An Introduction to Ricci flow

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Preliminary comments

These are the notes for the lecture course on Ricci flow given in the winter term 2019/2020 at the university of Freiburg. They are loosely based on Chow's and Knopf's book [5] and Topping's book [4].

The lecture will cover the following.

1. Definition of the Ricci flow
2. Short time existence of the Ricci flow (on a surface)
3. Evolution equations for the curvatures
4. Long time existence on a surface
5. Convergence of Riemannian manifolds and compactness of Ricci flows
6. Convergence of the Ricci flow on surfaces and the uniformization theorem

The reader should be familiar with smooth manifolds. More concretely, I will expect a familiarity with smooth manifolds, tangent spaces and the tangent bundle, differentials of functions and maps, vector fields and flows. A good introduction is [6].

It would also be desirable to have a basic understanding of tensor products and tensor fields on manifolds and integration on manifolds, although these concepts will be briefly discussed when they appear for the first time.

A working knowledge of Riemannian geometry would be highly advantageous to the reader. The parts that are strictly necessary will be introduced in the following chapter on prerequisites.

Chapter 0

Prerequisites: Riemannian geometry

This chapter collects the bare minimum of Riemannian geometry facts required to follow the course. Its purpose is not so much to serve as an introduction. Rather it is supposed to be a reminder and a reference. For a gentle introduction to Riemannian geometry I recommend [2]. For a more advanced but thorough treatment I recommend [3].

0.1 Vector bundles and multilinear algebra

We will start with the tangent bundle and related vector bundles.

Suppose M is a smooth n dimensional manifold. To every point $x \in M$, there is associated an n dimensional vector space T_xM. The *tangent bundle* TM is the collection of all those vector spaces

$$TM = \sqcup_{x\in M} T_xM.$$

There is a canonical projection

$$\pi : TM \to M$$

$$T_xM \ni v \mapsto x \in M.$$

The space TM can be equipped with the structure of a $2n$ dimensional manifold and such that $\pi : TM \to M$ is a smooth manifold.

Definition 0.1

A *vector bundle* E over a manifold M is a smooth manifold E together with a smooth map $\pi : E \to M$, so that

1. $\pi^{-1}(x)$ has a vector space structure for every $x \in M$,
2. for every $x \in M$ there exists a neighborhood $U \subset M$ and a diffeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$$

$$v \mapsto (\phi_1(v), \phi_2(v)),$$

such that $\phi_1(v) = \pi(v)$ and $\phi_2|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^r$ is a vector space isomorphism for every $x \in U$

The number r is called the *rank* of E.

Given $x \in M$, the *fiber* over x is the vector space $E_x = \pi^{-1}(x)$.

Definition 0.2

Let E be a vector bundle over a manifold M. A *section* of E is a smooth map $s : M \to E$ satisfying $s(x) \in E_x$ for every $x \in M$. (Or equivalently, $\pi \circ s = \mathrm{id}_M$.)

The set of sections of E is denoted by $\Gamma(E)$.

Definition 0.3

Let $V_1, \dots, V_m, W_1, \dots, W_n$ be $\mathbb{R}$ vector spaces. Then we define the *tensor product*

$$V_1 \otimes \dots V_m \otimes W_1^* \otimes \dots \otimes W_n^*$$

to be the vector space of multilinear maps

$$\alpha : V_1^* \times \dots \times V_m^* \times W_1 \times \dots W_n \to \mathbb{R}.$$

Definition 0.4

Let V be a vector space. The space of symmetric, bilinear forms on V is denoted by $\mathrm{Sym}^2 V$.

A form $\beta \in \mathrm{Sym}^2_+ V$ is positive definite, if $\beta(v, v) > 0$ for all $v \in V \backslash \{0\}$. The set $\mathrm{Sym}^2_+ V \subset \mathrm{Sym}^2 V$ is the subset of *positive definite* symmetric, bilinear forms.

Definition 0.5

Let V be an $\mathbb{R}$ vector space. A multilinear form $\omega : V \times \dots \times V \to \mathbb{R}$ is called *alternating*, if

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

for all $v_1, \dots, v_n \in V$.

The vector space of alternating multilinear k-forms on V is denoted by $\Lambda^k V^*$.

Definition 0.6

Suppose E is a vector bundle over M. Then we denote by E^* the vector bundle over M, which is defined to be

$$\sqcup_{x \in M} E_x^*$$

together with the obvious projection.

Definition 0.7

Suppose $E_1, \dots, E_n$ are vector bundles over M. Then we denote by

$$E_1 \otimes \dots \otimes E_n$$

the vector bundle over M, which is defined to be

$$\sqcup_{x\in M}(E_1)_x \otimes \ldots \otimes (E_n)_x$$

together with the obvious projection.

Remark 0.8. This definition is incomplete in the sense that a vector bundle has the structure of a smooth manifold. This structure can be chosen so that trivialisations of the vector bundles $E_1, \ldots, E_n$ induce a trivialisation of the tensor product bundle. Doing this properly involves some work. However, to use tensor products the above definition is sufficient in most cases. The same applies to the definition before.

Proposition 0.9

There is a one to one correspondence between $\mathcal{C}^\infty$-multilinear maps

$$\Gamma(E_1^*) \times \ldots \times \Gamma(E_m^*) \times \Gamma(F_1) \times \ldots \times \Gamma(F_n) \to \mathcal{C}^\infty(M)$$

and sections of $E_1 \otimes \ldots \otimes E_m \otimes F_1^* \otimes \ldots \otimes F_n^*$.

In one direction, this correspondence is given as follows: let $s \in \Gamma(E_1 \otimes \ldots \otimes E_m \otimes F_1^* \otimes \ldots \otimes F_n^*)$. Note that $s(x) \in (E_1)_x \otimes \ldots \otimes (E_m)_x \otimes (F_1^*)_x \otimes \ldots \otimes (F_n^*)_x$, i.e. $s(x)$ is a multilinear map

$$s(x) : (E_1)_x^* \times \ldots \times (E_m)_x^* \times (F_1)_x^* \times \ldots \times (F_n)_x^* \to \mathbb{R}.$$

Hence we can define a multilinear map

$$F : \Gamma(E_1^*) \times \ldots \times \Gamma(E_m^*) \times \Gamma(F_1) \times \ldots \times \Gamma(F_n) \to \mathcal{C}^\infty(M)$$

via

$$F(\alpha_1, \ldots, \alpha_m, s_1, \ldots, s_n)(x) = F(\alpha_1(x), \ldots, \alpha_m(x), s_1(x), \ldots, s_n(x)).$$

It is easy to check that F is $\mathcal{C}^\infty$ multilinear.

0.2 Riemannian metrics, connections and curvature

Definition 0.10

Let M be a manifold. A *Riemannian metric* g is a section of $\mathrm{Sym}^2 TM$, such that

$$g(x) \in Sym_+^2 TM_x$$

for every $x \in M$.

Remark 0.11. Thus for every $x \in M$, $g(x) : TM_x \times TM_x \to \mathbb{R}$ defines an inner product. In other words, a Riemannian metric is a tool to measure lengths of tangent vectors and angles between them.

Example 0.12. 1. $(\mathbb{R}^n, g_{Eucl})$ with $g_{Eucl}(x)(v,w) = \langle v, w\rangle$, where we identified $T_x\mathbb{R}^n$ and $\mathbb{R}^n$. The sharp brackets denote the standard inner product on $\mathbb{R}^n$

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i.$$

The Riemannian manifold $(\mathbb{R}^n, g_{Eucl})$ is usually called *Euclidean space.*

2. Let $N \subset \mathbb{R}^n$ be a smooth submanifold. Then for every point $x \in N$ the tangent space T_xN is a vector subspace of $T_x\mathbb{R}^n$. We can define a Riemannian metric on N by restricting the metric on $\mathbb{R}^n$ to N in the following way:

$$g_x : T_xN \times T_xN \to \mathbb{R}$$

$$g_x(v,w) = g_{Eucl,x}(v,w).$$

This works more generally for any submanifold $N \subset M$ of a Riemannian manifold (M, g).

3. A specific example of the previous construction is the standard sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\} \subset \mathbb{R}^{n+1}$. The metric obtained this way is called the *round metric* and we denote it by g_{Sph}.

4. Denote by $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ the upper halfspace. Then

$$g_{Hyp}(x)(v,w) = \frac{1}{x_n}\langle v, w\rangle$$

is a metric on $\mathbb{H}^n$. The manifold $(\mathbb{H}^n, g_{Hyp})$ is called *hyperbolic n-space.*

Definition 0.13

A *covariant derivative* or *connection* is a $\mathbb{R}$-linear operator

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

$$(X, Y) \mapsto \nabla_X Y$$

which is *tensorial* in the first component,

$$\nabla_{fX} Y = f\nabla_X Y \quad \text{for all } f \in \mathcal{C}^\infty(M), X, Y \in \Gamma(TM)$$

and satisfies the *Leibniz rule*

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y \quad \text{for all } f \in \mathcal{C}^\infty(M), X, Y \in \Gamma(TM).$$

Remark 0.14 (Induced connection). A given connection on TM induces a connection on any tensor bundle $TM \otimes \ldots \otimes TM \otimes T^*M \otimes \ldots \otimes T^*M$. By abuse of notation, this induced connection is also called ∇. Recall that a section μ of this bundle can be consideres a multilinear map

$$\mu : T^*M \times \ldots \times T^*M \times TM \times \ldots \times TM \to \mathbb{R}.$$

The induced connection is then defined by the following equation

$$\begin{aligned} X\left[\mu(\alpha_1,\dots,\alpha_r,V_1,\dots,V_s)\right] = &(\nabla_X\mu)(\alpha_1,\dots,\alpha_r,V_1,\dots,V_s)\\ &+\sum_{i=1}^{r}\mu(\alpha_1,\dots,\nabla_X\alpha_i,\dots,\alpha_r,V_1,\dots,V_s)\\ &+\sum_{j=1}^{s}\mu(\alpha_1,\dots,\alpha_r,V_1,\dots,\nabla_X V_i,\dots,V_r) \end{aligned}$$

In particular, for $\alpha\in\Gamma(T^*M)$, we have

$$X\alpha(V) = (\nabla_X\alpha)(V) + \alpha(\nabla_X V).$$

Definition 0.15

A connection ∇ is called *torsion free*, if

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

for all $X,Y\in\Gamma(TM)$.

Definition 0.16

Let (M,g) be a Riemannian manifold. A connection ∇ is called *metric*, if

$$Xg(V,W) = g(\nabla_X V,W) + g(V,\nabla_X W)$$

for all $X,V,W\in\Gamma(TM)$.

Theorem 0.17 [Fundamental theorem of Riemannian geometry]

Let (M,g) be a Riemannian manifold. There exists a unique torsion free connection, which is metric with respect to g.

Definition 0.18

Let (M,g) be a Riemannian manifold. The *Levi–Civita connection* ∇^g is the unique torsion free connection on (M,g).

The next proposition gives an explicit formula for the Levi–Civita connection of a Riemannian manifold.

Proposition 0.19 (Koszul formula)

Let (M,g) be a Riemannian manifold and let ∇^g be the Levi–Civita connection. Then

$$\begin{aligned} 2g(\nabla^g_X Y,Z) = \; &Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)\\ &+g([X,Y],Z)-g([X,Z],Y)-g([Y,Z],X) \end{aligned}$$

for any three vector fields $X,Y,Z\in\Gamma(TM)$.

Definition 0.20

The *curvature* of a connection ∇ is the operator

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$(X, Y, Z) \mapsto R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Proposition 0.21

The curvature is tensorial *in each component, i.e.*

$$R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ) = fR(X, Y)Z$$

and antisymmetric in the first two components, i.e.

$$R(X, Y)Z = -R(Y, X)Z$$

for all $X, Y, Z \in \Gamma(TM)$ *and* $f \in \mathcal{C}^\infty(M)$.

Definition 0.22

Let (M, g) be a Riemannian manifold.

The *Riemannian curvature tensor* is the section

$$\mathrm{Rm}^g \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes TM)$$

determined by the $\mathcal{C}^\infty$ multilinear map

$$\Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(T^*M) \to \mathcal{C}^\infty(M)$$

$$(X, Y, Z, \alpha) \mapsto \alpha(R(X, Y)Z),$$

where R is the curvature of the Levi–Civita connection ∇^g.

The *Ricci curvature tensor* is the symmetric bilinear form $\mathrm{Ric}_g \in \Gamma(\mathrm{Sym}^2 T^*M)$ defined by

$$\mathrm{Ric}_g(x)(v, w) = \mathrm{tr}(u \mapsto R^g(x)(u, v)w)$$

for $x \in M$ and $v, w \in T_xM$

The *scalar curvature* is the function $R_g \in \mathcal{C}^\infty(M)$ defined by

$$R_g(x) = \mathrm{tr}_g \mathrm{Ric}_g(x)$$

for $x \in M$.

Remark 0.23. The trace $\mathrm{tr}_g \beta \in \mathcal{C}^\infty(M)$ of a symmetric bilinear form $\beta \in \Gamma(\mathrm{Sym}^2 T^*M)$ is defined as

$$(\mathrm{tr}_g \beta)(x) = \sum_{i=1}^{n} \beta(e_i, e_i)$$

for $x \in M$ and an orthonormal basis $e_1, \ldots, e_n$ of T_xM. One can check that this definition is independent of the choice of the orthonormal basis.

In Riemannian geometry, the tangent bundle TM is often identifited implicitly with the cotangent bundle T^*M. This is done using the *musical isomorphisms*.

Definition 0.24

Let V be a real vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on V.

The *flat map* is given by

$$(\cdot)^\flat : V \to V^*$$
$$v \mapsto v^\flat = (w \mapsto \langle v, w \rangle) .$$

The *sharp map*

$$(\cdot)^\sharp : V^* \to V$$

is defined to be the inverse of the flat map.

Collectively, these maps are known as the *musical isomorphisms*.

Remark 0.25. Given a Riemannian metric, the musical isomorphisms can be used to identify TM with T^*M fibrewise and consequently we will write

$$(\cdot)^\flat : TM \to T^*M$$

and

$$(\cdot)^\sharp : T^*M \to TM.$$

Notice that these maps depend on the Riemannian metric, although it is suppressed in the notation.

A very important instance of the implicit identification of TM with T^*M is the Riemann curvature tensor. Using the flat map, we will quite often treat Rm as a section of $TM \otimes TM \otimes TM \otimes TM$. More explicitly:

$$\mathrm{Rm}(X, Y, Y, W) = \mathrm{Rm}(X, Y, Z, W^\flat) = g(R(X, Y)Z, W)$$

for $X, Y, Z, W \in TM$.

Proposition 0.26 (Symmetries of the Riemann tensor)

The Riemann tensor satisfies

1. $\mathrm{Rm}(X, Y, Z, W) = -\mathrm{Rm}(Y, X, Z, W)$
2. $\mathrm{Rm}(X, Y, Z, W) = -\mathrm{Rm}(X, Y, W, Z)$
3. $\mathrm{Rm}(X, Y, Z, W) = \mathrm{Rm}(Z, W, X, Y)$
4. $\mathrm{Rm}(W, X, Y, Z) + \mathrm{Rm}(X, Y, W, Z) + \mathrm{Rm}(Y, W, X, Z) = 0$

Proposition 0.27 (Differential Bianchi identity)

The Riemann tensor satisfies

$$(\nabla_U \mathrm{Rm})(X, Y, V, W) + (\nabla_V \mathrm{Rm})(X, Y, W, U) + (\nabla_W \mathrm{Rm})(X, Y, U, V) = 0.$$

Corollary 0.28 (Contracted Bianchi identity for the Ricci tensor)

The Ricci curvature Ric_g *and the scalar curvature* R_g *of a Riemannian metric satisfy*

$$\delta_g \operatorname{Ric}_g = \frac{1}{2} dR_g.$$

(The operator δ_g will be defined below, see definition 0.51.)

Proposition 0.29

Let M *be a manifold,* g *a Riemannian metric,* $f : M \to M$ *a diffeomorphism,* $\lambda \in \mathbb{R}$. *Then*

1. $\operatorname{Ric}_{f^* g} = f^* \operatorname{Ric}_g$
2. $\operatorname{Ric}_{\lambda^2 g} = \operatorname{Ric}_g$

0.3 Geodesics and the distance function

Let (M, g) be a Riemannian manifold. A Riemannian metric allows us to measure the *length* of paths in M, at least if they are sufficiently smooth. Let $\gamma : [a, b] \to M$ be a differentiable curve. Then we define the length of γ to be

$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

If we can measure the lengths of paths, there is also a natural definition for the distance between two points: the length of the shortest path between them. This idea leads to the definition of the *Riemannian distance function*

$$d^g(x, y) = \inf \{ L(\gamma) : \gamma : [0, 1] \to M \text{ is a differentiable path}, \gamma(0) = x, \gamma(1) = y \}.$$

It can be shown that (M, d^g) is a *metric space*, i.e. the distance function satisfies

1. $d^g(x, y) = d^g(y, x)$,
2. $d^g(x, y) = 0$ if and only if $x = y$,
3. $d^g(x, z) \leq d^g(x, y) + d^g(y, z)$.

These three properties are known as symmetry, definiteness and triangle inequality respectively.

Definition 0.30

A Riemannian manifold (M, g) is called *complete*, if the metric space (M, d^g) is complete.

0.4 Integration on Riemannian manifolds

Definition 0.31

Let M be a n-dimensional manifold. A *volume form* is a nowhere vanishing section of $\Lambda^n T^*M$.

Remark 0.32.

i) Volume forms bear their name, because they can be used to define a notion of volume for subsets of the manifold M. To see this, recall that on $\mathbb{R}^n$ an n-form ω is given by

$$\omega = f dx_1 \wedge \ldots \wedge dx_n \in \Gamma(\Lambda^n T^*\mathbb{R}^n)$$

for $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. If $d\lambda^n$ denotes the Lebesgue measure, we may define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f d\lambda^n.$$

The crucial point is that this definition is almost independent of the coordinates we choose on $\mathbb{R}^n$ to represent ω. Indeed, if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, we find

$$\varphi^*(\omega) = \varphi^*(f dx_1 \wedge \ldots \wedge dx_n) = (f \circ \varphi)(det d\varphi) dx_1 \wedge \ldots dx_n.$$

A somewhat subtle, but important point is that the determinant in the formula above appears with a sign, whereas in the transformation formula

$$\int_{F(U)} f d\lambda^n = \int_U f \circ F |\det dF| d\lambda^n,$$

the determinant appears as an absolute value. From this we conclude that

$$\int_{\mathbb{R}^n} f d\lambda^n = \int_{\mathbb{R}^n} f \circ \varphi |\det d\varphi| d\lambda^n,$$

and if $\det d\varphi > 0$

$$\int_{R^n} \omega = \int_{\mathbb{R}^n} f d\lambda^n = \int_{\mathbb{R}^n} f \circ \varphi \det d\varphi d\lambda^n = \int_{\mathbb{R}^n} \varphi^* \omega.$$

Now suppose that $\omega \in \Gamma(\Lambda^n T^*M)$ and $\operatorname{supp} \omega$ is contained in a subset of M, which is diffeomorphic to $\mathbb{R}^n$. Suppose that U is such a subset and $\psi : U \subset M \to \mathbb{R}^n$ is a diffeomorphism.

Then $\omega|_U = \psi^*(f dx_1 \wedge \ldots \wedge dx_n)$ and we can define

$$\int_M \omega = \int_{\mathbb{R}^n} f d\lambda^n.$$

Our previous thoughts then imply that this is actually independent of the choice of diffeomorphism ψ. A partition of unity allows us to write any section of $\Gamma(\Lambda^n T^*M)$

as a sum of sections, each supported in a subset of M diffeomorphic to $\mathbb{R}^n$, $\omega = \omega_1 + \ldots + \omega_N$. Since we expect the integral to be linear, we may then define

$$\int_M \omega = \int_M \omega_1 + \ldots \int_M \omega_N$$

and in this way we obtain a definition of the integral of any ω.

ii) The rank of the vector bundle $\Lambda^n T^* M$ is 1, i.e. it is a line bundle. This implies that if $\omega_1, \omega_2 \in \Gamma(\Lambda^n T^* M)$ are two nowhere vanishing sections, then there is a function $f \in \mathcal{C}^\infty(M)$, such that $\omega_2 = f\omega_1$ and f is never zero.

