi \wedge d*\xi \\ &= (-1)^n \int_{\partial B} \chi\, i^*_{\partial B}(d*\xi). \end{aligned}$$

The weak version of the Hodge gauge condition (1) implies $i^*_{\partial B}(d*\xi) = 0$. □

In dimension $n = 2$ we get $\xi = 0$ on the boundary. Namely for $\xi = \xi_0\, dx^1 \wedge dx^2$ we have $d*\xi = d\xi_0$, and the normalization becomes $\int_{\partial B} \xi_0\, d\theta = 0$. The presented gauge theorem is weak in the sense that no additional regularity of P and A is asserted. It is possible that P has singularities which change the topological type of the bundle. In dimensions $n \leq 4$ Uhlenbeck proved a stronger version where P is estimated in $W^{2,2}$, and accordingly A in $W^{1,2}$. These estimates depend on a smallness assumption for the L^2 norm of the curvature $F = dA + A\wedge A$. For $n \leq 3$ the smallness threshold can always be achieved by scaling, whereas in the critical dimension $n = 4$ it is a necessary, nontrivial condition.

5.2 Equations of the form $\Delta u = \Omega\nabla u$

Let $B \subset \mathbb{R}^2$ be the unit ball in $\mathbb{R}^2$. In this section we study the regularity properties of solutions of elliptic systems of the form

$$-\Delta u = \Omega\nabla u, \tag{5.4}$$

where $u \in W^{1,2}(B,\mathbb{R}^m)$ and $\Omega \in L^2(B, so(m)\otimes \wedge^1\mathbb{R}^2)$. Before coming to the detailed study let us give some examples for systems of the type (5.4).

1) From (**??**) we see that harmonic maps into spheres satisfy an equation of the form (5.4) with $(\Omega^{ij}) = (u^i\nabla u^j - u^j\nabla u^i) \in L^2(B, so(m)\otimes \wedge^1\mathbb{R}^2)$.

2) It is easy to see that surfaces with prescribed mean curvature $H \in L^\infty(\mathbb{R}^3)$ (i.e. solutions of (**??**)) solve a system of the form (5.4) with

$$\Omega = -2H(u)\begin{pmatrix} 0 & \nabla^\perp u^3 & -\nabla^\perp u^2 \\ -\nabla^\perp u^3 & 0 & \nabla^\perp u^1 \\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0 \end{pmatrix} \in L^2(B, so(3)\otimes\wedge^1\mathbb{R}^2).$$

3) Harmonic maps into general target manifolds.
Here we let $u \in W^{1,2}(B,N)$, where $N \hookrightarrow \mathbb{R}^m$ is a smooth and compact Riemannian manifold without boundary. Then we know from the discussions in chapter 2 that harmonic maps into N are critical points of the functional

$$E(u) = \frac{1}{2}\int_B |\nabla u|^2 dv_g.$$

To compute the critical points of E we let $\varphi \in C^1_c(B,\mathbb{R}^m)$ with $\varphi(x) \in T_{u(x)}N$ for all $x \in B$. Then we compute

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} E(u+t\varphi) \\ &= -\int_B \Delta u \varphi. \end{aligned}$$

Since this is true for all such φ we know that

$$\Delta u \perp T_u N.$$

Therefore if we let $\{\nu_{n+1},\dots,\nu_m\}$ be a smooth local orthonormal frame for the normal bundle near $u(x)$ we can write

$$\Delta u(x) = \sum_{i=n+1}^m \lambda_i(x)\nu_i(u(x)),$$

where the λ_i are scalar functions. Using the fact that $\langle \nabla u, \nu_i(u)\rangle = 0$ for every $i \in \{n+1,\dots,m\}$ we get

$$\begin{aligned} \lambda_i =& \langle \Delta u, \nu_i(u)\rangle \\ =& \operatorname{div}\langle \nabla u, \nu_i(u)\rangle - \langle \nabla_j u, (d_k\nu_i)(u)\nabla_j u^k\rangle \end{aligned}$$

and hence

$$\begin{aligned} \Delta u =& \sum_i \lambda_i \nu_i(u) \\ =& -\sum_{i=n+1}^m \sum_{k=1}^m \sum_{j=1}^2 \langle \nabla_j u, (d_k\nu_i)(u)\nabla_j u^k\rangle \nu_i(u) \\ =& -A(u)(\nabla u, \nabla u). \end{aligned}$$