W In particular, if M is connected, then f is either positive on all of M or negative. This allows us to define an equivalence relation on the nowhere vanishing sections of $\Lambda^n T^* M$

Definition 0.33

An *oriented manifold* is a pair of a n-dimensional manifold M and a nowhere vanishing form $\sigma \in \Gamma(\Lambda^n T^* M)$. An ordered basis $(v_1, \ldots, v_n)$ of $T_x M$ is called *orientied*, if

$$\sigma(v_1, \ldots, v_n) > 0.$$

Definition 0.34

Let (M, σ). A *volume form* on M is a nowhere vanishing form $\omega \in \Gamma(\Lambda^n T^* M)$. The volume form is called *compatible* with the orientation, if

$$\omega = f\sigma$$

for a positive function $f \in \mathcal{C}^\infty(M)$.

Definition 0.35

Let (M, σ) be an oriented Riemannian manifold. The *Riemannian volume form* $\mathrm{vol}_g \in \Gamma(\Lambda^n T^* M)$ is defined to be the unique volume form of unit length with respect to g, which is also compatible with the orientation, i.e.

$$|\mathrm{vol}_g|_g = 1.$$

Remark 0.36. The condition $|\mathrm{vol}_g|_g = 1$ is equivalent to the condition that

$$\mathrm{vol}_g(e_1, \ldots, e_n) = 1$$

for any oriented, orthonormal basis of $T_x M$.

0.5 Diffeomorphisms, Lie derivatives and isometries

Definition 0.37

Let $X, Y \in \Gamma(TM)$, $f \in \mathcal{C}^\infty(M)$. The *Lie derivative* of f in direction X is defined by

$$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} f(\phi_t(x)).$$

The *Lie derivative* of Y in direction X is defined by

$$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} d\phi_{-t}(x) Y(\phi_t(x)).$$

Remark 0.38 (Lie derivatives of tensor fields). As in the case of connections, the Lie derivatives can be extended to tensor field. Suppose μ is a section of the tensor bundle $TM \otimes \ldots \otimes TM \otimes T^*M \otimes \ldots \otimes T^*M$. We identify μ with a multilinear map

$$\mu : T^*M \times \ldots \times T^*M \times TM \times \ldots \times TM \to \mathbb{R}.$$

The Lie derivative $\mathcal{L}_X T$ is then defined by the following equation

$$\begin{aligned} X\left[\mu(\alpha_1, \ldots, \alpha_r, V_1, \ldots, V_s)\right] = (\mathcal{L}_X \mu)(\alpha_1, \ldots, \alpha_r, V_1, \ldots, V_s) \\ + \sum_{i=1}^{r} \mu(\alpha_1, \ldots, \mathcal{L}_X \alpha_i, \ldots, \alpha_r, V_1, \ldots, V_s) \\ + \sum_{j=1}^{s} \mu(\alpha_1, \ldots, \alpha_r, V_1, \ldots, \mathcal{L}_X V_i, \ldots, V_r) \end{aligned}$$

In particular, for $\alpha \in \Gamma(T^*M)$, we have

$$X\alpha(V) = (\mathcal{L}_X \alpha)(V) + \alpha(\mathcal{L}_X V).$$

Definition 0.39

Let (M, g) be a Riemannian manifold and let $f : N \to M$ be a diffeomorphism. The *pullback metric* f^*g is the Riemannian metric on N defined by the relation

$$f^*g(v, w) = g(df v, df w)$$

for all $v, w \in T_pN$.

Definition 0.40

An *isometry* between two Riemannian manifolds $(M_1, g_1), (M_2, g_2)$ is a diffeomorphism $f : M_1 \to M_2$, such that

$$f^*g_2 = g_1.$$

The name is isometry stems from the fact that an isometry preserves distances. This is stated more precisely in the following proposition.

Proposition 0.41

Let (M_1, g_1), (M_2, g_2) be two Riemannian manifolds and suppose that $f : M_1 \to M_2$ is an isometry. Then the following equation holds for any two points $p, q \in M_1$:

$$d^{g_1}(p, q) = d^{g_2}(f(p), f(q)).$$

The Levi–Civita connection also behaves well under pull back of the metric.

PROPOSITION 0.42
Let (M_1, g_1), (M_2, g_2) be two Riemannian manifolds and suppose that $f : M_1 \to M_2$ is an isometry. Then
$$f_* \nabla^{g_1}_X Y = \nabla^{g_2}_{f_* X} f_* Y.$$

0.6 Differential operators and partial integration

Let (M, g) be a Riemannian manifold.

Definition 0.43
The *gradient* of a function $f \in \mathcal{C}^\infty(M)$ is given by
$$\operatorname{grad}_g : \mathcal{C}^\infty(M) \to \Gamma(TM)$$
$$\operatorname{grad}_g f = (df)^\sharp.$$

Remark 0.44. An equivalent definition is that $\operatorname{grad}_g f(x)$ is the unique vector in T_xM satisfying
$$df(v) = g(\operatorname{grad}_g f(x), v)$$
for all $v \in T_xM$.

Definition 0.45
The *divergence* of a vector field $X \in \Gamma(TM)$ is
$$\operatorname{div}_g : \Gamma(TM) \to \mathcal{C}^\infty(M)$$
$$\operatorname{div}_g X = -\operatorname{tr} \nabla^g X.$$

Remark 0.46. Here $\nabla^g X$ is a $\mathcal{C}^\infty(M)$ linear map $\Gamma(TM) \to \Gamma(TM)$. Hence, $\nabla^g X$ can be considered as a tensor field of endomorphisms, in other words for every $x \in M$ there is an endomorphism
$$(\nabla^g X)(x) : T_xM \to T_xM.$$
We can take the trace of this endomorphism.

PROPOSITION 0.47
Let $X \in \Gamma$. Then
$$\mathcal{L}_X \operatorname{vol}_g = (\operatorname{div}_g X) \operatorname{vol}_g.$$

THEOREM 0.48 [Divergence theorem]
Let (M, g) be a closed manifold. Then
$$\int_M \operatorname{div}_g X \operatorname{vol}_g = 0.$$

Definition 0.49

The *Laplacian* is a differential operator given by

$$\Delta_g : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

$$\Delta_g f = \operatorname{div}_g \operatorname{grad}_g f.$$

Proposition 0.50

Let (M, g) be a closed Riemannian manifold. For any two functions $f_1, f_2 \in \mathcal{C}^\infty(M)$ the following identity holds

$$\int_M g(\operatorname{grad}_g f_1, \operatorname{grad}_g f_2) \operatorname{vol}_g = \int_M (\Delta_g f_1)\, f_2 \operatorname{vol}_g .$$

Definition 0.51

The *divergence* of a symmetric two tensor $h \in \Gamma(\operatorname{Sym}^2 TM)$ is

$$\delta_g : \Gamma(\operatorname{Sym}^2 TM) \to \Gamma(T^*M)$$

$$(\delta_g h)\,(v) = -\sum_i (\nabla^g h)(e_i, e_i, v),$$

where $v \in T_xM$ and $e_1, \ldots, e_n$ is any orthonormal basis of T_xM.

Remark 0.52. One needs to check that this definition is well-defined, i.e. independent of the choice of basis.

Chapter 1

Introduction

This lecture is an introduction to the Ricci flow.

A banal description of the Ricci flow is that it is a process, which changes a given Riemannian manifold (M, g) continually into a family of nicer Riemannian manifolds (M, g_t), where t varies in an interval.

To understand why this might be interesting or useful, let us consider the most famous application of the Ricci flow: the resolution of the Poincaŕe conjecture by Perelman.

To state the Poincaré conjecture, we first need to know what it means for a space to be simply connected. Intuitively, this means that any loop in the space can be pulled into a point.

Definition 1.1

Let X be a topological space.

A *loop* is a continuous map $\gamma : S^1 \to X$.

The space X is called *simply connected*, if for every loop γ there exists continuous map $F : D^2 \to X$, such that $F|_{S^1} = \gamma$.

Remark 1.2. Recall that

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$
$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Poincaré conjecture

Every connected, simply connected and compact manifold is homeomorphic to S^3.

A stronger version of this conjecture was stated as a fact by Poincaré in 1900. In 1905 he found a counterexample and instead posed as a question the statement we now know as the Poincaré conjecture. There were many attempts to answer this question, but it remained open for almost a century until 2002/2003, when it was finally resolved by

Perelman.

It is not immediately obvious that Riemannian geometry should play any role in this question.

Let us first recall some very basic definitions in Riemannian geometry.

Let (M, g) be a Riemannian manifold, i.e. a smooth manifold M together with a *Riemannian metric* g on M.

Definition 1.3

A *Riemannian metric* on a manifold M is a smooth family g of inner products on the tangent spaces of M, i.e.

1. For every point $x \in M$ there is an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$,
2. If $X_1, X_2 \in \Gamma(TM)$ are smooth vector fields, the function $x \mapsto g_x(X_1(x), X_2(x))$ is also smooth.

Definition 1.4

For a vector space V, denote by $\operatorname{Sym}^2 V$ the space of bilinear forms on V and by $\operatorname{Sym}^2_+ V \subset \operatorname{Sym}^2 V$ the space of inner products on V.

A *Riemannian metric* g on M is a section of the vector bundle $\operatorname{Sym}^2 TM$, such that $g(x) \in \operatorname{Sym}^2_+ V$ for every $x \in M$.

Given a Riemannian manifold, there is one and only one metric, torsion-free connection ∇^g. This connection is known as the *Levi–Civita connection*.

As to any connection, we may associate to ∇^g a curvature operator. Let $X, Y, Z \in \Gamma(TM)$. The curvature operator of ∇^g is defined by

$$R(X, Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z.$$

It can be shown that the value $[R(X, Y)Z](x) \in T_xM$ only depends on the values of X, Y, Z at x.

Traditionally, the *sectional curvature* a two-plane $E = \operatorname{span}\{v, w\} \subset T_xM$ is defined to be

$$\sec(v, w) = \frac{g(R(w, v)v, w)}{g(v, v)g(w, w) - g(v, w)^2}.$$

Clearly, by definition, if one knows R and g, then one can compute the sectional curvatures of every two-plane. Conversely, it can be shown that if one knows the sectional curvature of every two-plane E, then one can recover the curvature operator R.

Definition 1.5

(M, g) has *constant sectional curvature* $\kappa \in \mathbb{R}$, if for every point $x \in M$ and every two-plane $E = \operatorname{span}\{v, w\}$ the sectional curvature $\sec(v, w)$ is κ.

If all sectional curvatures are the same, then the following classical theorem tells you exactly which Riemannian manifolds can appear.

Classification of spaces of constant sectional curvature

Suppose (M, g) is a complete, simply connected Riemannian manifold.

If (M, g) has constant sectional curvature 1, then (M, g) is isometric to (S^n, g_{sph}).

If (M, g) has constant sectional curvature 0, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathrm{eucl}})$.

If (M, g) has constant sectional curvature -1, then (M, g) is isometric to $(\mathbb{H}^n, g_{\mathrm{hyp}})$.

Recalling that an isometry is a homeomorphism, this theorem suggests a possible avenue to prove the Poincaŕe conjecture. Let M be a connected, simply connected and compact manifold. Then show that you can find a Riemannian metric g, such that all the sectional curvatures of (M, g) are equal to 1.

Unfortunately, this does *not* make the problem easy. Nevertheless, it can be done, as Perelman proved.

How might one go about finding such a metric? Of course, using the Ricci flow!

The idea here is quite simple. Suppose you are given *any* metric g on your manifold M. Then you define a process, which changes your Riemannian metric g in a predicatble manner. Ideally, it should only stop changing your metric once the sectional curvatures are all equal to 1.

Our goal is to associate to a given metric g a family $(g_t)_{t\in[0,T]}$, such that $g_0 = g$ and so that in a quantifiable way our metric becomes closer to a sphere. One way to define such a family is to specify an equation, which is satisfied by the family:

$$\partial_t g_t(p) = Q(g_t)(p).$$

Ideally, $Q(g) = 0$ if and only if g has all sectional curvatures equal to 1. A three-dimensional accident aids us in this case. We first recall the definitions of the Ricci curvature and the scalar curvature.

Definition 1.6

The *Ricci curvature tensor* is the symmetric bilinear form $\mathrm{Ric}_g \in \Gamma(\mathrm{Sym}^2 T^*M)$ defined by

$$\mathrm{Ric}_g(p)(v, w) = \mathrm{tr}(u \mapsto R(v, u)w)$$

for $p \in M$ and $v, w \in T_pM$

The *scalar curvature* is the function $R_g \in \mathcal{C}^\infty(M)$ defined by

$$R_g(p) = \mathrm{tr}_g \mathrm{Ric}_g(p)$$

for $p \in M$.

Remark 1.7. The trace $\operatorname{tr}_g \beta \in \mathcal{C}^\infty(M)$ of a symmetric bilinear form $\beta \in \Gamma(\operatorname{Sym}^2 T^*M)$ is defined as

$$(\operatorname{tr}_g \beta)(p) = \sum_{i=1}^{n} \beta(e_i, e_i)$$

for $p \in M$ and an orthonormal basis $e_1, \dots, e_n$ of T_pM. One can check that this definition is independent of the choice of the orthonormal basis.

If M is two-dimensional, there is precisely one two-plane contained in T_pM, namely T_pM. Let $e_1, e_2 \in T_pM$ be an orthonormal basis (with respect to $g(p)$). By the definition of the scalar curvature, we have

$$R_g(p) = \operatorname{Ric}_g(p)(e_1, e_1) + \operatorname{Ric}_g(p)(e_2, e_2).$$

Now we compute

$$\operatorname{Ric}_g(p)(e_1, e_1) = g(R(e_1, e_1)e_1, e_1) + g(R(e_1, e_2)e_2, e_1) = K(e_1, e_2),$$

where $R(e_1, e_1) = 0$ because of antisymmetry. Analogously $\operatorname{Ric}_g(p)(e_2, e_2) = K(e_1, e_2)$. Thus

$$R_g(p) = 2K(e_1, e_2).$$

Since the sectional curvatures determine the curvature tensor, we have that in dimension two the scalar curvature R_g determines all curvatures of the manifold.

In dimension 3 this is no longer true. However, the Ricci tensor still determines all curvatures. (This will be shown in an exercise.) This lets us rephrase the condition that all sectional curvatures are equal, using the following proposition.

Proposition 1.8

Let (M, g) be a 2 or 3-dimensional Riemannian manifold. Then all sectional curvatures are equal to $\kappa \in \mathbb{R}$, if and only if $\operatorname{Ric}_g = \kappa(n-1)g$.

In higher dimensions, this fails. Nevertheless, the class of metrics satisfying this equation are very interesting.

Definition 1.9

A Riemannian metric g satisfying

$$\operatorname{Ric}_g = \lambda g$$

is called an *Einstein metric.*

The proposition before suggests we should define

$$Q(g) = \operatorname{Ric}_g - (n-1)g$$

or perhaps

$$Q(g) = -(\operatorname{Ric}_g - (n-1)g).$$

Note that if g_t satisfies

$$\partial_t g_t = \mu g_t$$

for some $\mu \in \mathbb{R}$, then

$$g_t = \exp(\mu t) g.$$

Hence the term $(n-1)g$ only changes g_t by rescaling. This means that the qualitative behavior does not depend on this term and thus we drop it. (At the cost that we may have to rescale the family g_t at some point.)

So we are left with two choices $Q(g) = \operatorname{Ric}_g$ or $Q(g) = -\operatorname{Ric}_g$. It is not exactly obvious, which one we should choose, but it turns out that only if we choose $Q(g) = -\operatorname{Ric}_g$, the family will be well defined for an arbitrary smooth initial metric g.

Definition 1.10

A family $(g_t)_{t\in[0,T)}$ is a solution of the *Ricci flow* with initial condition g, if $g_0 = g$ and

$$\partial_t g_t = -2 \operatorname{Ric}_{g_t}.$$

If g is an Einstein metric with Einstein constant λ, then

$$g(t) = (1 - 2\lambda t) g$$

is a solution of the Ricci flow with initial condition g.

The study of Ricci flow began with the following theorem.

Theorem 1.11 [Hamilton's theorem, 1982]

Suppose M is a compact 3 dimensional manifold and g is a Riemannian metric, such that

$$\operatorname{Ric}_g(v, v) > 0 \textit{ for every } v \in T_pM.$$

Then there exists a solution g_t of the Ricci flow with initial condition g on an interval $[0, T)$ and after rescaling g_t to unit volume g_t converges to a metric $\hat{g}$ as $t \to T$. The limit metric satisfies $\operatorname{Ric}_{\hat{g}} = (n-1)\hat{g}$.

The theorem does not remain true without the condition on Ric_g. Nevertheless, the Poincaré conjecture was eventually revolved by Perelman using the Ricci flow.

Starting from any compact, connected and simply connected 3 dimensional manifold, Perelman constructed a *generalised* Ricci flow (M_t, g_t), where the underlying manifold M_t can change its topology.

The reason to introduce changes in topologies is that the Ricci flow can run into *singularities*, i.e. it can stop being well-defined. In the case of Hamilton, there is one singularity at the finite time T, but it is over the whole manifold. However, a singularity may also affect only parts of the manifold. Perelman showed how to cut apart the manifold in such a case and construct a Riemannian metric on the two pieces. Then the Ricci flow can be restarted.

Crucially, the change of topology during these *surgeries* is very simple. This allows one to compare the topological type of M_t before and after the surgery. In particular, in the case of a simply connected manifold, the topology only changes in the sense that after the surgery there is a new component, which is diffeomorphic to S^3. The other piece retains the topological type from before the surgery.

Finally, Perelman shows that the generalised Ricci flow only exists for a finite time, as in Hamilton's theorem, and that after the last surgery, every connected component of M_t is diffeomorphic to S^3.

Perelman's proof is enormously complicated and would by far exceed what could be achieved in a lecture series such as this. Instead we will focus on the two dimensional situation.

Definition 1.12

Two metrics $g, \tilde{g}$ on a manifold M are *conformal*, if there exists a smooth function $\lambda : M \to \mathbb{R}_+$, such that

$$\tilde{g} = \lambda^2 g.$$

A classical and difficult theorem in the theory of surfaces is the *uniformization theorem*, first proven in 1907 independently by Koebe and Poincaré. In one version – the one that we are interested in – it can be stated as follows.

Uniformization theorem

Suppose (M, g) is a compact oriented Riemannian manifold of dimension 2. Then there exists a conformal metric $\bar{g}$, which has constant curvature.

Notice that this theorem immediately implies a "baby Poincaré conjecture":

Theorem

Suppose M is compact, simply connected surface. Then M is diffeomorphic to S^2.

Proof: Equip M with any Riemannian metric g. Then by the uniformization theorem we can conformally change g to a constant curvature metric $\bar{g}$. Now the curvature of $(M, \bar{g})$ is either constantly negative, zero or positive. By rescaling we may assume it is -1, 0 or 1. In the first case, $(M, \bar{g})$ would be isometric to $(\mathbb{H}^2, g_{\text{hyp}})$. In the second case, $(M, \bar{g})$ would be isometric to $(\mathbb{R}^n, g_{eucl})$. Both of these cases are excluded, because $\mathbb{H}^2$ and $\mathbb{R}^2$ are not compact.

Thus $(M, \bar{g})$ is isometric to (S^2, g_{sph}), which proves the theorem. □

It turns out that the Ricci flow also gives a method of proof of this theorem. Indeed, as we will see the Ricci flow preserves the conformal class of a metric, i.e. if $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow on a surface, then all g_t are conformal to g_0. Hence, if we can show that a solution of the Ricci flow starting from a Riemannian metric on a compact surface converges to a metric of constant curvature, we obtain a proof of the uniformization theorem.

This is the goal of this lecture course. It turns out that to do this we need to introduce many techniques useful to the study of the Ricci flow in any dimension. Whenever this is the case, we describe the technique for arbitrary dimension.

Chapter 2

Short time existence and uniqueness

It is very far from obvious that given an initial metric, there exists a Ricci flow starting at that metric. Indeed, the Ricci flow equation is a quasilinear parabolic partial differential equation. As such, one needs tools from the theory of PDE (partial differential equations) to treat this question. It will turn out that for any smooth Riemannian metric, a Ricci flow with this metric as initial condition exists. However, a very important feature is that this solution does not necessarily exist on the interval $[0, \infty)$, or in the parlance of flows, the solution may not exist *for all times*. Instead, at some finite time it may run into a *singularity*. We have already seen such an example: the sphere contracts under the Ricci flow into a point. At this time the Ricci flow stops.

What PDE theory enables us to prove is that there is a solution on *some* interval $[0, T)$ with $T > 0$. Such a result is known as *short time existence*. This is a foundational result for the Ricci flow, since, if it did not hold, the program towards the Poincaré conjecture would be in serious jeopardy.

Theorem 2.1 [Short time existence]

Let M be a closed manifold. Suppose g is a Riemannian metric. Then there exists $\epsilon > 0$ and a solution $(g_t)_{t\in[0,\varepsilon)}$ of the Ricci flow with $g_0 = g$.

We will not prove this theorem in this generality. We will give a proof in the two dimensional case.

Theorem 2.2 [Uniqueness of the Ricci flow]

Let M be a closed manifold. Suppose $(\hat{g}_t)_{t\in[0,\hat{T}]}$ and$(\check{g}_t)_{t\in[0,\check{T}]}$ are two solutions of the Ricci flow.

If $\hat{g}_0 = \check{g}_0$, then $\hat{g}_t = \check{g}_t$ for any $t \in [0, \min\{\hat{T}, \check{T}\}]$.

Definition 2.3

A solution $(g_t)_{t\in[0,T)}$ is called *maximal*, if there does not exist any solution $(\tilde{g}_t)_{t\in[0,T+\epsilon)}$

of the Ricci flow with $\epsilon > 0$ and the same initial conditions $g_0 = \tilde{g}_0$.

Corollary 2.4 (Unique maximal solutions)
Let M be a closed manifold and suppose g is a Riemannian metric on M. Then there exists a unique, maximal solution $(g_t)_{t\in[0,T_{\max})}$ of the Ricci flow with initial value g.

Proof: Exercise! □

2.1 The Ricci flow on a surface as a scalar equation

We are not going to prove these theorems for arbitrary dimensions, but we will prove them in the two dimensional case. Fortuitiously, the Ricci flow equation reduces to a scalar equation in this case.

If M is a surface, then the tangent space is two-dimensional. Thus on a surface at any point there is only one sectional curvature. For historical reasons, this curvature has a name, introduced in the next definition.

Definition 2.5
The *Gauss curvature* of a Riemannian surface (M, g) at the point $p \in M$ is $K_g(p) = \sec_g(v, w)$, where v, w is any basis of T_pM.

The Gauss curvature determines the full curvature tensor and consequently also the Ricci curvature. This is the content of the following proposition.

Proposition 2.6
Let (M, g) be a two-dimensional Riemannian manifold. Then for any $u, v, w \in T_pM$ the following identity holds

$$\mathrm{Rm}_g(X, Y, Z, W) = K_g\left(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\right)$$

Consequently, the Ricci curvature is given by

$$\mathrm{Ric}_g = K_g g$$

and the scalar curvature is $R_g = 2K_g$.

Proof: Exercise! □

The fact that the Ricci curvature has such a simple form in two dimensions allows us to reduce the Ricci flow equation to a scalar equation. Plugging in equation 2.6 in the Ricci flow equation yields

$$\partial_t g_t = -2\,\mathrm{Ric}_{g_t} = -2K_{g_t} g_t.$$

Thus, at any given point $p \in M$, the scalar product $g_t(p)$ changes only by rescaling, i.e.

$$g_t(p) = \lambda_t(p) g_0(p)$$

for some $\lambda_t(p) > 0$. We may thus assume $g_t = \lambda_t g$, where g is a Riemannian metric and $\lambda_t : M \to \mathbb{R}_+$ is a function. We then calculate

$$\partial_t g_t = (\partial_t \lambda_t) g = -2K_{\lambda_t g} g.$$

It turns out that it is more convenient to compute the curvature of $e^{2u}g$ in terms of u, rather than the curvature of λg in terms of λ. The next proposition provides this formula.

Proposition 2.7

Suppose (M, g) is a Riemannian manifold and $u \in C^2(M)$. Let $\tilde{g} = e^{2u}g$. Then for any vector fields $X, Y \in \Gamma(TM)$, we have

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + (Xu)Y + (Yu)X - g(X,Y) \operatorname{grad}_g u.$$

Proof: The Koszul formula applied to $\tilde{g}$ reads

$$\begin{aligned} 2\tilde{g}(\nabla^{\tilde{g}}_X Y, Z) = {} & X\tilde{g}(Y,Z) + Y\tilde{g}(X,Z) - Z\tilde{g}(X,Y) \\ & + \tilde{g}([X,Y],Z) - \tilde{g}([X,Z],Y) - \tilde{g}([Y,Z],X). \end{aligned}$$

If we multiply both sides by e^{-2u}, we get

$$\begin{aligned} 2g(\nabla^{\tilde{g}}_X Y, Z) = {} & e^{-2u}X(e^{2u}g(Y,Z)) + e^{-2u}Y(e^{2u}g(X,Z)) - e^{-2u}Z(e^{2u}g(X,Y)) \\ & + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned}$$

Now

$$e^{-2u}X(e^{2u}g(Y,Z)) = Xg(Y,Z) + 2(Xu)g(Y,Z).$$

Doing the same calculations for the two other terms involving derivatives, we obtain

$$2g(\nabla^{\tilde{g}}_X Y, Z) = 2g(\nabla^g_X Y, Z) + 2(Xu)g(Y,Z) + 2(Yu)g(X,Z) - 2(Zu)g(X,Y).$$

This we may simplify to

$$g(\nabla^{\tilde{g}}_X Y, Z) = g(\nabla^g_X Y + (Xu)Y + (Yu)X, Z) - (Zu)g(X,Y).$$

Using

$$Zu = g(\operatorname{grad}_g u, Z),$$

we obtain

$$g(\nabla^{\tilde{g}}_X Y, Z) = g(\nabla^g_X Y + (Xu)Y + (Yu)X - g(X,Y) \operatorname{grad}_g u, Z).$$

This proves the identity. □

Proposition 2.8

Let (M,g) be a Riemannian manifold and $u \in C^2(M)$. Let $\tilde{g} = e^{2u}g$. Then for any two g-orthonormal vectors $v, w \in T_pM$ the sectional curvature of the two plane $\operatorname{span}\{v,w\}$ *is given by*

$$\begin{aligned}\sec_{\tilde{g}}(v,w) &= e^{-4u}\operatorname{Rm}_{\tilde{g}}(w,v,v,w)\\ &= e^{-2u}\left(\sec_g(v,w) - \operatorname{Hess}_g u(v,v) - \operatorname{Hess}_g u(w,w) + du(v)^2 + du(w)^2 - |du|_g^2\right).\end{aligned}$$

Proof: Exercise! □

Corollary 2.9

Suppose (M,g) is a two-dimensional Riemannian manifold and suppose that $u \in C^2(M)$. Then the curvature of $K_{\tilde{g}} = e^{2u}g$ is given by

$$K_{\tilde{g}} = e^{-2u}\left(\Delta_g u + K_g\right).$$

Proof: Let e_1, e_2 be an orthonormal frame of T_pM. Then

$$|df|_g^2 = du(e_1)^2 + du(e_2)^2,$$

$$\Delta_g f = -\operatorname{tr}_g \operatorname{Hess}_g f = -\operatorname{Hess}_g f(e_1,e_1) - \operatorname{Hess}_g f(e_2,e_2)$$

and

$$K_g = \sec_g(e_1,e_2).$$

Thus, by the previous proposition

$$\begin{aligned}K_{\tilde{g}} = \sec_{\tilde{g}}(v,w) &= e^{-2u}\left(K_g + \Delta_g u + du(e_1)^2 + du(e_2)^2 - |du|_g^2\right)\\ &= e^{-2u}\left(\Delta_g u + K_g\right)\end{aligned}$$

□

The previous proposition enables a rewriting of the Ricci flow equation as a scalar partial differential equation.

Corollary 2.10

Let M be a closed surface and fix a metric g. The family $g_t = e^{2u_t}g$ solves the Ricci flow equation, if and only if

$$\partial_t u_t = -K_{g_t} = -e^{-2u_t}\left(\Delta_g u_t + K_g\right).$$

Proof: For the family g_t, the time derivative is

$$\partial_t g_t = \partial_t\left(e^{2u_t}g\right) = 2e^{2u_t}(\partial_t u_t)g = (2\partial_t u_t)g_t.$$

On the other hand, the Ricci flow equation is

$$\partial_t g_t = -2\operatorname{Ric}_{g_t} = -2K_{g_t}g_t.$$

Formula 2.9 then yields the claim, i.e.

$$\partial_t u_t = -K_{g_t} = -e^{-2u_t}\left(\Delta_g u_t + K_g\right)$$

□

It is significantly easier to establish short time existence in the two dimensional case than in general.

2.2 Proof of short time existence on surfaces

The Picard–Lindelöf theorem guarantees existence of solutions of initial value problems for ordinary differential equations. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function and consider the initial value problem

$$\begin{cases} \partial_t x(t) = F(x(t)), \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases}$$

The fundamental theorem of calculus implies that

$$\Phi : C^1([0,T], \mathbb{R}^n) \to \mathbb{R}^n \times C^0([0,T], \mathbb{R}^n)$$

$$x \mapsto (x(0), \partial_t x)$$

is a Banach space isomorphism. In fact, with the norm

$$\|x\|_{C^1([0,T],\mathbb{R}^n)} = |x(0)| + \max_{t\in[0,T]} |\partial_t x(t)|$$

it is an isometry.

Define a map

$$\Psi : C^1([0,T], \mathbb{R}^n) \to C^1([0,T], \mathbb{R}^n)$$

$$x \mapsto \Phi^{-1}(x_0, F \circ x).$$

Then $\Psi(x) = x$ if and only if $x(0) = x_0$ and $\partial_t x(t) = F(x(t)$ for all $t \in [0,T]$.

Thus, we have converted our initial value problem into a fixed point problem. What is left to do, is to show that the map actually posesses a fixed point. Here, the Banach fixed point theorem is invoked.

Theorem 2.11 [Banach fixed point theorem]
Suppose (X, d) is a complete metric space and $F : X \to X$ is a contracting map, i.e. there is a constant $\delta \in (0,1)$, such that the images of any two points $x_1, x_2 \in X$ satisfy the inequality

$$d(x_1, x_2) < \delta d(x_1, x_2).$$

Then F has a unique fixed point.

Thus, to finish the proof of the Picard–Lindelöf theorem, it suffices to prove that Ψ is a contraction. Whether Ψ is a contraction depends on the choice of F. In the exercises, we will see that if the Lipschitz constant of F is smaller than 1, then Ψ is a contraction. By modifying the "speed of time", we can convert any differential equation into an equivalent equation, where the Lipschitz constant is smaller than 1.

We will use a similar approach to produce solutions of the Ricci flow.

As we have seen above solutions of the Ricci flow on a surface are given by functions $u : M \times [0,T) \to \mathbb{R}$ satisfying

$$\partial_t u(x,t) = -e^{-2u(x,t)} \left(\Delta_g u(x,t) + K_g(x)\right)$$

The operator Φ above will be replaced by the map

$$u \mapsto (u(\cdot, 0), (\partial_t + \Delta_g)u).$$

Notice that to define $(\partial_t + \Delta_g)u$, we need $u : M \times [0,T) \to \mathbb{R}$ to be differentiable in time and two times differentiable in space. With the notation $M_T = M \times [0,T]$, we can define

$$C^{2,1}(M_T, g) = \{u : M_T \to \mathbb{R} : u \in C^0(M_T), \partial_t \in C^0(M_T), \operatorname{Hess}_g(f) \in C^0(M_T)\}$$

with the norm

$$\|u\|_{C^{2,1}(M_T,g)} = \|u\|_{C^0(M_T)} + \|\partial_t u\|_{C^0(M_T)} + \|\operatorname{Hess}_g(u)\|_{C^0(M_T)}.$$

This is a Banach space and the operator

$$C^{2,1}(M_T, g) \to C^0(M) \times C^{2,1}(M_T)$$
$$u \mapsto (u(\cdot, 0), (\partial_t + \Delta_g)u).$$

is a continuous linear map. Unfortunately, however, this map is not continuously invertible! This is the point of departure for Schauder theory, which replaces the space $C^{2,1}$ by functions whose derivatives are Hölder continuous. On these spaces the operator turns out to be continuously invertible.

Definition 2.12 (Hölder continuous functions)
Let $\alpha \in (0,1)$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

A function $f : \Omega \to \mathbb{R}$ is *α-Hölder continuous*, if there exists $C > 0$, such that for every $x, y \in \Omega$

$$|f(x) - f(y)| \leq C d(x,y)^\alpha.$$

The *Hölder coefficient* of f is

$$[f]_\alpha = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

The α-Hölder norm of f is

$$\|f\|_{C^\alpha(\Omega)} = \|f\|_{C^0(\Omega)} + [f]_\alpha.$$

The α-Hölder space is

$$C^{\alpha}(\Omega) = \{f \in C^0(\Omega) : \|f\|_{C^{\alpha}(\Omega)} < \infty\}.$$

Analogously to $C^k(\Omega)$, $k \in \mathbb{N}$, we define the $C^{k,\alpha}(\Omega)$ norm of f to be

$$C^{\alpha}(\Omega) = \{f \in C^k(\Omega) : \|f\|_{C^{k,\alpha}(\Omega)} < \infty\}$$

where

$$\|f\|_{C^{k,\alpha}} = \sum_{|\beta|\le k} \|\partial^\beta f\|_{C^0(\Omega)} + \sum_{|\beta|=k} [\partial^\beta f]_{\alpha,\Omega}.$$

Remark 2.13 (Hölder spaces on manifolds). Let M be a compact manifold. Then we can cover M by a finite number of charts (U_i, φ_i), $i = 1, \dots, m$, where $\varphi_i : U_i \subset M \to B_1 \subset \mathbb{R}^n$ are diffeomorphisms. Choose a partition of unity ρ_k, $k = 1, \dots, N$, subordinate to the cover $(U_i)_i$, i.e. such that $\operatorname{supp} \rho_k$ is contained in some U_i for every k.

We say $f : M \to \mathbb{R}$ is *α-Hölder continuous*, if $f \circ \varphi_i^{-1}$ is α-Hölder continuous. We define

$$[f]_\alpha = \max_{k \in \{1,\dots,N\}} [(\rho_k f) \circ \varphi_i^{-1}]_{\alpha,\alpha,2}.$$

Then

$$C^\alpha(M) = \|f\|_{C^0(M)} + [f]_\alpha.$$

We proceed similarly to define $C^{k,\alpha}(M)$.

Definition 2.14 (Parabolic Hölder continuous functions)
Let $\alpha \in (0,1)$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $T > 0$.

The *parabolic distance* on $\Omega_T = \Omega \times [0,T]$ is

$$d^p((x,s),(y,t)) = \sqrt{d(x,y)^2 + |s-t|}.$$

Given a function $f : M \times [0,T] \to \mathbb{R}$ the $(\alpha, \alpha/2)$-Hölder coefficient is

$$[f]_{\alpha,\alpha/2} = \sup_{\substack{(x,s),(y,t) \in \Omega \times [0,T] \\ (x,s) \neq (y,t)}} \frac{|f(x,s) - f(y,t)|}{d^p((x,s),(y,t))^\alpha}.$$

The $(\alpha, \alpha/2)$-Hölder norm is

$$\|f\|_{C^{\alpha,\alpha/2}(\Omega_T)} = \|f\|_{C^0(\Omega_T)} + [f]_{\alpha,\alpha/2}.$$

The space of $(\alpha, \alpha/2)$-Hölder continuous functions on Ω_T is

$$C^{\alpha,\alpha/2}(\Omega_T) = \{u \in C(\Omega_T) : \|u\|_{C^{\alpha,\alpha/2}(\Omega_T)} < \infty\}.$$

Analogously to $C^{2,1}(\Omega)$ we define $C^{2,\alpha,1,\alpha/2}(\Omega_T)$ to be

$$C^{2,\alpha,1,\alpha/2}(\Omega_T) = \{u \in C^{2,1}(\Omega_T) : \|u\|_{C^{2,\alpha,1,\alpha/2}(\Omega_T)} < \infty\}$$

where

$$\|u\|_{C^{2,\alpha,1,\alpha/2}(\Omega_T)} = \|u\|_{C^{2,1}(\Omega_T)} + [\partial_t u]_{\alpha,\alpha/2} + \sum_{|\beta|=2} [\partial^\beta u]_{\alpha,\alpha/2}.$$

Theorem 2.15

Suppose (M,g) is a closed Riemannian manifold, $D,T>0$. If $\varphi \in C^{2,\alpha}(M)$ and $f \in C^{\alpha,\alpha/2}(M_T)$, then there exists a unique solution $u \in C^{2,\alpha,1,\alpha/2}(M_T)$ of the initial value problem

$$\begin{cases} \partial_t u(x,t) + D\Delta_g u(x,t) = f(x,t) & \text{for all } (x,t) \in M_T \\ u(x,0) = \varphi(x) & \text{for all } x \in M \end{cases}$$

This solution satisfies the following inequality

$$\|u\|_{C^{2,\alpha,1,\alpha/2}(M_T)} \le C\left(\|\varphi\|_{C^{2,\alpha}(M)} + \|f\|_{C^{\alpha,\alpha/2}(M_T)}\right),$$

where $C_1 > 0$ is a constant depending on (M,g), D and T and $C_2 > 0$ is a constant depending only on (M,g)

More precisely, if $\varphi \equiv 0$, then there exist constants $C_1, C_2, C_3 > 0$ depending only on (M,g), such that

$$\|u\|_{C^0(M_T)} \le C_1 T \|f\|_{C^0(M_T)},$$
$$[\operatorname{Hess}_g(u)]_{\alpha,\alpha/2} \le C_2 \frac{1}{\sqrt{D}} [f]_{\alpha,\alpha/2},$$
$$[\partial_t u]_{\alpha,\alpha/2} \le C_3 \sqrt{D} [f]_{\alpha,\alpha/2}.$$

Remark 2.16. This is a deep theorem from the theory of partial differential equations. Estimates of this kind are called Schauder estimates. This theorem will not be proven in this class.

Corollary 2.17

Suppose (M,g) is a closed Riemannian manifold. The linear map

$$\eta_D : C^{2,\alpha,1,\alpha/2}(M_T) \to C^{2,\alpha}(M) \times C^{\alpha,\alpha/2}(M_T)$$
$$u \mapsto (u(\cdot,0), (\partial_t + D\Delta_g)u)$$

is an isomorphism of Banach spaces.

Using this isomorphism allows us to reformulate the Ricci flow equation as a fixed point problem.

Before we do this, we need one more observation.

Proposition 2.18

Let $D > 0$.

A familiy $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow, if and only if $\hat{g}_t = g_{Dt}$ solves the equation

$$\partial_t \hat{g}_t = -2D \operatorname{Ric}_{\hat{g}_t}$$

on the interval $[0, T/D]$.

Let M be a closed surface and suppose g is any Riemannian metric. We established above that $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow equation, if and only if $g_t = e^{2u_t}g$ and $u_t : M \times [0,T) \to \mathbb{R}$ satisfies

$$\begin{cases} \partial_t u_t = -e^{-2u_t}\left(\Delta_g u_t + K_g\right) \\ u_0 = 0 \end{cases}$$

By the proposition above, $\hat{u}_t = u_{Dt}$ then satisfies

$$\partial_t u_t = -De^{-2u_t}\left(\Delta_g u_t + K_g\right). \tag{2.1}$$

This equation may be rewritten as

$$\partial_t u_t + D\Delta_g u_t = D(1 - e^{-2u_t})\Delta_g u_t - DK_g e^{-2u_t}.$$

Define

$$N : C^{2,\alpha,1\alpha/2}(M_T) \to C^{\alpha,\alpha/2}(M_T)$$

$$u \mapsto (1 - e^{-2u})\Delta_g u + K_g e^{-2u}.$$

Using N and η_D, we can rewrite the above equation as

$$\eta_D(u) = (0, DN(u)).$$

From now on we fix $T = 1$. Since η is an isomorphism, we can define the map

$$\Psi_D : C^{2,\alpha,1,\alpha/2}(M_1) \to C^{2,\alpha,1,\alpha/2}(M_1)$$

$$u \mapsto \eta_D^{-1}(0, DN(u)).$$

Finally, we have arrived at a reformulation of the problem as a fixed point problem, because u solves the equation if and only if

$$\Psi_D(u) = u.$$

We sum up the discussion above in the following proposition

Proposition 2.19

Suppose $u \in C^{2,\alpha,1,\alpha/2}(M_1)$ is a fixed point of Ψ_D. Then

$$v(x,t) = u(x, t/D)$$

solves

$$\begin{cases} \partial_t v_t = -e^{-2v_t}\left(\Delta_g v_t + K_g\right) \\ u_0 = 0 \end{cases}$$

on $M \times [0, D]$. Consequently, $\tilde{g}_t = e^{2v_t}g$ is a solution of the Ricci flow on $[0, D]$.