Moreover, using the definition of A, we see that (using that $\sum_k \nabla u^k \nu_i^k(u) = 0$ for every i)

$$\begin{aligned}\Delta u^s &= -\sum_{i,k}\langle \nabla u, (d_k\nu_i)(u)\nabla u^k\rangle \nu_i^s(u)\\ &= -\sum_{i,k,l}\nabla u^k(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l),\end{aligned}$$

and hence u solves an equation of the form (5.4) with

$$(\Omega_{sk}) = (\sum_{i,l}(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l)) \in L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2).$$

4) Conformally invariant variational problems.
We consider the functional

$$E_\omega(u) = \frac{1}{2}\int_B(|\nabla u|^2 + \omega(u)(\partial_x u, \partial_y u))dx,$$

where ω is a C^1 two-form on $\mathbb{R}^m$ such that the L^∞-norm of $d\omega$ is bounded. By Theorem 2.4.1 we see that every conformally invariant energy in two-dimensions can be written in this way. The Euler-Lagrange equation of E_ω can easily be computed to be

$$\Delta u^i + A^i(u)(\nabla u, \nabla u) + \lambda^i_{jl}(u)\partial_x u^j\partial_y^l = 0,$$

where $\lambda^i_{jl}(u) = d\omega(u)(e_i, e_j, e_l)$ and where $\{e_i\}_{\{i=1,\dots,m\}}$ is the standard basis of $\mathbb{R}^m$. Using that $\lambda^i_{jl} = -\lambda^j_{il}$ we calculate

$$\lambda^i_{jl}(u)\partial_x u^j\partial_y^l = \frac{1}{4}(\lambda^i_{jl}(u) - \lambda^j_{il}(u))\nabla^\perp u^l\nabla u^j.$$

Combining this with the result of 3) we see that the Euler-Lagrange equation of every conformally invariant energy in two dimensions can be written in the form (5.4) with

$$\begin{aligned}\Omega_{sk} =& \sum_{i,l}(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l)\\ &-\sum_l\frac{1}{4}(\lambda^s_{kl}(u) - \lambda^k_{sl}(u))\nabla^\perp u^l\\ &\in L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2).\end{aligned}$$

After having collected all these examples of systems of the type (5.4) we now state the main Theorem of this chapter. This Theorem was only recently proved by Tristan Rivière [37] (see also [30], [38] and [49] for related results).

Theorem 5.2.1. *Let $u \in W^{1,2}(B, \mathbb{R}^m)$ be a solution of* (5.4) *with $\Omega \in L^2(B, so(m)\otimes\wedge^1\mathbb{R}^2)$. Then u is continuous and therefore by Theorem* **??** *as smooth as the data permits.*

Proof. The Theorem will be proved in three steps.
<u>Step 1:</u>

Lemma 5.2.2. *Let $m \in \mathbb{N}$ and $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$. Let $A \in L^\infty \cap W^{1,2}(B, M(m))$ and $B \in W^{1,2}(B, M(m))$ be solutions of*

$$\nabla A - A\Omega = \nabla^\perp B. \tag{5.5}$$

Then $u \in W^{1,2}(B, \mathbb{R}^m)$ is a solution of (5.4) with Ω iff

$$\operatorname{div}(A\nabla u + B\nabla^\perp u) = 0. \tag{5.6}$$

Proof. By a direct calculation (using that $\operatorname{div}\nabla^\perp = 0$ and $\nabla u \nabla^\perp v = -\nabla^\perp u \nabla v$) and using (5.5) we get

$$\begin{aligned}\operatorname{div}(A\nabla u + B\nabla^\perp u) =&(\nabla A - \nabla^\perp B)\nabla u + A\Delta u\\ =&A(\Delta u + \Omega\nabla u).\end{aligned}$$

This proves the Lemma. □

Step 2:

Lemma 5.2.3. *There exists $\varepsilon > 0$, $c > 0$ such that for every $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$ with*

$$\int_B |\Omega|^2 dx < \varepsilon, \tag{5.7}$$

there exist $A \in L^\infty \cap W^{1,2}(B, Gl(m))$ and $B \in W^{1,2}(B, M(m))$ satisfying

$$\int_B (|\nabla A|^2 + |\nabla B|^2)dx + ||\operatorname{dist}(A, SO(n))||^2_{L^\infty} \le c\int_B |\Omega|^2 \quad \textit{and} \tag{5.8}$$

$$\nabla A - A\Omega - \nabla^\perp B = 0. \tag{5.9}$$

Proof. For $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$ with $\int_B |\Omega|^2 dx < \varepsilon$ we apply Theorem **??** to get the existence of $P \in W^{1,2}(B, SO(m))$ and $\xi \in W^{1,2}(B, so(m))$ such that $\xi = 0$ on ∂B,

$$\nabla^\perp \xi = P^{-1}\nabla P + P^{-1}\Omega P. \tag{5.10}$$

and

$$||\xi||_{W^{1,2}} + ||\nabla P||_{L^2} + ||\nabla P^{-1}||_{L^2} \le c||\Omega||_{L^2}. \tag{5.11}$$

Claim 1: There exist $\hat{A} \in W^{1,2} \cap L^\infty(B, M(m))$ and $B \in W^{1,2}(B, M(m))$ solving

$$\Delta\hat{A} = \nabla\hat{A}\nabla^\perp\xi + \nabla^\perp B\nabla P \quad \text{in} \quad B, \tag{5.12}$$

$$\Delta B = -\nabla^\perp\hat{A}\nabla P^{-1} - \operatorname{div}(\hat{A}\nabla\xi P^{-1} + \nabla\xi P^{-1}) \quad \text{in} \quad B, \tag{5.13}$$

$$\frac{\partial\hat{A}}{\partial\nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial B, \tag{5.14}$$

$$\int_B \hat{A} = 0. \tag{5.15}$$

To prove this claim we apply Theorem **??** (combined with remark **??**) and standard L^2-theory to get

$$||\hat{A}||_{W^{1,2}} + ||\hat{A}||_{L^\infty} \le c||\nabla\xi||_{L^2}||\nabla\hat{A}||_{L^2} + c||\nabla P||_{L^2}||\nabla B||_{L^2} \quad \text{and} \tag{5.16}$$

$$||B||_{W^{1,2}} \le c||\nabla P^{-1}||_{L^2}||\nabla\hat{A}||_{L^2} + c||\nabla\xi||_{L^2}||\hat{A}||_{L^\infty} + c||\nabla\xi||_{L^2}. \tag{5.17}$$

Using (5.11) and choosing ε small enough we combine (5.16) and (5.17) to get

$$||\hat{A}||_{W^{1,2}} + ||\hat{A}||_{L^\infty} + ||B||_{W^{1,2}} \le c||\Omega||_{L^2}. \tag{5.18}$$

The existence of the desired solution of (5.12)-(5.15) (and hence the proof of Claim 1) now follows from a standard fixed-point argument.
Next we define $\tilde{A} = \hat{A} + id$ and we see from (5.12)-(5.15) that $\tilde{A}$ and B solve

$$\Delta\tilde{A} = \nabla\tilde{A}\nabla^\perp\xi + \nabla^\perp B\nabla P \quad \text{in} \quad B, \tag{5.19}$$
$$\Delta B = -\nabla^\perp\tilde{A}\nabla P^{-1} - \operatorname{div}(\tilde{A}\nabla\xi P^{-1}) \quad \text{in} \quad B, \tag{5.20}$$
$$\frac{\partial\tilde{A}}{\partial\nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial B, \tag{5.21}$$
$$\int_B \tilde{A} = |B|. \tag{5.22}$$

Moreover we get from (5.18) that

$$||\nabla\tilde{A}||_{L^2} + ||\operatorname{dist}(\tilde{A}, SO(m))||_{L^\infty} + ||B||_{W^{1,2}} \le c||\Omega||_{L^2}. \tag{5.23}$$