Proposition 2.20

For any $M > 0$, $0 < \delta < 1$, there exists $D > 0$, such that for $u, v \in C^{2,\alpha,1,\alpha/2}(M_1)$ with

$$\|u\|_{C^{2,\alpha,1,\alpha/2}(M_1)}, \|v\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \leq M,$$

we have

$$\|\Psi_D(u)\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \leq M$$

and

$$\|\Psi_D(u) - \Psi_D(v)\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \leq \delta \|u - v\|_{C^{2,\alpha,1,\alpha/2}(M_1)}.$$

Proof: In the exercises, we will prove the following two properties of N:

1. $\|N(u)\|_{C^{\alpha,\alpha/2}(M_1)} \leq C \exp(\|u\|_{C^0(M_1)}) \left(1 + \|u\|_{C^\alpha(M_1)}\right) (1 + \|u\|_{C^{2,\alpha,1,\alpha/2}}),$
2. $\|N(u) - N(v)\|_{C^{\alpha,\alpha/2}(M_1)} \leq CF(\|u\|_{C^{\alpha,\alpha/2}} + \|v\|_{C^{\alpha,\alpha/2}})\|u - v\|_{C^{2,\alpha,1,\alpha/2}}$

for some $C > 0$ and a continuous function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

On the other hand we have the estimate

$$\|\eta_D^{-1}(0, f)\|_{C^{2,\alpha,1,\alpha/2}(M_T)} \leq \hat{C} \max\{\sqrt{D}, \sqrt{D}^{-1}\} \|f\|_{C^{\alpha,\alpha/2}(M_1)}.$$

Thus

$$\begin{aligned}
\|\Psi_D(u)\|_{C^{2,\alpha,1,\alpha/2}(M_1)} &= \|\eta_D^{-1}(0, DN(u))\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \\
&\leq \hat{C} \max\{\sqrt{D}, \sqrt{D}^{-1}\} D \|N(u)\|_{C^{\alpha,\alpha/2}(M_T)} \\
&\leq \hat{C} D \max\{\sqrt{D}, \sqrt{D}^{-1}\} C \exp(\|u\|_{C^0}) \left(1 + \|u\|_{C^\alpha(M_1)}\right) (1 + \|u\|_{C^{2,\alpha,1,\alpha/2}})
\end{aligned}$$

In particular, if $\|u\|_{C^{2,\alpha,1,\alpha/2}(M_T)} \leq M$, by choosing $D > 0$ to be small, we can arrange that $\|\Psi_D(u)\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \leq M$.

In the same vein, we can show that

$$\|\Psi_D(u) - \Psi_D(v)\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \leq \tilde{C} D \max\{\sqrt{D}, \sqrt{D}^{-1}\} \|u - v\|_{C^{2,\alpha,1,\alpha/2}(M_T)}.$$

Again, by choosing $D > 0$ small, we can arrange

$$\tilde{C} D \max\{\sqrt{D}, \sqrt{D}^{-1}\} < \delta$$

and so

$$\|\Psi_D(u) - \Psi_D(v)\|_{C^{2,\alpha,1,\alpha/2}(M_1)} < \delta \|u - v\|_{C^{2,\alpha,1,\alpha/2}(M_T)}.$$

□

Corollary 2.21

Let M be a compact surface. Let g be a Riemannian metric on M. Then there exists $T > 0$ and a solution of the Ricci flow $(g_t)_{t \in [0,T]}$ with initial condition g.

Proof: Choose some $M > 0$ and $\delta < 1$. Then let $D > 0$ such that the conditions of the previous proposition are met. The map Ψ_D restricted to the set

$$\{u \in C^{2,\alpha,1,\alpha/2}(M_1) : \|u\|_{C^{2,\alpha,1,\alpha/2}(M_1)} \leq M\}$$

then satisfies the condition of the Banach fixed point theorem. Thus there exists $u \in C^{2,\alpha,1,\alpha/2}(\Omega_1)$ satisfying $\Psi_D(u) = u$. Proposition 2.19 says that such a fixed point corresponds to a solution of the Ricci flow with initial value g on the interval $[0, D]$. □

Corollary 2.22

Let M be a compact surface. Let g be a Riemannian metric on M. Suppose $(g_t)_{t\in[0,T]}, (\hat{g}_t)_{t\in[0,T]}$ are solutions of the Ricci flow with initial condition g. Then $g_t = \hat{g}_t$ for $t \in [0, T]$.

Proof: This follows from the uniqueness in the Banach fixed point problem. By proposition 2.19, we know that both g_t and $\hat{g}_t$ correspond to fixed points of Ψ_D for some appropiate $D > 0$. By decreasing D if necessary, we may assume that both fixed pointds are in the range where Ψ_D is a contraction. Because of the uniqueness of fixed points, this implies $g_t = \hat{g}_t$ for $t \in [0, D]$.

To prove it for every $t \in T$, note that we can "restart" the Ricci flow at time D with initial condition $g_D = \hat{g}_D$. Then by the same argument $g_{D+t} = \hat{g}_{D+t}$ for all $t \in [0, D']$ for some $D' > 0$. This shows that the set $I = \{t \in [0, T] : g_t = \hat{g}_t\}$ is open in $[0, T]$. It is also closed, because the solution is continuous in time. Thus $I = [0, T]$. □

Remark 2.23. There is a gap in the above proof. We have shown that for a $\mathcal{C}^\infty$ Riemannian metric g we can find solutions of the Ricci flow. At this time, we only know that the metric g_D is in $C^{2,\alpha}$. So we either need to show the short time existence for $C^{2,\alpha}$ metrics or we need to check that g_D is indeed $\mathcal{C}^\infty$.

In fact, the solution is $\mathcal{C}^\infty$ both in time and space. This can be proven by the "bootstrapping method" using regularity theory of partial differential equations.

Chapter 3

Evolution of the curvature and extension criteria

In the previous chapter we established that for any smooth metric g on a compact surface, we can find a solution of the Ricci flow with initial condition g on *some* time interval $[0,T)$. By an abstract argument we can extend this solution to a unique, maximal solution on an interval $[0,T_{\max})$. Our existence proof yielded no information on the size of $T_{\max}$, neither a lower nor an upper bound. It turns out that a very useful tool to get information on the size of $T_{\max}$ is to consider how certain quantities associated to g_t change in time. The first quantity we consider is the Gaussian curvature.

Proposition 3.1

Let M be a surface and suppose $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow. Then the Gaussian curvature K_{g_t} satisfies the partial differential equation

$$\partial_t K_{g_t} + \Delta_{g_t} K_{g_t} = 2K_{g_t}^2.$$

Proof: Let $g = g_0$. With $g_t = e^{2u_t} g_0$ we have

$$\partial_t u_t = -K_{g_t} = -e^{-2u_t}\left(\Delta_g u_t + K_g\right).$$

Moreover,

$$\Delta_{g_t} f = e^{-2u_t}\Delta_g f.$$

Thus, we compute

$$\begin{aligned}\partial_t K_{g_t} &= -2(\partial_t u_t)K_{g_t} - e^{-2u_t}\Delta_g \partial_t u_t\\ &= 2K_{g_t}^2 - \Delta_{g_t} K_{g_t}.\end{aligned}$$

□

This equation is very useful, because for equations of this type the maximum principle allows us to compute bounds by solving an ordinary differential equation. This is made precise in the next proposition.

Theorem 3.2 [Parabolic maximum principle]

Suppose M is a compact manifold, $(g_t)_{t\in[0,T]}$ is a smooth family of Riemannian metrics on M, $(X_t)_{t\in[0,T]}$ is a smooth family of vector fields on M, $F:\mathbb{R}\times[0,T]\to\mathbb{R}$ is a smooth function. Suppose further that $\phi:[0,T]\to\mathbb{R}$ is a solution of the ODE

$$\frac{d}{dt}\phi(t) = F(\phi(t),t).$$

Any $u\in\mathcal{C}^\infty(M\times[0,T])$ satisfying

$$\begin{cases}\partial_t u(x,t)+\Delta_{g_t}u(x,t)+g_t\left(X_t(x),\operatorname{grad}_{g_t}u(x,t)\right)\leq F(u(x,t),t)\\ \max_M u_0\leq\phi(0)\end{cases}$$

is bounded by ϕ on the whole interval, i.e.

$$\max_{x\in M}u(x,t)\leq\phi(t)\text{ for all }t\in[0,T].$$

Proof: The idea for the proof is that if x is a maximum of a smooth function, then its gradient vanishes at x, its Hessian at x is negative semi-definite and consequently the Laplacian is non-negative.

Let $t\in[0,T]$. Because M is compact, then $u(\cdot,t)$ assumes its maximum at a point $x\in M$. At this point we have

$$\Delta_{g_t}u(x,t)+g_t(X_t(x),\operatorname{grad}_{g_t}u(x,t))\geq 0.$$

On the other hand, by assumption

$$\partial_t u(x,t)+\Delta_{g_t}u(x,t)+g_t\left(X(t),\operatorname{grad}_{g_t}u_t\right)\leq F(u(x,t),t).$$

Substracting the first inequality from the second, we obtain

$$\partial_t u(x,t)\leq F(u(x,t),t).$$

To show that u lies below the solution of the comparison ODE, we first apply a trick to make the inequality strict. For $\epsilon>0$, let ϕ_ϵ be the solution of the ODE

$$\begin{cases}\frac{d}{dt}\phi_\epsilon(t)=F(\phi_\epsilon(t),t)+\epsilon,\\ \phi_\epsilon(0)=\phi(0)+\epsilon\end{cases}$$

From the theory of ODEs, we know that for every ϵ a unique solution exists and moreover, ϕ_ϵ converges uniformly to ϕ on $[0,T]$.

We will now argue that

$$u(x,t)<\phi_\epsilon(t)$$

for every $\epsilon>0$ and $(x,t)\in M\times[0,T]$. Letting ϵ go to 0, we obtain

$$u(x,t)\leq\phi(t).$$

To see this, we argue by contradiction: if this was false, then for every $\epsilon > 0$, we could choose $(x_0, t_0) \in M \times [0, T]$, such that

$$u(x_0, t_0) = \phi_\epsilon(u(x_0, t_0), t_0).$$

Moreover, we may assume that t_0 is the first time, such that this equation holds. This means

$$u(x, t) < \phi_\epsilon(u(x, t), t)$$

for every $x \in M$, $t < t_0$. In particular $u(x_0, t) - \phi_\epsilon(u(x_0, t), t)$ is negative for $t < t_0$ and zero at t_0. This implies

$$\partial_t(u(x_0, t_0) - \phi_\epsilon(u(x_0, t_0), t_0)) \geq 0.$$

By definition of ϕ_ϵ, this implies

$$\partial_t u(x_0, t_0) \geq \partial_t \phi_\epsilon(u(x_0, t_0), t_0) = F(u(x_0, t_0), t_0) + \epsilon$$

and consequently, because $u(x_0, t_0)$ is a maximum of $u(\cdot, t_0)$,

$$\partial_t u(x_0, t_0) + \Delta_{g_{t_0}} u(x_0, t_0) + g_{t_0}(X_{t_0}(x), \operatorname{grad}_{g_{t_0}} u(x_0, t_0)) \geq F(u(x_0, t_0), t_0) + \epsilon.$$

This is clearly a contradiction to the assumption. □

Remark 3.3. By considering $-u$, the maximum principle turns into a minimum principle, i.e. we can replace $\leq$ by $\geq$ and max by min in the theorem above.

Theorem 3.4

Suppose M is a compact surface and $(g_t)_{t \in [0, T_{\max})}$ is the maximal solution of the Ricci flow. With

$$\kappa_- = \min_M K_{g_0} \leq \max K_{g_0} = \kappa_+,$$

it follows that

$$\frac{\kappa_-}{1 - 2\kappa_- t} \leq K_{g_t} \leq \frac{\kappa_+}{1 - 2\kappa_+ t}$$

for every $t \in [0, T_{\max})$.

Proof: The evolution equation for K_{g_t} is

$$\partial_t K_{g_t} + \Delta_{g_t} K_{g_t} = 2K_{g_t}^2.$$

We apply the maximum principle with $F(x, t) = 2x^2$. Thus we need to solve the ODE

$$\frac{d}{dt}\phi(t) = 2\phi(t)^2$$

with inital condition $\phi(0) = \kappa$. The solution is given by

$$\phi(t) = \frac{\kappa}{1 - 2\kappa t}$$

and thus

$$\min_{x \in M} K_{g_t}(x) \geq \phi(t) = \frac{\kappa}{1 - 2\kappa t}.$$

□

Corollary 3.5

If $|K_{g_0}| \leq M$, then if $\frac{1}{4M} < T_{\max}$, it follows that

$$|K_{g_{1/(4M)}}| \leq 2M.$$

Remark 3.6. Soon, we will see that $\max |K_{g_t}| \to \infty$ as t approaches the maximal time of existence. This implies $\frac{1}{4M} < T_{\max}$. In particular, there is a lower bound for the maximal time of existence in terms of the maximum of the curvature of the initial metric.

Corollary 3.7

For any $\kappa \in \mathbb{R}$, the condition $K_g \geq \kappa$ is preserved along the Ricci flow, i.e. if $(g_t)_{t\in[0,T_{\max})}$ is a solution of the Ricci flow, then $K_{g_t} \geq \kappa$ for all $t \in [0, T_{\max})$.

Corollary 3.8

The condition $K_g \leq 0$ is preserved along the Ricci flow.

For the Ricci flow in arbitrary dimensions there is a similar equation.

Theorem 3.9

If M is a n-dimensional manifold and g_t is a solution of the Ricci flow, then

$$\partial_t R_{g_t} + \Delta_{g_t} R_{g_t} = 2|\operatorname{Ric}_{g_t}|^2.$$

Remark 3.10. This equation is not quite as useful in higher dimensions as in two dimensions. The reason is that the right hand side depends on the full Ricci tensor, not only on the scalar curvature. However, since

$$|\operatorname{Ric}_g|^2 = \sum_{i=1}^{n} \operatorname{Ric}_g(e_i, e_i)^2$$

and

$$R_g = \sum_{i=1}^{n} \operatorname{Ric}_g(e_i, e_i),$$

we obtain (by Cauchy–Schwarz inequality)

$$R_g^2 \leq n|\operatorname{Ric}_g|^2$$

and thus

$$\partial_t R_{g_t} + \Delta_{g_t} R_{g_t} \geq \frac{2}{n} R_g^2.$$

Using the minimum principle as in theorem 3.4, this still allows us to derive the lower bound

$$R_{g_t} \geq \frac{\sigma}{1 - \frac{2\sigma}{n} t}, \quad \text{where } \sigma = \min_M R_{g_0}.$$

Our goal is to understand, at what time the Ricci flow stops being well-defined. Note that a solution $(g_t)_{t\in[0,T)}$ of the Ricci flow is per definition a smooth family of metrics satisfying

$$\partial_t g_t(x) = -2\operatorname{Ric}_{g_t}(x).$$

This implies, on a compact manifold, that at any fixed time t there is an upper bound on

$$|\partial_t g_t(x)|^2_{g_t} = 4|\operatorname{Ric}_{g_t}(x)|^2.$$

This is a priori only a necessary condition, not a sufficient condition, since by the same argument we also have a bound on all the derivatives of Ric_{g_t}.

We have seen how to obtain a bound on K_{g_t}. The obvious next step is to find bounds for the derivatives of K_{g_t}.

Let $X, Y \in \Gamma(TM)$ and $f \in \mathcal{C}^\infty(M)$. If we consider XYf, then this expression also depends on the derivatives of Y. For this reason, we introduced the Hessian $\operatorname{Hess}_g(f) = \nabla^g df$. For higher derivatives we face the same problem and we define the higher derivatives inductively via

$$\begin{aligned}\nabla^1 f &= df,\\ \nabla^k f &= \nabla^g \nabla^{k-1} f.\end{aligned}$$

In particular, $\nabla^2 f = \operatorname{Hess}_g(f)$. Note that ∇^g denotes the induced connection, see remark 0.14. The k-th covariant derivative of f is thus a $(0,k)$-tensor field, i.e. $\nabla^k f \in \Gamma(T^*M^{\otimes k})$.

We will also need to take norms of such tensors. To do this, we define

$$|T|^2_g = \sum_{i_1,\ldots,i_k\in\{1,\ldots,n\}} T(e_{i_1},\ldots,T_{e_{i_k}})^2$$

for $T \in \Gamma(T^*M^{\otimes k})$ and $e_1,\ldots,e_n$ an orthonormal basis of T_xM.

We start with the first derivative and we calculate

$$\begin{aligned}\partial_t dK_{g_t} &= d\partial_t K_{g_t}\\ &= d\left(-\Delta_{g_t}K_{g_t} + 2K^2_{g_t}\right)\\ &= -d\Delta_{g_t}K_{g_t} + 4K_{g_t}dK_{g_t}.\end{aligned}$$

To employ the maximum principle, we need an evolution equation for a real valued function. We can consider $|dK_{g_t}|^2_{g_t}$. Then

$$\begin{aligned}\partial_t|dK_{g_t}|^2_{g_t} &= (\partial_t g_t)(dK_{g_t}, dK_{g_t}) + 2g_0(d\partial_t K_{g_t}, K_{g_t})\\ &= (\partial_t g_t)(dK_{g_t}, dK_{g_t}) - 2g_t(d\Delta_{g_t}K_{g_t}, K_{g_t}) + 2g_t(4K_{g_t}dK_{g_t}, dK_{g_t})\end{aligned}$$

The maximum principle does not immediately apply, because there is no $\Delta_{g_t}|dK_{g_t}|^2$ term. With the aid of the next two propositions, the term $g_t(d\Delta_{g_t}K_{g_t}, K_{g_t})$ can be rewritten into a sum containing $\Delta_{g_t}|dK_{g_t}|^2$.

The connection Laplacian is defined by

$$\nabla^{g*}\nabla^g\alpha = -\sum_{i=1}^n(\nabla^g\nabla^g\alpha)(e_i, e_i),$$

where $e_1,\ldots,e_n$ is any orthonormal basis.

Proposition 3.11

Let (M, g) be a Riemannian manifold and let $\alpha \in \Gamma(T^*M)$. Then

$$\frac{1}{2}\Delta_g|\alpha|_g^2 = g(\nabla^{g*}\nabla^g\alpha, \alpha) - |\nabla^g\alpha|^2.$$

Proof: Let us first compute the Hessian of $|\alpha|_g^2$. We have

$$\begin{aligned}(\mathrm{Hess}_g\,|\alpha|_g^2)(V, W) &= VW|\alpha|_g^2 - (d|\alpha|_g^2)(\nabla^g_V W)\\ &= V(2g(\nabla^g_W\alpha, \alpha)) - (d|\alpha|_g^2)(\nabla^g_V W)\\ &= 2g(\nabla^g_V\nabla^g_W\alpha, \alpha) + 2g(\nabla^g_W\alpha, \nabla^g_V\alpha) - 2g(\nabla^g_{\nabla^g_V W}\alpha, \alpha)\end{aligned}$$

Let $e_1, \ldots, e_n$ be an orthonormal frame. Then

$$\begin{aligned}\Delta_g|\alpha|_g^2 &= -\sum_{i=1}^n \mathrm{Hess}_g(|\alpha|_g^2)(e_i, e_i)\\ &= -2\sum_{i=1}^n \Big(g(\nabla^g_{e_i}\nabla^g_{e_i}\alpha, \alpha) + g(\nabla^g_{e_i}\alpha, \nabla^g_{e_i}\alpha) - g(\nabla^g_{\nabla^g_{e_i}e_i}\alpha, \alpha)\Big)\\ &= -2\sum_{i=1}^n g\left(\nabla^g_{e_i}\nabla^g_{e_i}\alpha - \nabla^g_{\nabla^g_{e_i}e_i}\alpha, \alpha\right) - 2\sum_{i=1}^n |\nabla_{e_i}\alpha|_g^2\\ &= -2\sum_{i=1}^n g\left((\nabla^g\nabla^g\alpha)(e_i, e_i), \alpha\right) - 2|\nabla^g\alpha|_g^2\\ &= 2g(\nabla^{g*}\nabla^g\alpha, \alpha) - 2|\nabla^g\alpha|_g^2.\end{aligned}$$

□

Proposition 3.12 (Bochner formula)

Let (M, g) be a Riemannian manifold and let $f \in \mathcal{C}^\infty(M)$. Then

$$d\Delta_g f = \nabla^{g^*}\nabla^g df + \mathrm{Ric}_g(df),$$

where for $\alpha \in \Gamma(T^*M)$ the one form $\mathrm{Ric}_g(\alpha) \in \Gamma(T^*M)$ is defined by

$$[\mathrm{Ric}_g(\alpha)](X) = \mathrm{Ric}_g(\alpha^\sharp, X).$$

Lemma 3.13

Let (M, g) be a Riemannian manifold.