Now it is easy to see that (5.19) can be rewritten as

$$\operatorname{div}(\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp BP) = 0 \tag{5.24}$$

and hence, by Lemma **??**, there exists $C \in W^{1,2}(B, M(m) \otimes \wedge^1\mathbb{R}^2)$ such that

$$\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp BP = \nabla^\perp C. \tag{5.25}$$

Since by (5.21) and the definition of ξ we have

$$\begin{aligned}(\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp BP)\cdot\nu =& \frac{\partial\tilde{A}}{\partial\nu} - \tilde{A}\nabla^\perp\xi\cdot\nu - \nabla^\perp BP\cdot\nu \\ =& 0\end{aligned}$$

on ∂B we can moreover assume that $C = 0$ on ∂B. Using a rotation by $\frac{\pi}{2}$ (one can also view $\nabla^\perp$ as $\star d$ and then the rotation by $\frac{\pi}{2}$ is just another application of $\star$) we see that (5.25) is equivalent to

$$-\nabla CP^{-1} = \nabla^\perp\tilde{A}P^{-1} + \tilde{A}\nabla\xi P^{-1} + \nabla B,$$

and hence, using (5.20), we calculate

$$\begin{aligned}-\operatorname{div}(\nabla CP^{-1}) &= \nabla^\perp\tilde{A}\nabla P^{-1} + \operatorname{div}(\tilde{A}\nabla\xi P^{-1}) + \Delta B \\ &= 0.\end{aligned} \tag{5.26}$$

<u>Claim 2:</u> Every solution C of (5.26) with $C = 0$ on ∂B vanishes identically.
To see this we apply again Lemma **??** and get the existence of $D \in W^{1,2}(B, M(m) \otimes \wedge^1\mathbb{R}^2)$ such that

$$\nabla^\perp D = \nabla CP^{-1}. \tag{5.27}$$

Since $C = 0$ on ∂B we easily see that $\frac{\partial D}{\partial \nu} = 0$ on ∂B and we can also assume that $\int_B D = 0$. Hence C and D solve

$$\Delta C = \nabla^\perp D \nabla P \quad \text{in} \quad B, \tag{5.28}$$

$$\Delta D = \nabla C \nabla^\perp P^{-1} \quad \text{in} \quad B, \tag{5.29}$$

$$C = 0 \quad \text{and} \quad \frac{\partial D}{\partial \nu} = 0 \quad \text{on} \quad \partial B, \tag{5.30}$$

$$\int_B D = 0. \tag{5.31}$$

From this we see that we can apply Theorem **??** for (5.28) and (5.29) (in this case combined with remark **??**) to get

$$||\nabla C||_{L^2} + ||\nabla D||_{L^2} \le c(||\nabla P||_{L^2}||\nabla D||_{L^2} + ||\nabla P^{-1}||_{L^2}||\nabla C||_{L^2}). \tag{5.32}$$

By choosing ε small enough we get from (5.11) that $C = D = 0$ and this shows the claim. From (5.25) we now see that $\tilde{A}$ and B solve

$$\nabla \tilde{A} - \tilde{A}\nabla^\perp \xi - \nabla^\perp B P = 0. \tag{5.33}$$

Defining $A = \tilde{A}P^{-1}$ we see that

$$\begin{aligned} ||\nabla A||_{L^2} + ||\operatorname{dist}(A, SO(m))||_{L^\infty} &\le c(||\nabla \tilde{A}||_{L^2} + ||\tilde{A}||_{L^\infty}||\nabla P^{-1}||_{L^2} + ||\operatorname{dist}(\tilde{A}, SO(m))||_{L^\infty}) \\ &\le c||\Omega||_{L^2}, \end{aligned} \tag{5.34}$$

where we used (5.11) and (5.23). Moreover we use (5.10) and (5.33) to calculate

$$\begin{aligned} 0 = \nabla \tilde{A} - \tilde{A}\nabla^\perp \xi - \nabla^\perp B P &= \nabla A P + A \nabla P - A P \nabla^\perp \xi - \nabla^\perp B P \\ &= A\nabla P - A \nabla P + (\nabla A - A\Omega - \nabla^\perp B) P \end{aligned}$$

and therefore

$$\nabla A - A\Omega - \nabla^\perp B = 0. \tag{5.35}$$

This finishes the proof of the Lemma. □

<u>Step 3:</u>
For every point $x \in B$ we choose a radius $r_x > 0$ such that $\int_{B_{r_x}(x)} |\Omega|^2 < \varepsilon$, where ε is the same as in Lemma 5.2.3. In the following we write $B_{r_x}(x) = B$. Then we can apply Lemma 5.2.3 to get the existence of A and B solving (5.5). Hence we can apply Lemma 5.2.2 to see that

$$\operatorname{div}(A\nabla u) = \nabla B \nabla^\perp u = -\nabla^\perp B \nabla u \quad \text{and} \tag{5.36}$$

$$\nabla^\perp (A \nabla u) = \nabla^\perp A \nabla u. \tag{5.37}$$

Now we apply Lemma **??** to get the existence of $\alpha \in W^{1,2}(B, \mathbb{R}^m)$ and $\beta \in W^{1,2}(B, \mathbb{R}^m \otimes \wedge^1 \mathbb{R}^2)$ such that

$$A \nabla u = \nabla \alpha + \nabla^\perp \beta. \tag{5.38}$$

Using (5.36) we see that α solves

$$\Delta\alpha = \operatorname{div}(A\nabla u) = -\nabla^{\perp} B\nabla u. \tag{5.39}$$

Now we denote by $\overline{u}$ the mean value of u on $B_{\frac{1}{2}}$ and let $\tilde{u} \in W_0^{1,2}(\mathbb{R}^2, \mathbb{R}^m)$ be the extension with compact support of $u - \overline{u}$. Then we have that $\nabla\tilde{u} = \nabla u$ on $B_{\frac{1}{2}}$. Moreover we use Poincare's inequality to get

$$||\nabla\tilde{u}||_{L^2(\mathbb{R}^2)} \le c||u - \overline{u}||_{W^{1,2}(B_{\frac{1}{2}})} \le c||\nabla u||_{L^2(B)}.$$

We extend B in the same way and denote the resulting map by $\tilde{B} \in W_0^{1,2}(\mathbb{R}^2, M(m))$. Then we let $\tilde{\alpha}$ be the solution of

$$\Delta\tilde{\alpha} = -\nabla^{\perp}\tilde{B}\nabla\tilde{u} \tag{5.40}$$

on $\mathbb{R}^2$. Since by Corollary **??** $-\nabla^{\perp}\tilde{B}\nabla\tilde{u} \in \mathcal{H}^1(\mathbb{R}^2)$ with

$$|| - \nabla^{\perp}\tilde{B}\nabla\tilde{u}||_{\mathcal{H}^1(\mathbb{R}^2)} \le c||\nabla B||_{L^2(B)}||\nabla u||_{L^2(B)}$$

we can apply Theorem 4.3.8 to get that $\tilde{\alpha} \in W^{2,1}(\mathbb{R}^2)$. Since

$$\Delta(\alpha - \tilde{\alpha}) = 0$$

on $B_{\frac{1}{2}}$ we get that $\alpha \in W^{2,1}(B_{\frac{1}{4}})$ (harmonic functions are smooth). Next we observe that β solves

$$\Delta\beta = \nabla^{\perp} A\nabla u \tag{5.41}$$

and hence we can argue as before to get that $\beta \in W^{2,1}(B_{\frac{1}{4}})$. Therefore we see from (5.38) that

$$A\nabla u \in W^{1,1}(B_{\frac{1}{4}}) \tag{5.42}$$

and therefore (using (5.8))

$$\nabla u \in W^{1,1}(B_{\frac{1}{4}}) \quad \text{or} \quad u \in W^{2,1}(B_{\frac{1}{4}}). \tag{5.43}$$

Combining this with Corollary **??** we finish the proof of the Theorem. □

With the following counterexamples of Frehse [16] we show that one can not drop the condition that Ω has to be antisymmetric.