1. *For any $\alpha \in \Gamma(T^*M)$ and $X, Y, Z \in \Gamma(TM)$, the following formula holds*

$$[R(X, Y)\alpha](Z) = -\alpha(R(X, Y)Z),$$

where on the left hand side R is the curvature of the induced connection on T^*M and on the right hand side R is the curvature of the Levi–Civita connection.

2. *For $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$, the following formula holds*

$$(\nabla^g\nabla^g\alpha)(X,Y,Z) - (\nabla^g\nabla^g\alpha)(Y,X,Z) = [R(X,Y)\alpha](Z).$$

Proof: We leave the first claim as an exercise. For the second, we calculate

$$\begin{aligned}
(\nabla^g\nabla^g\alpha)(X,Y,Z) &= (\nabla^g_X\nabla^g\alpha)(Y,Z)\\
&= X[(\nabla^g\alpha)(Y,Z)] - (\nabla^g\alpha)(\nabla^g_X Y, Z) - (\nabla^g\alpha)(Y, \nabla^g_X Z)\\
&= X[Y\alpha(Z) - \alpha(\nabla^g_Y Z)] - (\nabla^g\alpha)(\nabla^g_X Y, Z) - (\nabla^g\alpha)(Y, \nabla^g_X Z)\\
&= XY\alpha(Z) - (\nabla^g\alpha)(X, \nabla^g_Y Z) - \alpha(\nabla^g_X\nabla^g_Y Z)\\
&\quad - (\nabla^g\alpha)(\nabla^g_X Y, Z) - (\nabla^g\alpha)(Y, \nabla^g_X Z).
\end{aligned}$$

Thus

$$\begin{aligned}
&(\nabla^g\nabla^g\alpha)(X,Y,Z) - (\nabla^g\nabla^g\alpha)(Y,X,Z)\\
=&(XY - YX)\alpha(Z) - \alpha(\nabla^g_X\nabla^g YZ - \nabla^g_Y\nabla^g_X Z) - (\nabla^g\alpha)(\nabla^g_X Y - \nabla^g_Y X, Z)\\
=&-\alpha(\nabla^g_X\nabla^g YZ - \nabla^g_Y\nabla^g_X Z) + [X,Y]\alpha(Z) - (\nabla^g_{[X,Y]}\alpha)(Z)\\
=&-\alpha(\nabla^g_X\nabla^g YZ - \nabla^g_Y\nabla^g_X Z) + \alpha(\nabla^g_{[X,Y]}Z)\\
=&-\alpha(R(X,Y)Z)\\
=&[R(X,Y)\alpha](Z).
\end{aligned}$$

□

Lemma 3.14
Let $e_1, \ldots, e_n$ be an orthonormal frame of TM, i.e. local vector fields, such that $g(e_i, e_j) \equiv 1$. Then

$$\sum_{i=1}^n \mathrm{Hess}_g(f)(\nabla^g_X e_i, e_i) = 0$$

for any $X \in \Gamma(TM)$.

Proof: Exercise! □

Proof of the Bochner formula: Suppose $e_1, \ldots, e_n$ is an orthonormal frame. Then we compute

$$\begin{aligned}
(d\Delta_g f)(X) &= -\sum_{i=1}^n X\left((\nabla^g df)(e_i,e_i)\right)\\
&= -\sum_{i=1}^n \left((\nabla^g\nabla^g df)(X,e_i,e_i) - (\nabla^g df)(\nabla^g_X e_i, e_i) - (\nabla^g df)(e_i,\nabla^g_X e_i)\right)\\
&\overset{3.14}{=} -\sum_{i=1}^n (\nabla^g\nabla^g df)(X,e_i,e_i)\\
&\overset{3.13,2.}{=} -\sum_{i=1}^n \left((\nabla^g\nabla^g df)(e_i,X,e_i) + (R^g(X,e_i)df)(e_i)\right)\\
&\overset{*}{=} -\sum_{i=1}^n (\nabla^g\nabla^g df)(e_i,e_i,X) - \sum_{i=1}^n (R^g(X,e_i)df)(e_i),\\
&= \left(\nabla^{g*}\nabla^g df\right)(X) + \sum_{i=1}^n df(R^g(X,e_i)e_i),\\
&= (\nabla^{g^*}\nabla^g df)(X) + \mathrm{Ric}_g(X, df^\sharp)\\
&= \left[\nabla^{g^*}\nabla^g df + \mathrm{Ric}_g(df)\right](X).
\end{aligned}$$

In the equation marked $*$, the symmetry of $\nabla^g df$ was used. □

Corollary 3.15

Let M be a surface and suppose $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow. Then

$$\partial_t dK_{g_t} = -\nabla^{g_t*}\nabla^{g_t} dK_{g_t} + 3K_{g_t} dK_{g_t}.$$

Corollary 3.16

Let (M,g) be a Riemannian surface and $f \in C^2(M)$. Then

$$\frac{1}{2}\Delta_g |df|_g^2 = g(d\Delta_g f, df) - K_g|df|_g^2 - |\nabla^g df|_g^2.$$

Proof: Exercise! □

Lemma 3.17

Suppose $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow on a surface and suppose $\alpha \in \Gamma(T^*M)$. Then

$$\partial_t|\alpha|_{g_t}^2 = 2K_{g_t}|\alpha|_{g_t}^2.$$

Proof: By definition

$$|\alpha(x)|_g^2 = \alpha(e_1)^2 + \alpha(e_2)^2$$

for a g-orthonormal basis $e_1, e_2 \in T_xM$.

Since we work on a surface, we know that

$$g_t = e^{2u_t}g_0 \text{ where } \partial_t u_t = -K_{g_t}.$$

Now suppose e_1, e_2 is a g_0-orthonormal basis. Clearly, $e_1^t = e^{-u_t(x)}e_1$ and $e_2^t = e^{-u_t(x)}e_2$ form a g_t orthonormal basis.

Now we calculate

$$\partial_t\alpha(e_i^t) = \partial_t\alpha(e^{-u_t(x)}e_i) = -\left(\partial_t u_t(x)\right)\alpha(e_i^t) = K_{g_t}\alpha(e_i^t)$$

and the claimed formula follows. □

Now we can conclude the following evolution equation for $|dK_g|^2$.

PROPOSITION 3.18

Suppose $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow on a surface. Then

$$\partial_t|dK_{g_t}|^2 + \Delta_{g_t}|dK_{g_t}|^2 = -2|\nabla^{g_t}dK_{g_t}|^2 + 8K_{g_t}|dK_{g_t}|^2$$

Proof: Clearly,

$$\partial_t|dK_{g_t}|_g^2 = (\partial_t g_t)(dK_{g_t}, dK_{g_t}) + 2g(dK_{g_t}, d\partial_t K_{g_t}).$$

For the first term, we get by the previous lemma $2K_{g_t}|dK_{g_t}|^2$ and for the second term we compute

$$\begin{aligned} g(dK_{g_t}, -d\Delta_{g_t}K_{g_t} + 2d(K_{g_t}^2)) &= -g(d_{K_{g_t}}, d\Delta_g K_{g_t}) + 4K_{g_t}|dK_{g_t}|^2 \\ &= -\frac{1}{2}\Delta_{g_t}|dK_{g_t}|^2 - K_{g_t}|dK_{g_t}|^2 - |\nabla^g dK_{g_t}|_{g_t}^2 + 4K_{g_t}|dK_{g_t}|^2 \\ &= -\frac{1}{2}\Delta_{g_t}|dK_{g_t}|^2 - |\nabla^g dK_{g_t}|_{g_t}^2 + 3K_{g_t}|dK_{g_t}|^2. \end{aligned}$$

Thus,

$$\partial_t|dK_{g_t}|_g^2 = -\Delta_{g_t}|dK_{g_t}|^2 - 2|\nabla^g dK_{g_t}|_{g_t}^2 + 8K_{g_t}|dK_{g_t}|^2.$$

□

PROPOSITION 3.19

Let $A > 0$. Suppose M is compact surface and $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow. If

$$\sup_{t\in[0,T]}|K_{g_t}| \le A,$$

then

$$|dK_{g_t}|^2 \le \frac{A^2 + 2A^3T + 16A^4T^2}{t}.$$

In particular, if $T = 1/A$, we have

$$|dK_{g_t}| \le \frac{\sqrt{19}A}{\sqrt{t}}.$$

Proof: Let $u(x,t) = t|dK_{g_t}(x)|^2 + \alpha|K_{g_t}|^2$, where $\alpha \in \mathbb{R}_+$. Using

$$\Delta_g(f^2) = 2f\Delta_g f - 2|df|_g^2,$$

we see that

$$\begin{aligned}\partial_t |K_{g_t}|^2 &= 2K_{g_t}\partial_t K_{g_t} \\ &= 2K_{g_t}(-\Delta_{g_t} K_{g_t} + 2K_{g_t}^2) \\ &= -\Delta_{g_t}|K_{g_t}|^2 - 2|dK_{g_t}|^2 + 4K_{g_t}^3.\end{aligned}$$

Then it follows that

$$\begin{aligned}\partial_t u &= |dK_{g_t}|^2 + t\left(-\Delta_{g_t}|dK_{g_t}|^2 - 2|\nabla^{g_t} dK_{g_t}|^2 + 8K_{g_t}|dK_{g_t}|^2\right) + \alpha(-\Delta_{g_t}|K_{g_t}|^2 - 2|dK_{g_t}|^2 + 4K_{g_t}^3) \\ &= -\Delta_{g_t}\left(t|dK_{g_t}|^2 + \alpha|K_{g_t}|\right)^2 + (1 + 8tK_{g_t} - 2\alpha)|dK_{g_t}|^2 + 4\alpha K_{g_t}^3 - 2t|\nabla^{g_t} dK_{g_t}|^2.\end{aligned}$$

This can be rewritten as

$$\partial_t u + \Delta_{g_t} u = (1 + 8tK_{g_t} - 2\alpha)|dK_{g_t}|^2 + 4\alpha K_{g_t}^3 - 2t|\nabla^{g_t} dK_{g_t}|^2.$$

Since $t \in [0, T]$ and $|K_{g_t}| \leq A$ by assumption, it follows that

$$\begin{aligned}\partial_t u + \Delta_{g_t} u &\leq (1 + t8K_{g_t} - 2\alpha)|dK_{g_t}|^2 + 4\alpha K_{g_t}^3 \\ &\leq (1 + 8tA - 2\alpha)|dK_{g_t}|^2 + 4\alpha A^3 \\ &\leq (1 + 8TA - 2\alpha)|dK_{g_t}|^2 + 4\alpha A^3.\end{aligned}$$

Thus, choosing $\alpha = \frac{1+8TA}{2}$, we obtain

$$\partial_t u + \Delta_{g_t} u \leq 2(1 + 8TA)A^3.$$

Now we apply the maximum principle $F(y, t) = 2(1+8TA)A^3$. Note that $u(x, 0) = |K_{g_0}(x)|^2 \leq A^2$. Thus we have to solve the initial value problem

$$\dot{\phi}(t) = 2(1 + 8TA)A^3, \quad \phi(0) = A^2.$$

The solution is $\phi(t) = A^2 + 2(1 + 8TA)A^3 t$ and we obtain

$$t|dK_{g_t}|^2 + |K_{g_t}|^2 = u(x, t) \leq A^2 + 2(1 + 8TA)A^3 t \leq A^2 + 2A^3 T + 16A^4 T^2$$

and thus

$$|dK_{g_t}|^2 \leq \frac{A^2 + 2A^3 T + 16A^4 T^2}{t}.$$

□

Lemma 3.20

Let (M, g) be a Riemannian manifold and $u_t \in \mathcal{C}^\infty(M)$ a smooth family of functions. Let $X, Y \in \Gamma(TM)$. Let $g_t = e^{2u_t} g$. Then

$$\partial_t \nabla^{g_t}_X Y = (X\partial_t u)Y + (Y\partial_t u) - g_t(X, Y)\operatorname{grad}_{g_t} \partial_t u_t.$$

Proof: Recall that

$$\nabla^{g_t}_X Y = \nabla^g_X Y + (Xu)Y + (Yu)X - g(X, Y)\operatorname{grad}_g u.$$

From this it follows that

$$\partial_t \nabla^{g_t}_X Y = (X\partial_t u)Y + (Y\partial_t u) - g(X,Y)\,\mathrm{grad}_g\,\partial_t u_t.$$

Now observe that

$$g(X,Y)\,\mathrm{grad}_g\,\partial_t u = g_t(X,Y)\,\mathrm{grad}_{g_t}\,\partial_t u,$$

since by an easy calculation

$$\mathrm{grad}_{g_t} f = e^{-2u_t}\,\mathrm{grad}_g f.$$

□

Theorem 3.21

There exist smooth functions $F, G : \mathbb{R}^{k-1} \to \mathbb{R}$, such that for any solution $(g_t)_{t\in[0,T]}$ of the Ricci flow on a surface, the following inequality holds

$$\partial_t|\nabla^k K_{g_t}|^2_{g_t} + \Delta_{g_t}|\nabla^k K_{g_t}|^2_{g_t} \le F(|K_{g_t}|, |dK_{g_t}|, \dots, |\nabla^{k-1}K_{g_t}|)|\nabla^k K_{g_t}|^2 + G(|K_{g_t}|, |dK_{g_t}|, \dots, |\nabla^{k-1}K_{g_t}|).$$

Proof: Throughout the following proof, we will use the following notation: $H(A_1, \dots, A_n)$ will denote some algebraic combination of the terms $A_1, \dots, A_n$. The important point is not the precise form of this combination, but the fact that H depends smoothly on the terms $A_1, \dots, A_n$. Moreover, all the terms appearing will have the property that if $A_1, \dots, A_n$ are tensors, then

$$|H(A_1, \dots, A_n)| \le F(|A_1|, \dots, |A_n|),$$

for some smooth function $F : \mathbb{R}^n \to \mathbb{R}$.

First note that $\nabla^k = \nabla^{g_t} \dots \nabla^{g_t} d$ depends on t.

We calculate

$$\partial_t|\nabla^k K_{g_t}|^2_{g_t} = (\partial_t g_t)(\nabla^k K_{g_t}, \nabla^k K_{g_t}) + g_t(\left(\partial_t \nabla^k\right) K_{g_t}, \nabla^k K_{g_t}) + g_t(\nabla^k(\partial_t K_{g_t}), \nabla^k K_{g_t})$$

By a similar argument as in lemma 3.17, we find that

$$(\partial_t g_t)(\nabla^k K_{g_t}, \nabla^k K_{g_t}) = 2kK_{g_t} g_t(\nabla^k K_{g_t}, \nabla^k K_{g_t}) = 2kK_{g_t}|\nabla^k K_{g_t}|^2.$$

(Cf. exercises.) To treat the last term, first observe that

$$\begin{aligned}
\nabla^k \Delta_g f &= \nabla^g \dots \nabla^g (d\Delta_g f) \\
&= \nabla^g \dots \nabla^g \left(\nabla^{g*}\nabla^g df + K_g df\right) \\
&= \nabla^k \nabla^{g*}\nabla^g df + \nabla^g \dots \nabla^g (K_g df).
\end{aligned}$$

For the term $\nabla^k \nabla^{g*}\nabla^g df$, observe that

$$\nabla^g \nabla^{g*}\nabla^g df - \nabla^{g*}\nabla^g(\nabla^g df) = H_1(K_g)\nabla^g df + H_2(\nabla^g K_g)df.$$

Repeated use of the Leibniz rule to the term $\nabla^g \dots \nabla^g (K_g df)$, shows that

$$\nabla^g \dots \nabla^g (K_g df) = K_g \nabla^k df + \dots + (\nabla^{k-1} K_g) \otimes df.$$

Thus, we obtain

$$\begin{aligned}
g_t(\nabla^k(\partial_t K_{g_t}), \nabla^k K_{g_t}) &= g_t(\nabla^k(-\Delta_{g_t} K_{g_t} + 2K_{g_t}^2), \nabla^k K_{g_t}) \\
&= -g_t(\nabla^{g*}\nabla^g(\nabla^k df) + \nabla^{g_t} \dots \nabla^{g_t}(K_{g_t} dK_{g_t}) + \nabla^k(2K_{g_t}^2), \nabla^k K_{g_t}) \\
&= -g_t(\nabla^{g_t *}\nabla^{g_t}\nabla^k K_{g_t}, \nabla^k K_{g_t}) \\
&+ H_2(K_{g_t}, dK_{g_t}, \dots, \nabla^{k-1} K_{g_t})|\nabla^k K_{g_t}|^2 \\
&+ g_t(H_3(K_{g_t}, dK_{g_t}, \dots, \nabla^{k-1} K_{g_t}), \nabla^k K_{g_t}).
\end{aligned}$$

For the middle term, we appeal to the lemma 3.20. As a simplification, we assume $k = 2$. Then the middle term becomes

$$g_t((\partial_t \nabla^{g_t}) dK_{g_t}, \nabla^{g_t} dK_{g_t}).$$

We compute

$$\begin{aligned}
\partial_t(\nabla^{g_t} df)(X, Y) &= \partial_t \left(XYf - df(\nabla^{g_t}_X Y) \right) \\
&= -df((X\partial_t u)Y + (Y\partial_t u)X - g_t(X, Y) \operatorname{grad}_{g_t} \partial_t u).
\end{aligned}$$

With this in mind, we calculate

$$\begin{aligned}
g_t((\partial_t \nabla^{g_t}) dK_{g_t}, \nabla^{g_t} dK_{g_t}) &= \sum_{i,j=1}^n \left(\partial_t \nabla^{g_t} dK_{g_t}\right)(e_i, e_j) \left(\nabla^{g_t} dK_{g_t}\right)(e_i, e_j) \\
&= \sum_{i,j=1}^n - \left(dK_{g_t}(e_j) dK_{g_t}(e_i) + dK_{g_t}(e_i) dK_{g_t}(e_j)\right) \left(\nabla^{g_t} dK_{g_t}\right)(e_i, e_j) \\
&= g_t(H_4(dK_{g_t}), \nabla^{g_t} dK_{g_t}).
\end{aligned}$$

Similarly, we find for $k \geq 2$

$$g_t(\left(\partial_t \nabla^k\right) K_{g_t}, \nabla^k K_{g_t}) = g_t(H_5(dK_{g_t}, \dots, \nabla^{k-1} K_{g_t})), \nabla^k K_{g_t}).$$

In conclusion, we have seen that

$$\begin{aligned}
\partial_t |\nabla^k K_{g_t}|^2_{g_t} &= -g_t(\nabla^{g_t *}\nabla^{g_t}\nabla^k K_{g_t}, \nabla^k K_{g_t}) \\
&+ H_6(K_{g_t}, dK_{g_t}, \dots, \nabla^{k-1} K_{g_t})|\nabla^k K_{g_t}|^2 \\
&+ g_t(H_7(K_{g_t}, dK_{g_t}, \dots, \nabla^{k-1} K_{g_t}), \nabla^k K_{g_t}).
\end{aligned}$$

Now we conclude the theorem from

$$\Delta_g |\nabla^k f|^2 = g(\nabla^{g*}\nabla^g \nabla^k f, \nabla^k f) - |\nabla^g \nabla^k f|^2$$

and the inequality

$$g(\mu_1, \mu_2) \leq |\mu_1|_g |\mu_2|_g \leq \frac{1}{2}|\mu_1|_g^2 + \frac{1}{2}|\mu_2|_g^2$$

applied to the last term. □

Theorem 3.22

Let M be a compact surface. Suppose that $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow and that

$$\sup_{(x,t)\in M\times[0,T)} |K_{g_t}(x)| < \infty.$$

Then T is not the maximal time of existence, i.e. the maximal solution of the Ricci flow with the same initial metric g_0 is defined on an interval $[0, T_{\max})$ with

$$T_{\max} > T.$$

Lemma 3.23

Let (M, g) be a compact manifold and $T > 0$. Suppose that $f : M \times [0, T) \to \mathbb{R}$ is a smooth function and suppose that

$$\sup_{t\in[0,T)} \|\partial_t^k \nabla^l f\|_{C^0(M)} < \infty$$

for every $k, l \in \mathbb{N}$. Then f can be extended to a smooth function on $M \times [0, T]$.