Remark 5.2.4. *Let $u = (u_1, u_2) \in W^{1,2}(B, S^1 \subset \mathbb{R}^2)$ be defined by*

$$u_1(x) = \sin\ln\ln\frac{2}{|x|},$$
$$u_2(x) = \cos\ln\ln\frac{2}{|x|}$$

then it is easy to check that u *solves the elliptic system* $-\Delta u = \Omega \nabla u$ *with*

$$\Omega = \begin{pmatrix} (u_1 + u_2)\nabla u_1 & (u_1 + u_2)\nabla u_2 \\ (u_2 - u_1)\nabla u_1 & (u_2 - u_1)\nabla u_2 \end{pmatrix}.$$

So in this case Ω *is not antisymmetric and* u *is bounded but not continuous.*
For $u \in W^{1,2}(B, \mathbb{R}^2)$ *given by*

$$u_1(x) = \ln\ln\frac{2}{|x|},$$
$$u_2(x) = \ln\ln\frac{2}{|x|}$$

we have that u *solves the elliptic system* $-\Delta u = \Omega \nabla u$ *with*

$$\Omega = \begin{pmatrix} \nabla u_1 & 0 \\ 0 & \nabla u_2 \end{pmatrix}.$$

Here we even don‘t have that u *is bounded.*

Bibliography

[1] L. Almeida. The regularity problem for generalized harmonic maps into homogeneous spaces. *Calc. Var.*, 3:193–242, 1995.

[2] H.W. Alt. *Lineare Funktionalanalysis.* Springer-Verlag, Berlin, 1992.

[3] J. Bourgain and H. Brezis. New estimates for elliptic equations and Hodge type systems. *J. Eur. Math. Soc.*, 9:277–315, 2007.

[4] H. Brezis and J.-M. Coron. Multiple solutions of H-systems and Rellich's conjecture. *Comm. Pure Appl. Math.*, 37:149–187, 1984.

[5] H. Brezis and J. van Schaftingen. Boundary estimates for elliptic systems with L^1-data. *Calc. Var.*, 30:369–388, 2007.

[6] A. P. Calderon and A. Zygmund. On the existence of certain singular integrals. *Acta Math.*, 88:85–139, 1952.

[7] A. Chang, L. Wang, and P. Yang. A regularity theory of harmonic maps. *Comm. Pure Appl. Math.*, 52:1099–1111, 1999.

[8] Y. Chen. The weak solutions to the evolution problems of harmonic maps. *Math. Z.*, 201:69–74, 1989.

[9] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl.*, 72:247–286, 1993.

[10] K. Deimling. *Nonlinear functional analysis.* Springer Verlag, Berlin, 1985.

[11] U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab. *Minimal Surfaces I, II.* Springer Verlag, Berlin Heidelberg 1992.

[12] L. C. Evans. Partial regularity for stationary harmonic maps into spheres. *Arch. Rat. Mech. Anal.*, 116:101–163, 1991.

[13] L. C. Evans and R. F. Gariepy *Measure theory and fine properties of functions* CRC Press, Boca Raton 1992.

[14] C. Fefferman and E. Stein. H^p spaces of several variables. *Acta Math.*, 129:137–193, 1972.

[15] I. Fonseca and W. Gangbo. *Degree theory in analysis and applications.* Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, New York, 1995.

[16] J. Frehse. A discontinuous solution to a mildly nonlinear elliptic system. *Math. Z.*, 134:229–230, 1973.

[17] Y. Ge. A remark on generalized harmonic maps into spheres. *Nonlinear Anal.*, 36:495–506, 1999.

[18] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Lecture Notes, SNS Pisa. Edizioni della Normale, Pisa, 2005.

[19] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer Verlag, Berlin, 2 edition, 2001.

[20] E. Giusti. *Minimal surfaces and functions of bounded variation.* volume 80 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 1984.

[21] M. Grüter. Regularity of weak H-surfaces. *J. Reine Angew. Math.*, 329:1–15.

[22] M. Grüter. Conformally invariant variational integrals and the removability of isolated singularities. *Manuscripta Math.*, 47:85–104, 1984.

[23] F. Hélein. Régularité des applications faiblement harmoniques entre une surface et une sphère. *C. R. Acad. Sci. Paris Sér. I Math.*, 312:519–524, 1990.

[24] F. Hélein. Regularity of weakly harmonic maps from a surface into a manifolds with symmetries. *Manuscripta Math.*, 70:203–218, 1991.

[25] F. Hélein. Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne. *C. R. Acad. Sci. Paris Sér. I Math.*, 312:591–596, 1991.

[26] F. Hélein. *Harmonic maps, conservation laws and moving frames*, volume 150 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2002.

[27] S. Hildebrandt and K.-O. Widman Some regularity results for quasilinear elliptic systems of second order. *Math. Z.*, 142:67–86, 1975.

[28] W. V. D. Hodge. A Dirichlet problem for a harmonic functional. *Proc. London Math. Soc.*, 2:257–303, 1933.

[29] J. Jost. *Two-dimensional geometric variational problems.* John Wiley and Sons, Chichester, 1991.

[30] T. Lamm and T. Rivière. Conservation laws for fourth order systems in four dimensions. Preprint, 2006 (to appear in *Comm. PDE*).

[31] J. Maly. Advanced theory of differentiation - Lorentz spaces. *Preprint.*

[32] C. Morrey. *Multiple integrals in the calculus of variations*, volume 130 of *Grundlehren der mathematischen Wissenschaften.* Springer, New York, 1966.

[33] S. Müller. Higher integrability of determinants and weak convergence in L^1. *J. Reine Angew. Math.*, 412:20–34, 1990.

[34] L. Nirenberg. *Topics in nonlinear functional analysis.* Courant Lecture Notes in Mathematics, American Mathematical Society, Providence, R.I., 2001.

[35] S. Poornima. An embedding theorem for the Sobolev space $W^{1,1}$. *Bull. Sc. Math.*, 107:253–259, 1983.

[36] T. Rivière. Everywhere discontinuous harmonic maps onto spheres. *Acta Math.*, 175:197–226, 1995.

[37] T. Rivière. Conservation laws for conformally invariant variational problems. *Invent. Math.*, 168:1–22, 2007.

[38] T. Rivière and M. Struwe. Partial regularity for harmonic maps, and related problems. Preprint, 2006 (to appear in *Comm. Pure and Appl. Math.*).

[39] T. Rivière. Analysis aspects of Willmore surfaces. Preprint, 2007.

[40] M. Ruzicka. *Nichtlineare Funktionalanalysis.* Springer, Berlin, 2004.

[41] S. Semmes. A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller. *Comm. Partial Differential Equations*, 19:277-319, 1994.

[42] J. Shatah. Weak solutions and development of singularities of the SU(2) σ-model. *Comm. Pure Appl. Math.*, 41:459–469, 1988.

[43] K. Steffen. Isoperimetric inequalities and the problem of Plateau. *Math. Ann.*, 222:97–144, 1976.

[44] E. M. Stein. Note on the class $L \log L$. *Studia Mathematica*, 32:305–310, 1969.

[45] E. M. Stein. *Singular integrals and differentiability properties of functions*, volume 30 of *Princeton Mathematical Series.* Princeton University Press, 1970.

[46] E. M. Stein. *Harmonic analysis*, volume 43 of *Princeton Mathematical Series.* Princeton University Press, 1993.

[47] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on euclidean spaces*, volume 32 of *Princeton Mathematical Series.* Princeton University Press, 1971.

[48] M. Struwe. *Variational Methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer Verlag, Berlin, third edition, 2000.

[49] M. Struwe. Partial regularity for biharmonic maps, revisited. Preprint, 2007.

[50] K. Uhlenbeck. Connections with L^p bounds on curvature. *Commun. Math. Phys.*, 83:31–42, 1982.

[51] L. Wang. A geometric approach to the Calderon-Zygmund estimates. *Acta Math. Sin.*, 19:381–396, 2003.

[52] K. Wehrheim. *Uhlenbeck compactness*, *EMS Series of Lectures in Mathematics* European Mathematical Society (EMS), Zürich, 2004.

[53] H. Wente. An existence theorem for surfaces of constant mean curvature. *J. Math. Anal. Appl.*, 26:318–344, 1969.

[54] M. Wiegner. über die Regularität schwacher Lösungen gewisser elliptischer Systeme. *Manuscripta Math.* , 15:365–384, 1975.

[55] W. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.