Remark 3.24. A function $f : M \times [a, b] \to \mathbb{R}$ is smooth, if there exists a smooth extension of f to $M \times (\tilde{a}, \tilde{b})$ with $\tilde{a} < a < b < \tilde{b}$.

Sketch of the proof: First, one observes that if we choose a coordinate system $\varphi : U \subset M \to V \subset \mathbb{R}^n$, then

$$\sup_{\substack{x\in M\\ t\in[0,T)}} |\partial_t^k \partial_x^\alpha f(\varphi^{-1}(x), t)| < \infty$$

for every $k \in \mathbb{N}$ and every multi-index α. Thus we may as well assume our function is defined on $\mathbb{R}^n$ to begin with.

For simplicity, we treat only the case $f : [0, T) \to \mathbb{R}$. The function and its first derivative are bounded. Hence there exists $C > 0$, such that

$$\frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|} < C$$

for all $t_1, t_2 \in [0, T)$.

Now consider a sequence $t_n \to T$. Then

$$|f(t_m) - f(t_n)| < C|t_m - t_n|.$$

This implies that $f(t_n)$ is a Cauchy sequence and thus the limit

$$\lim_{n\to\infty} f(t_n)$$

exists. The inequality also implies that the limit does not depend on the sequence. Thus we can extend f to a function on $[0, T]$. Applying the inequality once more shows that the extension is continuous.

Arguing the same way, we see that all the derivatives $\partial_t^k f$ also extend continously to $[0, T]$. We can then define a smooth extension $\tilde{f} : [0, T + \epsilon, \to)\mathbb{R}$

$$\begin{cases} f(t), & t \in [0, T] \\ \sum_{k=0}^{\infty} (\partial_t^k f)(T) \psi_k(t)(t-T)^k, & t > T. \end{cases}$$

Here $\psi_k : [0, T+\epsilon) \to \mathbb{R}$ is a bump function, such that $\psi_k(t) \equiv 1$ in a neighborhoos of T and

$$\max_{t \in [0, T+\epsilon)} |(\partial_t^k f)(T)\psi_k(t)(t-T)^k| \leq 2^{-k}.$$

The proof generalizes to the case $f : M \times [0, T) \to \mathbb{R}$ by considering the uniform convergence of the functions $f(\cdot, t)$. $\square$

Proof of the theorem: We have seen that for any smooth metric g on a compact surface, there exists a unique maximal solution of the Ricci flow on M with initial condition g.

Now suppose that $(g_t)_{t \in [0,T)}$ is a solution of the Ricci flow with

$$\sup_{(x,t) \in M \times [0,T)} |K_{g_t}(x)| < \infty.$$

Recall that the Ricci flow is equivalent to the scalar equation

$$\partial_t u_t = -K_{g_t}, \qquad u_0 \equiv 0$$

for the conformal metrics $e^{2u_t} g_0$.

If we manage to show that

$$\sup_{t \in [0,T)} \|\partial_t^k \nabla^l u\|_{C^0(M, g_0)} < \infty \tag{$\star$}$$

for every $k, l \in \mathbb{N}$, then according to the lemma, we can extend u smoothly to $M \times [0, T]$. Then we can consider the Ricci flow with initial condition $g_T = e^{2u_T} g_0$. By our short time existence result, we know that this flow $\hat{g}_t$ exists on an interval $[0, \epsilon)$, $\epsilon > 0$.

We can then define a new solution of the Ricci flow with initial condition g_0 via

$$\tilde{g}_t = \begin{cases} g_t, & t \in [0, T] \\ \hat{g}_{t-T}, & t \in [T, T+\epsilon). \end{cases}$$

This shows that $T_{\max} \geq T + \epsilon$, proving the theorem.

To show that all the derivatives of the conformal factor remain bounded, i.e. that $(\star)$ holds, we first observe that

$$u_t(x) = \int_0^t K_{g_s}(x) ds.$$

Thus it is sufficient to show that all derivatives of K_{g_t} remain bounded.

By proposition 3.18, we know that

$$|dK_{g_t}|^2 \leq \frac{K^2 + 2K^3T + 16K^4T^2}{t}$$

for $t \in [0, T)$. Moreover, because g_t is smooth on $[0, T)$, we know apriori that there is a bound on any subinterval $[0, T_0]$. Hence there exists a constant $C_1 > 0$, such that

$$|dK_{g_t}|^2_{g_t} < C_1.$$

Starting with this estimate, we can show inductively that there exists $C_k > 0$, such that

$$|\nabla^k K_{g_t}|^2_{g_t} < C_k \tag{$\star\star$}$$

on $[0, T)$. Indeed, suppose that $k \in \mathbb{N}$ and inequality $(\star\star)$ holds for all $j \leq k$. Then by theorem 3.21, there exists a smooth function $F : \mathbb{R}^k \to \mathbb{R}$, such that

$$\partial_t |\nabla^{k+1} K_{g_t}|^2_{g_t} + \Delta_{g_t} |\nabla^k K_{g_t}|^2_{g_t} \leq F(|K_{g_t}|, |dK_{g_t}|, \ldots, |\nabla^k K_{g_t}|) |\nabla^{k+1} K_{g_t}|^2.$$

Since inequality $(\star\star)$ holds for all $j \leq k$ and F and G are continuous, it follows that there exists $A, B > 0$, such that

$$F(|K_{g_t}|, |dK_{g_t}|, \ldots, |\nabla^k K_{g_t}|) \leq A$$

and

$$G(|K_{g_t}|, |dK_{g_t}|, \ldots, |\nabla^k K_{g_t}|) \leq B$$

on $M \times [0, T]$.

Thus

$$\partial_t |\nabla^{k+1} K_{g_t}|^2_{g_t} + \Delta_{g_t} |\nabla^k K_{g_t}|^2_{g_t} \leq A |\nabla^{k+1} K_{g_t}|^2 + B.$$

To this equation we can apply the maximum principle using the comparison ODE

$$\phi'(t) = A\phi(t) + B,$$

$$\phi(0) = D := \max |\nabla^{k+1} K_{g_0}|^2_{g_0}.$$

This ODE has the solution

$$\phi(t) = D \exp(At) + Bt$$

and we conclude that

$$|\nabla^{k+1} K_{g_t}|^2_{g_t} \leq B \exp(AT) + BT =: C_{k+1}$$

on $M \times [0, T]$.

This does not yet show inequality $(\star)$, because the metric depends on t. The next lemma shows that $(\star)$ follows from this. □

Lemma 3.25

Suppose M is a compact manifold and suppose that $g_t = e^{2u_t} g$, $t \in (0,T)$ is a smooth family of metrics on M.

If

$$\sup_{x\in M, t\in(0,T)} |u_t(x)| < \infty \text{ and } \sup_{x\in M, t\in(0,T)} |\nabla^{g_t,k}\partial_t u_t(x)| < \infty$$

for every k, then

$$\sup_{x\in M, t\in(0,T)} |\nabla^{g_0,k}\partial_t u_t(x)| < \infty$$

for every k.

For a (partial) proof we refer to the exercises.

Chapter 4

The Ricci flow on surfaces: long time existence

4.1 The normalized Ricci flow on a surface

It turns out to consider convergence and long time existence it is more beneficial to consider a variant of the Ricci flow, which fixes the total area of the surface.

As we have seen in the exercises, if g_t satisfies the Ricci flow equation, then

$$\partial_t \operatorname{vol}_{g_t} = -2K_{g_t} \operatorname{vol}_{g_t} .$$

In particular on a surface, we have

$$\frac{d}{dt} \int_M \operatorname{vol}_{g_t} = -2 \int_M K_{g_t} \operatorname{vol}_{g_t} .$$

According to the following lemma, the integral on the right hand side is actually independent of t.

Lemma 4.1

Let M be a compact surface and g, $\tilde{g}$ be two conformal metrics on M. Then

$$\int_M K_{\tilde{g}} \operatorname{vol}_{\tilde{g}} = \int_M K_g \operatorname{vol}_g .$$

Proof: Suppose $\tilde{g} = e^{2u} g$. Then the curvature $K_{\tilde{g}}$ is given by the formula $e^{-2u}\left(\Delta_g u + K_g\right)$ and the volume form $\operatorname{vol}_{\tilde{g}}$ is given by $e^{2u} g$. Thus

$$\begin{aligned} \int_M K_{\tilde{g}} \operatorname{vol}_{\tilde{g}} &= \int_M e^{-2u}\left(\Delta_g u + K_g\right) e^{2u} \operatorname{vol}_g \\ &= \int_M \Delta_g u \operatorname{vol}_g + \int_M K_g \operatorname{vol}_g . \end{aligned}$$

The integral of $\Delta_g u$ vanishes because of the divergence theorem. □

Denoting the surface area of (M, g_t) by

$$V(t) = \int_M \mathrm{vol}_{g_t},$$

we obtain

$$V(t) = V(0) - Kt,$$

where $K = \int_M K_{g_0} \mathrm{vol}_{g_0}$.

Thus, if $K > 0$, the total area will become zero at time $V(0)/K$. On the other hand, if $K < 0$, the total area will increase as long as the solution exists. In either case we cannot hope for convergence.

The normalized Ricci flow is a way to evolve a Riemannian metric by Ricci flow, but rescaling it so that its volume is fixed.

Suppose g is any Riemannian metric on a surface. Then we define $\bar{K}$ to be the *average* Gaussian curvature, i.e.

$$\bar{K} = \left(\int_M \mathrm{vol}_g\right)^{-1} \int_M K_g \,\mathrm{vol}_g .$$

Definition 4.2

A family of Riemannian g_t metrics is called a solution of the *normalized Ricci flow* with initial condition g, if

$$\partial_t g = 2(\bar{K} - K_{g_t})g_t$$

and $g_0 = g$.

Proposition 4.3

If g_t solves the normalized Ricci flow, then the volume $V(t) = \int_M \mathrm{vol}_{g_t}$ is constant.

Proof: Exercise. □

Together with the previous lemma, this implies that the average scalar curvature is also constant along the normalized Ricci flow.

Note that if $\bar{K} = 0$, the normalized Ricci flow coincides with the standard Ricci flow. In the other cases, the following proposition describes the relationship between the two Ricci flows.

Proposition 4.4

Let g be a Riemannian metric with $\bar{K} \neq 0$. Let $(g_t)_{t\in[0,T_{\max})}$ be the solution of the Ricci flow with initial condition g. Then

$$\tilde{g}_t = e^{2\bar{K}t} g_{\phi(t)}$$

with $\phi(t) = \left(1 - e^{-2\bar{K}t}\right)/(2\bar{K})$ is a solution of the normalized Ricci flow on the interval $[0, \tilde{T}_{\max})$, where $\tilde{T}_{\max} = -\log(1 - 2\bar{K}T_{\max})/(2\bar{K})$.

Conversely, if $(g_t)_{t\in[0,T_{\max})}$ is the solution of the normalized Ricci flow with initial condition g, then

$$\hat{g}_t = (1-2\bar{K}t)g_{\psi(t)}$$

with $\psi(t) = -\log(1-2\bar{K}t)/(2\bar{K})$ is a solution of the Ricci flow on the interval $[0, \hat{T}_{\max})$ where $\hat{T}_{\max} = (1-\exp(-2\bar{K}T_{\max}))/(2\bar{K})$.

Proof: Suppose $(g_t)_{t\in[0,T_{\max})}$ is a solution of the Ricci flow.

Let us assume $\tilde{g}_t = f(t)g_{\phi(t)}$ for some functions f, ϕ. Then we compute

$$\begin{aligned}
\partial_t \tilde{g}_t &= f'(t)g_{\phi(t)} + f(t)(\partial_t g)_{\phi(t)}\phi'(t) \\
&= \frac{f'(t)}{f(t)}\tilde{g}_t + f(t)\phi'(t)\left(-2K_{g_{\phi(t)}}g_{\phi(t)}\right) \\
&= \frac{f'(t)}{f(t)}\tilde{g}_t - 2f(t)\phi'(t)K_{\tilde{g}_t}\tilde{g}_t \\
&= \left(\frac{f'(t)}{f(t)} - 2f(t)\phi'(t)K_{\tilde{g}_t}\right)\tilde{g}_t,
\end{aligned}$$

where we used that $\mathrm{Ric}_{\lambda^2 g} = \mathrm{Ric}_g$ and $\mathrm{Ric}_g = K_g g$. We conclude that if $f'(t)/f(t) = 2\bar{K}$ and $\phi'(t)f(t) = 1$, the family of metrics $\tilde{g}_t$ is a solution of the normalized Ricci flow.

Since $g_0 = \tilde{g}_0$, we moreover have $f(0) = 1$ and $\phi(0) = 0$.

Solving the ODEs yields

$$f(t) = e^{2\bar{K}t} \quad \text{and} \quad \phi(t) = \left(1 - e^{-2\bar{K}t}\right)/(2\bar{K}).$$

Note that independent of the sign $\phi(t)$ is an increasing function and $\tilde{g}_t$ is well-defined, whenever $\phi(t) < T_{\max}$.

This is the case when

$$t < -\log(1-2\bar{K}T_{\max})/(2\bar{K}).$$

The converse statement follows similarly and is left as an exercise. □

We can restate the previous main results for the normalized Ricci flow.

Theorem 4.5

Let M be a compact surface and suppose that g is a smooth metric. Then there exists a unique maximal solution $(g_t)_{t\in[0,T_{\max})}$ of the normalized Ricci flow with initial condition $g_0 = g$.

Proof: Exercise. □

Theorem 4.6

Suppose M is a compact surface and suppose that $(g_t)_{t\in[0,T)}$ is a solution of the normalized Ricci flow. If

$$\sup_{M\times[0,T)} |K_{g_t}(x)| < \infty,$$

then T is not the maximal time of existence.

Proof: The solution $(g_t)_t$ corresponds to the solution

$$\hat{g}_t = (1 - 2\bar{K}t)g_{\psi(t)}$$

with $\psi(t) = -\log(1 - 2\bar{K}t)/(2\bar{K})$ on the interval $[0, \hat{T})$ where

$$\hat{T} = (1 - \exp(-2\bar{K}T))/(2\bar{K}).$$

Note that the factor $(1 - 2\bar{K}t)$ is bounded away from zero on the interval $[0, \hat{T})$. Thus the curvature of $\hat{g}_t$ is also bounded, i.e.

$$\sup_{x\in M, 0\leq t<\hat{T}} |K_{\hat{g}_t}(x)| < \infty.$$

Hence the Ricci flow $\hat{g}_t$ extends beyond the time $\hat{T}$. Translating this solution back to a solution of the normalized Ricci flow, we see that the normalized Ricci flow also extends beyond time T. □

The goal of this chapter is to prove the following theorem.

Theorem 4.7

Let M be compact, oriented surface and suppose g is a Riemannian metric on M. Then the normalized Ricci flow with initial condition g exists for all time, i.e. $T_{\max} = \infty$.

4.2 A lower bound on the curvature

As a very first step towards these theorems, let us consider the evolution of the scalar curvature R_g under the Ricci flow.

Proposition 4.8

Suppose g_t is a solution of the normalized Ricci flow. Then

$$\partial_t K_{g_t} + \Delta_{g_t} K_{g_t} = 2K_{g_t}(K_{g_t} - \bar{K}).$$

Corollary 4.9

Suppose $(g_t)_{t\in[0,T)}$ is a solution of the normalized Ricci flow on a compact surface. Denote by $K_{\min}$ the minimum of the Gauß curvature of g_0. Then depending on the sign of the average scalar curvature, we get the following lower bounds on the Gauß curvatures:

$\bar{K} < 0:$

$$K_{g_t} - \bar{K} \geq \frac{\bar{K}}{1 - \left(1 - \bar{K}/K_{\min}\right) e^{2\bar{K}t}} - \bar{K} \geq (K_{\min} - \bar{K}) e^{2\bar{K}t}$$

$\bar{K} = 0:$

$$K_{g_t} \geq \frac{K_{\min}}{1 - K_{\min} t} > -\frac{1}{t}$$

$\bar{K} > 0, K_{\min} < 0:$

$$K_{g_t} \geq \frac{\bar{K}}{1 - \left(1 - \bar{K}/K_{\min}\right) e^{2\bar{K}t}} \geq K_{\min} e^{-\bar{K}t}$$

Proof: Applying the maximum principle, we find that $K_{g_t} \geq \phi(t)$, where $\phi(t)$ solves

$$\begin{cases} \phi(0) = K_{\min} = \min_{x\in M} K_{g_0}(x), \\ \phi'(t) = 2\phi(t)(\phi(t) - \bar{K}). \end{cases}$$

If $\bar{K} \neq 0$, we find that

$$\phi(t) = \frac{\bar{K}}{1 - \left(1 - \bar{K}/\phi(0)\right) \exp(2\bar{K}t)}.$$

If $\bar{K} = 0$, we find that

$$\phi(t) = \frac{\phi(0)}{1 - \phi(0)t}.$$

In either case, the statement follows. □

In the cases $\bar{K} < 0$ and $\bar{K} = 0$, these estimates say that the deviation from the average $\bar{K}$ from below becomes exponentially small. On the other hand, until now we know nothing about the deviation from above.

4.3 An upper bound on the curvature

Definition 4.10

A *Ricci soliton* is a Riemannian manifold (M, g) and a vector field X, such that

$$\mathrm{Ric}_g = \lambda g - \frac{1}{2}\mathcal{L}_X g$$

for some constant $\lambda \in \mathbb{R}$.

Ricci solitons with $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ are called *shrinking, steady and expanding* respectively.

A *gradient Ricci soliton* is a Ricci soliton, such that the vector field X is a gradient vector field, i.e. $X = \operatorname{grad}_g f$ for some $f \in \mathcal{C}^\infty(M)$.

Proposition 4.11

Suppose (M, g, X) is a Ricci soliton and let $\phi_t : M \to M$ be the family of diffeomorphisms satisfying

$$\frac{d\phi_t(x)}{dt} = \frac{1}{1 - 2\lambda t} X(\phi_t(x)).$$

Then

$$g(t) = (1 - 2\lambda t)\phi_t^* g$$

is a solution of the Ricci flow. The interval of existence is $(-1/(2\lambda), \infty)$ if the soliton is expanding, $\mathbb{R}$ if the soliton is steady and $(-\infty, 1/(2\lambda))$ if the soliton is shrinking.

The solution evolves by a change by diffeomorphism and rescaling.

Assume that (M, g, X) is a Ricci soliton and assume that M is a closed surface. Then let us take the trace of the Ricci soliton equation:

$$2K_g = 2\lambda - \frac{1}{2} \operatorname{div}_g X.$$

Rearranging this equation and integrating over M, we obtain

$$\lambda \int_M \operatorname{vol}_g = \int_M K_g \operatorname{vol}_g - \frac{1}{2} \int_M \operatorname{div}_g X \operatorname{vol}_g.$$

By the divergence theorem, the last term is zero. Thus we obtain $\lambda = \bar{K}$ and so the Ricci soliton equation for surfaces reads

$$(K_g - \bar{K})g = -\frac{1}{2}\mathcal{L}_X g.$$

Note that

$$\mathcal{L}_{\operatorname{grad}_g f} g = 2 \operatorname{Hess}_g(f).$$

So for a gradient Ricci soliton, the equation is

$$(K_g - \bar{K})g = -\operatorname{Hess}_g(f).$$

Taking the trace of this equation yields

$$\Delta_g f = 2(K_g - \bar{K}).$$

Thus we have

$$\operatorname{Hess}_g(f) + \frac{1}{2}(\Delta_g f)g = 0.$$

It turns out that we obtain an interesting equation if we take the divergence ∇^{g*} of this equation. The divergence on a tensor field T is defined to be

$$\nabla^{g*}T = -\sum_{i=1}^{n}(\nabla^g T)(e_i, e_i, \cdot, \dots, \cdot).$$

In particular, we have

$$\begin{aligned}
\nabla^{g*}(fg) &= -\sum_{i=1}^{n}(\nabla^g fg)(e_i, e_i, \cdot, \dots, \cdot) \\
&= -\sum_{i=1}^{n}\left((e_i f)g(e_i, \cdot) + f(\nabla^g g)(e_i, e_i)\right) \\
&= -\sum_{i=1}^{n} g((e_i f)e_i, \cdot) \\
&= -g(\operatorname{grad}_g f, \cdot) \\
&= -df.
\end{aligned}$$

Together with the Bochner formula this implies

$$\begin{aligned}
\nabla^{g*}\left(\operatorname{Hess}_g(f) + \frac{1}{2}(\Delta_g f)g\right) &= \nabla^{g*}\nabla^g df - \frac{1}{2}d\Delta_g f \\
&= \frac{1}{2}d\Delta_g f - K_g df \\
&= dK_g - K_g df = 0.
\end{aligned}$$

Since $\Delta_g f = 2(K_g - \bar{K})$, we obtain

$$dK_g - \frac{1}{2}(\Delta_g f)df + \bar{K}df = 0.$$

Since $\operatorname{Hess}_g(f) + \frac{1}{2}(\Delta_g f)g$, it follows that

$$\begin{aligned}
V|df|^2 &= 2g(\nabla^g_V df, df) \\
&= 2g(\operatorname{Hess}_g(f)(V, \cdot, df) \\
&= -g((\Delta_g f)g(V, \cdot), df) \\
&= -(\Delta_g f)g(V^\flat, df) \\
&= -(\Delta_g f)Vf
\end{aligned}$$

or

$$d|df|^2 = -(\Delta_g f)df.$$

And so

$$dK_g + d\frac{1}{2}|df|^2 + \bar{K}df = d(K_g + \frac{1}{2}|df|^2 + \bar{K}f) = 0.$$

The following proposition summarizes this calculation.

Proposition 4.12

Suppose M is a connected, compact surface and (M, g, f) is gradient soliton. Then

$$2K_g + |df|^2 + 2\bar{K} f$$

is constant on M.

Our goal remains to understand *general* solutions of the Ricci flow. It turns out that even for Riemannian metrics, which are not gradient solitons, there is a useful analogue. The foundation for our reasoning is the following theorem from analysis.

Theorem 4.13

Suppose (M, g) is a connected, compact Riemannian manifold. Let $h \in \mathcal{C}^\infty(M)$. If

$$\int_M h \operatorname{vol}_g = 0,$$

then there exists a function $f \in \mathcal{C}^\infty(M)$, such that

$$\Delta_g f = h.$$

Up to an additive constant, the function f is uniquely determined by h.

From this theorem we infer that for any connected compact Riemannian manifold (M, g), we may define a function f, such that

$$\Delta_g f = 2(K_g - \bar{K}),$$

since the integral of the right hand side vanishes. Now supposing that $(g_t)_{t \in [0,T)}$ is a solution of the normalized Ricci flow, we also find a family $\hat{f}_t$, such that

$$\Delta_{g_t} \hat{f}_t = 2(K_{\bar{g}_t} - \bar{K}).$$

We expect that

$$F_t = 2K_{g_t} + 2\bar{K} \hat{f}_t + |d\hat{f}_t|^2$$

satisfies a nice evolution equation, because the function is constant on solitons.

Note that we have the choice to add a constant to $\hat{f}_t$ at any time t. The evolution equation depends on that choice. If we carefully choose such a constant, then we have a nice evolution equation for this function. This is the content of the following lemma.

Lemma 4.14

Let M, $(g_t)_{t \in [0,T)}$ and $\hat{f}_t$ as above. Then there exists a function $c : [0, T) \to \mathbb{R}$, such that $f_t = \hat{f}_t + c(t)$ satisfies

$$\partial_t f_t + \Delta_{g_t} f_t = 2\bar{K} f_t.$$

Proof: We can compute the time derivative of $\Delta_{g_t}\hat{f}_t$ in two ways. First, we compute

$$\partial_t\Delta_{g_t}\hat{f}_t = 2(K_{g_t} - \bar{K})\Delta_{g_t}\hat{f}_t + \Delta_{g_t}\partial_t\hat{f}_t,$$

where we used

$$\partial_t\Delta_{g_t}h = 2(K_{g_t} - \bar{K})\Delta_{g_t}h$$

from the exercises.

On the other hand, $\Delta_{g_t}f_t = 2(K_{g_t} - \bar{K})$ and we thus calculate

$$\partial_t\Delta_{g_t}\hat{f}_t = -2\Delta_{g_t}(K_{g_t} - \bar{K}) + 4K_{g_t}(K_{g_t} - \bar{K}) = -\Delta_{g_t}\Delta_{g_t}\hat{f}_t + 2K_{g_t}\Delta_{g_t}\hat{f}_t$$

from the evolution equation for the Gauß curvature along the normalized Ricci flow.

Rearranging these two equations gives us

$$\Delta_{g_t}\left(\partial_t\Delta_{g_t}\hat{f}_t + \Delta_{g_t}\hat{f}_t\right) = (2K_{g_t} - 2K_{g_t} + 2\bar{K})\Delta_{g_t}f$$

or

$$\Delta_{g_t}\left(\partial_t\Delta_{g_t}\hat{f}_t + \Delta_{g_t}\hat{f}_t - \bar{K}\hat{f}_t\right) = 0.$$

Since M is assumed connected, only constant functions are harmonic and this implies

$$\partial_t\Delta_{g_t}\hat{f}_t + \Delta_{g_t}\hat{f}_t - 2\bar{K}\hat{f}_t = \alpha(t)$$

for some function $\alpha : [0, T) \to \mathbb{R}$.

If we choose $c(t) = -e^{2\bar{K}t}\int_0^t e^{-\bar{K}s}\alpha(s)ds$, then it follows that

$$\partial_t\Delta_{g_t}f_t + \Delta_{g_t}f_t = 2\bar{K}f_t,$$

because

$$\partial_t c(t) = -2\bar{K}c(t) - \alpha(t).$$

□

It is now natural to investigate the evolution of $F_t = 2K_{g_t} + 2\bar{K}f_t + |df_t|^2$. Since we already know how f_t evolves, we may as well just study $2K_{g_t} + |df_t|^2$. This will lead us to our curvature estimate.

Theorem 4.15

Let M be a compact surface and suppose $(g_t)_{t\in[0,T)}$ is a solution of the normalized Ricci flow. Consider the function $H_t = 2(K_{g_t} - \bar{K}) + |df_t|^2$, where f_t is as in the lemma. Then

$$\partial_t H_t + \Delta_{g_t}H_t \leq 2\bar{K}H_t.$$

Proof: First, we calculate

$$\begin{aligned}\partial_t|df_t|^2 &= 2(K_{g_t}-\bar K)|df_t|^2 + 2g_t(\partial_t df_t, df_t)\\ &= 2(K_{g_t}-\bar K)|df_t|^2 + 2g_t(-d\Delta_{g_t}f_t, df_t) + 4\bar K|df_t|^2,\end{aligned}$$

or equivalently

$$\partial_t|df_t|^2 + 2g_t(d\Delta_{g_t}f_t, df_t) = 2(K_{g_t}+\bar K)|df_t|^2.$$

Recall that

$$g(d\Delta_{g_t}f_t, df_t) = \frac{1}{2}\Delta_{g_t}|df_t|^2 + K_{g_t}|df_t|^2 + |\nabla^{g_t}df_t|^2.$$

Thus

$$\partial_t|df_t|^2 + \Delta_{g_t}|df_t|^2 = 2\bar K|df_t|^2 - 2|\nabla^{g_t}df_t|^2.$$

On the other hand,

$$\begin{aligned}\partial_t(K_{g_t}-\bar K) + \Delta_{g_t}(K_{g_t}-\bar K) &= 2K_{g_t}(K_{g_t}-\bar K)\\ &= 2(K_{g_t}-\bar K)^2 + 2\bar K(K_{g_t}-\bar K)\\ &= \frac{1}{2}(\Delta_{g_t}f_t)^2 + 2\bar K(K_{g_t}-\bar K).\end{aligned}$$

Now by an exercise

$$(\Delta_g f)^2 - n|\operatorname{Hess}_g f|^2 \le 0$$

for any $f \in \mathcal{C}^\infty(M)$, where n is the dimension of the manifold.

Thus

$$\begin{aligned}\partial_t H_t + \Delta_{g_t}H_t &= (\Delta_{g_t}f_t)^2 + 4\bar K(K_{g_t}-\bar K) + 2\bar K|df_t|^2 - 2|\nabla^{g_t}df_t|^2\\ &\le 2\bar K(2(K_{g_t}-\bar K) + |df_t|^2)\\ &= 2\bar K H_t\end{aligned}$$

as claimed. □

Corollary 4.16

Let M be a compact surface and let $(g_t)_{t\in[0,T)}$ be a solution of the normalized Ricci flow and let H_t be as in the theorem. Then

$$2(K_{g_t}-\bar K) \le H_t \le Ce^{2\bar K t}.$$

Proof: This follows immediately from the maximum principle. □

Theorem 4.17

Let M be a compact surface and let g be a Riemannian metric on M. Then the normalized Ricci flow with initial condition g exists for all time.

Proof: Let $(g_t)_{t\in[0,T)}$ be a solution with initial condition g and $T < \infty$. By the previous theorem, we know that

$$K_{g_t} \leq \bar{K} + Ce^{2\bar{K}t}.$$

Since T is finite, this implies

$$\sup_{x\in M, t\in[0,T)} K_{g_t} < \infty.$$

On the other hand, we have seen that in each of the three cases $\bar{K} < 0, \bar{K} = 0$ and $\bar{K} > 0$, there exists a lower bound, so that

$$\inf_{x\in M, t\in[0,T)} K_{g_t} > -\infty.$$

Thus

$$\sup_{x\in M, t\in[0,T)} |K_{g_t}| < \infty$$

and this implies that T can not be the maximal time. Since this holds for any $T \in \mathbb{R}_+$, we conclude that the maximal time of existence must be infinite, proving the theorem. □

Chapter 5

Convergence of the normalized Ricci flow on a surface

5.1 The cases $\bar{K} < 0$ and $\bar{K} = 0$

Suppose M is a compact surface and g is a Riemannian metric on M with

$$\bar{K} = \left(\int_M \mathrm{vol}_g\right)^{-1} \int_M K_g \, \mathrm{vol}_g \leq 0.$$

In the last chapter, we established that the normalized Ricci flow with initial condition g exists for all times. Let $(g_t)_{t\in[0,\infty)}$ be this solution.

Our aim is to prove that the Ricci flow converges to a metric of constant curvature as $t \to \infty$.

Recall that $g_t = e^{2u_t} g$ for some function $u : M \times [0, \infty) \to \mathbb{R}$ with $u_0 \equiv 0$. Rephrasing convergence of the Ricci flow in terms of u_t, we say that the Ricci flow converges to a metric of constant curvature, if u_t converges to a function u_∞ as $t \to \infty$ and $e^{2u_\infty} g$ is a metric of constant curvature.

We are going to see that u_t converges to a smooth function u_∞ in every C^k norm.

For the rest of the section we assume that

$$\bar{K} = \left(\int_M \mathrm{vol}_g\right)^{-1} \int_M K_g \, \mathrm{vol}_g \leq 0.$$

We treat the case $\bar{K} < 0$ first.

We established the following curvature estimate for the normalized Ricci flow.

Proposition 5.1

Let $(g_t)_{t\in[0,\infty)}$ be the solution of the normalized Ricci flow with initial condition g. If $\bar{K} < 0$, then there exists a constant $C > 0$ depending only on the initial metric g, such that

$$|K_{g_t} - \bar{K}| \leq C e^{2\bar{K}t}.$$

Thus, the curvature K_{g_t} converges uniformly to $\bar{K}$ with exponential rate as $t \to \infty$. Since

$$u_t(x) = u_0(x) - \int_0^t (K_{g_t}(x) - \bar{K})ds$$

it also follows that u_t converges uniformly to some function u_∞ as $t \to \infty$.

Since we want to show *smooth convergence*, we also need to show that the higher derivatives of u_t converge, or equivalently, that the higher derivatives of K_{g_t} decay sufficiently fast.

For this, we first adapt proposition 3.19 to the case of the volume normalized Ricci flow.

PROPOSITION 5.2

Let $(g_t)_{t\in[0,T)}$ be a solution of the volume Ricci flow on a surface. Then

$$\partial_t |dK_{g_t}|^2 + \Delta_{g_t} |dK_{g_t}|^2 = -2|\nabla^{g_t} dK_{g_t}|^2 + (8K_{g_t} - 2\bar{K})|dK_{g_t}|^2.$$

Proof: The proof is the same as for proposition 3.19, except that now

$$(\partial_t g_t)(dK_{g_t}, dK_{g_t}) = 2(K_{g_t} - \bar{K})|dK_{g_t}|^2.$$

□

COROLLARY 5.3

Suppose M is a compact surface and $(g_t)_{t\in[0,\infty)}$ is a solition of the normalized Ricci flow with $\bar{K} < 0$.

There exists a constant $C > 0$, such that

$$|dK_{g_t}|^2 \leq Ce^{2\bar{K}t}.$$

Proof: We have the estimate

$$|K_{g_t} - \bar{K}| \leq Ce^{2\bar{K}t}$$

for some $C > 0$. The previous proposition now yields

$$\begin{aligned} \partial_t |dK_{g_t}|^2 + \Delta_{g_t} |dK_{g_t}|^2 &\leq (8K_{g_t} - 2\bar{K})|dK_{g_t}|^2 \\ &= [8(K_{g_t} - \bar{K}) + 6\bar{K}]|dK_{g_t}|^2 \\ &\leq \left(Ce^{2\bar{K}t} + 6\bar{K}\right)|dK_{g_t}|^2. \end{aligned}$$

For sufficiently large t we have

$$Ce^{2\bar{K}t} + 6\bar{K} < 2\bar{K}$$

and so

$$\partial_t |dK_{g_t}|^2 + \Delta_{g_t} |dK_{g_t}|^2 \leq 2\bar{K}|dK_{g_t}^2.$$

Now the result follows immediately from the maximum principle. □

Higher derivative estimates can be obtained by refining the estimates in theorem **??**. Doing this, one obtains the following result.

Theorem 5.4

Suppose M is a compact surface and $(g_t)_{t\in[0,\infty)}$ is a solition of the normalized Ricci flow with $\bar{K} < 0$.

There exist constants $C_k > 0$, such that

$$|\nabla^k K_{g_t}|^2 \leq C_k e^{2\bar{K}t}.$$

And we conclude the following theorem.

Theorem 5.5 [Uniformization for $\bar{K} < 0$]

Suppose (M, g) is a compact Riemannian surface with $\bar{K} < 0$. Then the normalized Ricci flow $(g_t)_{t\in[0,\infty)}$ with initial condition g converges exponentially fast to a metric of constant negative curvature.

More precisely, let u_t be such that that $g_t = e^{2u_t}g$. Then there exists a uniform limit $u_\infty = \lim_{t\to\infty} u_t$ and

$$\|u_t - u_\infty\|_{C^k(M)} \leq C_k e^{2\bar{K}t}.$$

Next we consider the case $\bar{K} = 0$. In this case, refining the analysis in theorem **??** yields the following theorem.

Theorem 5.6

Suppose (M, g) is a compact Riemannian surface with $\bar{K} = 0$. Let $(g_t)_{t\in[0,\infty)}$ be the solution of the Ricci flow with initial condition g. Then there exists a $C > 0$, such that

$$|K_{g_t}| < \frac{C}{1+t}.$$

From this, one can obtain along the lines of **??** gives estimates for the higher order derivatives of K_{g_t}.

Proposition 5.7

Suppose (M, g) is a compact Riemannian surface with $\bar{K} = 0$. Let $(g_t)_{t\in[0,\infty)}$ be the solution of the Ricci flow with initial condition g. Then there exists a $C = C(k) > 0$, such that

$$|\nabla^k K_{g_t}|^2 < \frac{C}{(1+t)^{k+2}}.$$

Arguing as in the $\bar{K} < 0$ case, these estimates imply the uniformization theorem for $\bar{K} = 0$.

Theorem 5.8 [Uniformization for $\bar{K} = 0$]

Suppose (M, g) is a compact Riemannian surface with $\bar{K} = 0$. Then the Ricci flow $(g_t)_{t\in[0,\infty)}$ with initial condition g converges polynomially fast to a flat metric.

More precisely, let u_t be such that that $g_t = e^{2u_t} g$. Then there exists a uniform limit $u_\infty = \lim_{t\to\infty} u_t$ and

$$\|u_t - u_\infty\|_{C^k(M)} \leq \frac{C_k}{(1+t)^{k+2}}.$$

5.2 The case $\bar{K} > 0$

This case is significantly more difficult than the other two cases. Clearly, the upper bound on the curvature

$$K_{g_t} < Ce^{2\bar{K}t}$$

is of no use to establish convergence. We will see that in addition to analyzing the curvature we will need to study the behavior of certain global invariants of the metric evolving under the Ricci flow.

We will discuss two different proofs. First, we will sketch an argument given by Hamilton. This argument is a particular example of a very powerful strategy in geometric flows and relies on many results, which we are not able to discuss in detail. The second proof, due to Andrews and Bryan, relies on a similar idea. However, their proof gives finer control of the global invariant of the metric. This can in turn be used to control the curvature in a way that is sufficient to establish exponential convergence, similarly as in the case $\bar{K} < 0$.

Hamilton's approach

Compactness for Ricci flows and blow up limits

The extension theorem for Ricci flows has a straightforward generalization to arbitrary dimension.

Theorem 5.9

Suppose M is a compact manifold, $T \in \mathbb{R}_+$ and $(g_t)_{t\in[0,T)}$ is a solution of the Ricci flow on M.

If

$$\sup_{x\in M, 0\leq t<T} |\operatorname{Rm}^{g_t}(x)| < \infty,$$

then T is not the maximal time of existence.

Now suppose that M is a compact manifold and $(g_t)_{t\in[0,T_{\max})}$ is a maximal solution of the Ricci flow. If $T_{\max} < \infty$, it follows that

$$\sup_{x\in M, 0\leq t<T_{\max}} |\operatorname{Rm}^{g_t}(x)| = \infty.$$

Pick an increasing sequence $t_n \to T_{\max}$ and points x_n such that

$$|\operatorname{Rm}^{g_{t_n}}(x_n)| = \sup_{x\in M, t\in[0,t_n]} |\operatorname{Rm}^{g_t}(x)|.$$

Evidently, $|\operatorname{Rm}^{g_{t_n}}(x_n)|$ diverges as $n \to \infty$.

In the exercises, we saw that if $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow and $\lambda \in \mathbb{R}_+$, then

$$\tilde{g}_t = \lambda^2 g_{t/\lambda}$$

is also a solution of the Ricci flow on the interval $[0, \lambda T]$.

We then choose $\lambda_n = |\operatorname{Rm}^{g_{t_n}}(x_n)|$ and define for every n a new Ricci flow

$$g_t^n = \lambda_n^2 g_{\lambda_n(t+t_n)}$$

for $t \in [-t_n\lambda_n, (T - t_n)\lambda_n,)$.

Definition 5.10

Given a compact manifold M and $(g_t)_{t\in[0,T_{\max})}$, we will call the sequence of Ricci flows $(g_t^n)_t$ a *blow up sequence*.

Notice that while we can always construct a blow up sequence, it is in no way unique!

Pictorially speaking, we are "zooming in" to the singularity using a microscope. This can be helpful in understanding the behavior of the Ricci flow. Ideally, we would like to take a limit of the solutions g^n. We then expect that these limits are special solutions of the Ricci flow, for example gradient solitons. If we can also classify such solitons, we have a rather complete understanding of the Ricci flow.

We will see how this strategy can be applied for S^2.

As a first step, we will mention a result due to Hamilton, which gives a sufficient condition for when a sequence of Ricci flow solutions has a convergent subsequence.

To state this theorem, we first need to define convergence of Ricci flows.

Definition 5.11

Let $-\infty \leq T_- \leq 0 \leq T_+ \leq \infty$ and $T_- < T_+$.

Let $(M_n, (g_t^n)_{t\in(T_-,T_+)})$ be a sequence of solutions of the Ricci flow and let $x_n \in M_n$. Suppose that all of the metrics g_t^n are complete.

We say that $(M_n(g_t^n)_t, x_n)$ *converges* to $(M, (g_t)_{t\in(T_-,T_+)}, x)$, if there exists

1. a sequence of open, precompact sets $\Omega_n \subset M$ exhausting M with $x \in \Omega_n$,

2. smooth diffeomorphisms $\varphi_n : \Omega_n \to \varphi(\Omega_n) \subset M_n$ with $\varphi_n(x_n) = x$,

such that

$$\partial_t^l \nabla^k [\varphi_n^* g_t^n - g_t]$$

converges uniformly to 0 for all $k, l \in \mathbb{N}_0$ on every compact subset of $M \times (T, T_+)$.

Remark 5.12. This definition should be compared to the definition of pointed Cheeger–Gromov convergence:

A sequence (M_n, g_n, x_n) of complete Riemannian manifolds converges to (M, g, x) if there exists

1. a sequence of open, precompact sets $\Omega_n \subset M$ exhausting M with $x \in \Omega_n$,
2. smooth diffeomorphisms $\varphi_n : \Omega_n \to \varphi(\Omega_n) \subset M_n$ with $\varphi_n(x_n) = x$,

such that

$$\nabla^k [\varphi_n^* g^n - g]$$

converges uniformly to 0 for all $k \in \mathbb{N}_0$ on every compact subset of M.

Hamilton proved the following theorem.

Theorem 5.13

Let $-\infty \leq T_- \leq 0 \leq T_+ \leq \infty$ and $T_- < T_+$.

Let $(M_n, (g_t^n)_{t \in (T_-, T_+)})$ be a sequence of solutions of the Ricci flow and let $x_n \in M_n$. Suppose every g_t^n is complete and suppose that

$$\sup_n \sup_{x \in M_n, t \in (T_-, T_+)} |\operatorname{Rm}^{g_t^n}(x)| < \infty,$$

$$\inf_n \operatorname{inj}(M_n, g_0^n, x_n) > 0.$$

Then there exists a manifold M, a solution of the Ricci flow $(g_t)_{t \in (T_-, T_+)}$ and $x \in M$, and a subsequence, still denoted by $(M_n, (g_t^n)_t, x_n)$, such that

$$(M_n, (g_t^n)_t, x_n) \to (M, g, x)$$

as $n \to \infty$.

Corollary 5.14

Let M be a compact Riemannian manifold and suppose that $(g_t)_{t \in [0, T_{\max})}$ is a maximal solution of the Ricci flow with $T_{\max} < \infty$. Suppose $(g_t^n)_{t \in (T_-^n, T_+^n)}$ is a blow up sequence for $(g_t)_t$. Then if

$$\inf_n \operatorname{inj}(M_n, g_0^n, x_n) > 0,$$

then there exists a subsequence converging to a Ricci flow $(M_\infty, (g_t)_{t \in (-\infty, T)}, x)$, where $T > 0$.

Remark 5.15.

1. Notice that while M is compact, M_∞ need not be compact.
2. We have the inequality

$$T \leq \sup_{t\in[0,T_{\max})} \| \operatorname{Rm}_t^g \|_{L^\infty(M)} (T_{\max} - t).$$

   Moreover, if the right hand side is infinite, then $T = \infty$.

   If $T < \infty$, the singularity is called a type I singularity.

   If $T = \infty$, the singularity is called a type II singularity.
3. If $T < \infty$, the solution is called *ancient*.
4. If $T = \infty$, then the solution is called *eternal*.

A key difficulty in using this theorem is to establish a lower bound for the injectivity radius. This is difficult, because the injectivity radius is a global invariant. In contrast, the maximum principle only gives information about local invariants. Indeed, one of the key steps in Hamilton's approach is to establish an estimate on the injectivity radius and the related isoperimetric constant.

Let g be any metric on S^2. Suppose that $\gamma : S^1 \to S^2$ is smooth, simple curve. By Jordan's theorem, the complement of the curve γ in S^2 consists of two connected components Ω_1 and Ω_2. We then define the isoperimetric ratio (of the curve γ) by

$$C_S(\gamma) = L_g(\gamma)^2 \left(\frac{1}{\operatorname{Vol}_g(\Omega_1)} + \frac{1}{\operatorname{Vol}_g(\Omega)} \right).$$

(This is equivalent, up to a multiplicative constant, to the more standard $C_S(\gamma) = \frac{L_g(\gamma)^2}{\min\{\operatorname{Vol}_g(\Omega_1), \operatorname{Vol}_g(\Omega_2)\}}$.)

Definition 5.16

The isoperimetric constant of (S^2, g) is

$$C_S(g) = \inf\{C_S(\gamma) : \gamma : S^1 \to S^2 \text{ smooth, simple curve}\}.$$

Note that $C_S(g)$ is scale invariant, i.e.

$$C_S(\lambda^2 g) = C_S(g)$$

for any $\lambda > 0$.

Theorem 5.17 [Hamilton]

Let $(g_t)_t$ be a solution of the Ricci flow on S^2. Then $C_S(g_t)$ is an increasing function.

Proposition 5.18

Let g be a Riemannian metric on S^2. Then

$$\operatorname{inj}(S^2, g) \geq \frac{\pi}{\sqrt{K_{\max}}} \min\left\{1, \sqrt{\frac{C_S(g)}{4\pi}}\right\},$$

where

$$K_{\max} = \max_{x \in S^2} K_g(x).$$

Suppose g is a Riemannian metric on S^2 and consider the associated Ricci flow $(g_t)_{t\in[0,T_{\max})}$. By Hamilton's result $C_S(g_t) \geq C_S(g)$ for all $t \in [0, T_{\max})$.

Let us now consider a blow up sequence of $(g_t)_t$. Then $C_S(g_t^n) \geq C_S(g)$, because C_S is a scale invariant quantity. Moreover, at time 0, we have

$$\max_{x\in M} |K_{g_0^n}(x)| = 1$$

by construction.

In particular, there is a constant $\epsilon > 0$ independent of n, such that

$$\operatorname{inj}(S^2, g_0^n) > \epsilon.$$

Thus we can extract a blow up limit $(M_\infty, (\hat{g}_t)_{t\in(-\infty,T)}, x)$. We now consider the possibilities $T < \infty$ and $T = \infty$ separately.

If $T = \infty$, then

$$\sup_{t\in[0,T_{\max})} K_{\max}(g_t)(T_{\max} - t) = \infty.$$

Note that

$$\operatorname{Vol}(S^2, g_t) = \operatorname{Vol}(S^2, g_0) - 8\pi t = 8\pi(T_{\max} - t).$$

This implies

$$\sup_{t\in[0,T_{\max})} K_{\max}(g_t) \operatorname{Vol}(S^2, g_t) = \infty.$$

In particular, the blow up limit has $\operatorname{Vol}(M_\infty, \hat{g}_0) = \infty$ and is thus non-compact.

Moreover, since there is a lower bound $K_{g_t} \geq \kappa$ for some $\kappa \in \mathbb{R}$, the blow up sequence has a lower bound

$$K_{g_t^n} = \frac{\kappa}{\lambda_n},$$

where $\lambda_n \to \infty$. Hence the blow up limit has $K_{\hat{g}_t} \geq 0$.

A strong version of the maximum principle implies that if $K_{\hat{g}_t}(x) = 0$ for some $(x, t) \in M_\infty \times (-\infty, \infty)$, then in fact $K_{\hat{g}_t} \equiv 0$. However, this is impossible, because by construction

$$K_{\max}(\hat{g}_0) = 1.$$

Summing up, $(M_\infty, (\hat{g}_t)_{t\in(-\infty,\infty)})$ is an eternal solution of the Ricci flow with strictly positive curvature. For such solutions, Hamilton proved the following theorem.

Theorem 5.19

Let M be a simply-connected manifold and let $(g_t)_{t\in(-\infty,\infty)}$ be a solution of the Ricci flow on M with uniformly bounded curvature and strictly positive curvature operator.

Then $(g_t)_t$ is a steady gradient soliton.

In our case M_∞ is also simply connected, because S^2 is, and the curvature is bounded by $0 < K_{g_t} \leq 1$. Thus the theorem applies and our blow up limit is in fact a gradient soliton.

It turns out that this solution has to be the *cigar soliton.* The cigar soliton is given by the metric

$$\frac{1}{1+x^2+y^2}(dx^2+dy^2)$$

on $\mathbb{R}^2$.

Proposition 5.20

Any two-dimensional steady gradient soliton is isometric to the cigar soliton.

This is shown by first showing that the solution must be rotationally invariant, i.e. it has an isometric action of S^1. Then the soliton equation simplifies to an ordinary differential equation. The solution of this equation can be explicitly calculated and corresponds to the cigar soliton.

The cigar soliton is asymptotic to a cylinder and as such the isoperimetric constant of the cigar soliton is 0. As we saw, the isoperimetric constant of the blow up limit $(\hat{g}_t)_t$ is bounded below by $C_S(g) > 0$. This is a contradiction and so we are in fact dealing with a type I singularity.

In this case, one can show that the blow up limit $(\hat{g}_t)_t$ is a compact shrinking gradient soliton.

Proposition 5.21

If (M, g, X) is a compact, two-dimensional Ricci soliton, then $X = 0$ and g is a constant curvature metric.

Thus $\hat{g}_0$ is in fact a round spherical metric on S^2.

Theorem 5.22

For any initial metric g on the sphere S^2, any blow up limit of the Ricci flow $(g_t)_{t\in[0,T_{\max})}$ is a shrinking round sphere.

In particular if $t_n \to T_{\max}$, then $\mathrm{Vol}(S^2, g_{t_n})^{-1} g_{t_n}$ converges uniformly in every C^k norm to the round sphere of volume 1.

We also mention the following result, which gives further insight into the behavior of the 2D Ricci flow.

Theorem 5.23 [Daskalopoulos–Hamilton–Sesum, 2012]
Let $(g_t)_{t\in(-\infty,T)}$ be a solution of the Ricci flow on a compact manifold M. Then $M = S^2$ $(g_t)_t$ is either the shrinking round sphere or the King–Rosenau solution.

The King–Rosenau solution will play an important role in the next section and will be given explicitly there.

Andrews' and Bryan's approach

We now present an alternative, more detailled proof of convergence. In a way, the main idea is very similar in that we also establish control on the isoperimetric constant. However, the information we obtain is stronger than in Hamilton's proof and immediately implies very strong curvature bounds, which allow us to proceed as in the case $\bar{K} < 0$.

Indeed, rather than studying the isoperimetric constant, we are going to look at the isoperimetric profile.

Definition 5.24
Let (M, g) be a compact surface. The *isoperimetric profile* of (M, g) is the function $h : (0, 1) \to \mathbb{R}_+$ defined via

$$h_g(\xi) = \inf\{L_g(\partial\Omega) : \Omega \subset M, \mathrm{Vol}_g(\Omega) = \xi \,\mathrm{Vol}_g(M)\},$$

where Ω runs over all domains in M with smooth boundary.

An obvious property of the isoperimetric profile function h is symmetry around $1/2$:

$$h_g(\xi) = h_g(1 - \xi).$$

Assuming M is a sphere and using the definition $C_S(\gamma) = \frac{L_g(\gamma)^2}{\min\{\mathrm{Vol}_g(\Omega_1),\mathrm{Vol}_g(\Omega_2)\}}$, we also find that

$$C_S(g) = \inf\{h_g(\xi)^2/(\xi \,\mathrm{Vol}_g(M)) : 0 < \xi \leq 1/2\}.$$

It is also useful to consider the isoperimetric profile for the round sphere. For this we note that the infimum $h(\xi)$ is for $\xi \in (0, 1)$ is attained by a domain whose boundary has constant geodesic curvature. A priori the domain need not be connected. For a sphere however, the optimal domains are spherical caps. An elementary exercise then shows

$$h(\xi) = 4\pi\sqrt{\xi(1 - \xi)}.$$

Andrews' and Bryan's approach may be summarised as follows:

1. The isoperimetric profile of a metric evolving according to the normalized Ricci flow can be compared to a solution of a certain one-dimensional parabolic PDE.

2. A *rotationally invariant* solution of the normalized Ricci flow gives rise to a solution of this PDE.

3. There is an explicit ancient rotationally invariant solution of the normalized Ricci flow, called the *King–Rosenau solution* or *Sausage model.* The solution of the comparison PDE can also be explicitly determined and this yields a bound for the isoperimetric profile of *any* solution of the normalized Ricci flow.

4. There is an approximation of the isoperimetric profile for small ξ in terms of the maximum of the curvature and the volume of the surface. Feeding the lower bound for the isoperimetric profile back into the approximation, we obtain an upper bound on the curvature of the solution. This upper bound is sufficiently tight to imply convergence.

We begin with the comparison principle for the isoperimetric profile.

Theorem 5.25

Let $\varphi : (0,1) \times [0,\infty) \to \mathbb{R}_+$ be a smooth function, which satisfies

1. *$\xi \mapsto \varphi(\xi, t)$ is strictly concave for every $t \in [0,\infty)$,*
2. *$\limsup_{\xi \searrow 0} \frac{\varphi(\xi,t)}{4\pi\sqrt{\xi}} < 1$ for every $t \in [0,\infty)$,*
3. *$\partial_t \varphi < \frac{1}{(4\pi)^2}\left(\varphi^2 \partial_\xi \partial_\xi \varphi - \varphi (\partial_\xi \varphi)^2\right) + \varphi + (\partial_\xi \varphi)(1 - 2\xi)$ on $(0,1) \times [0,\infty)$.*

Suppose $(g_t)_{t\in[0,\infty)}$ is a solution of the normalized Ricci flow on S^2 with $\mathrm{Vol}(S^2, g_t) = 4\pi$. If

$$h_{g_0}(\xi) > \varphi(\xi, 0) \text{ for every } \xi \in (0,1),$$

then

$$h_{g_t}(\xi) > \varphi(\xi, t) \text{ for every } (\xi, t) \in (0,1) \times [0,\infty).$$

The proof is somewhat lengthy, as may be expected from the definition of the isoperimetric profile. To manipulate this quantity, we need to understand how areas of domains and lengths of curves change under variation of the domain or curve and also under variation of the Riemannian metric. This is contained in the following two lemmas.

Lemma 5.26

Let M be a surface and let $(g_t)_{t\in(-\epsilon,\epsilon)}$ be a smooth family of Riemannian metrics.

If Ω is a domain in M, then

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{g_t}(\Omega) = \frac{1}{2}\int_\Omega \mathrm{tr}_{g_0}(\partial_t|_{t=0} g_t)\,\mathrm{vol}_{g_0}\,.$$

In particular, if g_t is a solution of the normalized Ricci flow

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}_{g_t}(\Omega) = -\int_\Omega (K_{g_0} - \bar{K}) \, \mathrm{vol}_{g_0} .$$

If $\gamma : [a, b] \to M$ is a smooth curve, then

$$\frac{d}{dt}\Big|_{t=0} L_{g_t}(\gamma) = \int_a^b \frac{\partial_t|_{t=0} g_t(\dot\gamma(s), \dot\gamma(s))}{2\sqrt{g_0(\dot\gamma(s), \dot\gamma(s))}} ds.$$

In particular, if g_t is a solution of the normalized Ricci flow

$$\frac{d}{dt}\Big|_{t=0} L_{g_t}(\gamma) = -\int_a^b (K_{g_0} - \bar{K})\sqrt{g_0(\dot\gamma, \dot\gamma)} ds = -\int_\gamma (K_{g_0} - \bar{K}) ds_{g_0}.$$

Lemma 5.27

Let (M, g) be a Riemannian surface. Suppose Ω is a domain in M with C^1 boundary $\gamma : S^1 \to M$. Let ν be the outward pointing vector field along γ. If $f : S^1 \to \mathbb{R}$ is a smooth function, then we can define a family of curves γ_t via

$$\gamma_t(s) = \exp_{\gamma(t)}(tf(s)\nu(s)).$$

Let Ω_t be the corresponding domain bounded by γ_t.

Then

$$\frac{d}{ds}\Big|_{s=0} L_g(\gamma_s) = \int_\gamma k_\gamma f ds$$

and

$$\frac{d}{ds}\Big|_{s=0} \mathrm{Vol}_g(\Omega_s) = \int_\gamma f ds.$$

Proof of theorem 5.25: We argue by contradiction.

If the theorem were false, there would exist a function $\varphi : (0, 1) \times [0, \infty) \to \mathbb{R}$ satisfying conditions 1.–3. and $(g_t)_{t\in[0,\infty)}$ a solution of the normalized Ricci flow on S^2 with function h_{g_0} is strictly larger than the function $\varphi(\xi, 0)$ and nevertheless

$$h_{g_{t_0}}(\xi_0) \le \varphi(\xi_0, t_0)$$

for some $\xi_0 \in (0, 1)$ and $t_0 > 0$. Since $h_{g_t}(\xi)$ depends continuously on both variables, we may assume $h_{g_{t_0}}(\xi_0) = \varphi(\xi_0, t_0)$. Moreover, we assume that for all $t < t_0$, the inequality

$$h_{g_t}(\xi) > \varphi(\xi, t)$$

holds, i.e. that t_0 is the first time there is equality.

This will lead to a contradiction. To this end, let Ω_0 be a domain of M with $\mathrm{Vol}_{g_{t_0}}(\Omega_0) = 4\pi\xi_0$ and $L_{g_{t_0}}(\partial\Omega_0) = \varphi(\xi_0, t_0)$. One can show that such a domain exists, is connected and has smooth boundary. On S^2, such a domain will also be simply connected. (See theorem 1 of [1].) Let $\gamma_0 : S^1 \to S^2$ be the boundary curve of Ω_0.

We can now get information about φ in two ways: one, we will consider variations of t, i.e. of the metric. Two, we will consider variations of the domain, i.e. of the boundary curve γ. In the first case, this means using the Ricci flow equation, in the second this means using the definition of the isoperimetric profile. These two sources are responsible for the differential inequality.

By assumption $h_{g_t}(\xi) \geq \varphi(\xi, t)$ for all $t \leq t_0$. Together with the definition of the isoperimetric profile, this implies

$$L(t) \geq h_{g_t}(V(t)) \geq \varphi(V(t), t)$$

for all $t \leq t_0$ and by choice of Ω_0

$$\boxed{L(t_0) = h_{g_{t_0}}(V(t_0)) = \varphi(V(t_0), t_0),}$$

where $L(t) = L_{g_t}(\gamma_0)$ and $V(t) = \mathrm{Vol}_{g_t}(\Omega_0)/(4\pi)$. We record also that

$$\boxed{V(t_0) = \xi_0.}$$

Taking the derivative, we obtain

$$L'(t_0) \leq \partial_t|_{t=t_0}\varphi(V(t), t) = (\partial_t\varphi)(V(t), t) + (\partial_\xi\varphi)(V(t), t)V'(t_0).$$

Using the lemma, we have

$$L'(t_0) = -\int_{\gamma_0} (K_{g_{t_0}} - \bar{K})ds_{\gamma_0} = L(t_0) - \int_{\gamma_0} K_{g_{t_0}} ds_{\gamma_0}$$

and

$$V'(t_0) = -\frac{1}{2\pi}\int_{\Omega_0} (K_{g_{t_0}} - \bar{K})\,\mathrm{vol}_{g_{t_0}} = \frac{1}{2\pi}\mathrm{Vol}_{g_{t_0}}(\Omega_0) - \frac{1}{2\pi}\int_{\Omega_0} K_{g_{t_0}}\,\mathrm{vol}_{g_{t_0}}.$$

Note that the assumption $\mathrm{Vol}_{g_t}(S^2) = 4\pi$ implies $\bar{K} = 1$. We remarked that Ω_0 is connected and simply connected, so that $\chi(\Omega_0) = 1$. Thus, by Gauß–Bonnet theorem we have

$$\int_{\Omega_0} K_{g_{t_0}}\,\mathrm{vol}_{g_{t_0}} = 2\pi - \int_{\gamma_0} k_{\gamma_0} ds_{g_{t_0}}.$$

Since $V(t_0) = \xi_0$ and $L(t_0) = \varphi(\xi_0, t_0)$, we have the inequality

$$\boxed{\varphi(\xi_0, t_0) - \int_{\gamma_0} K_{g_{t_0}} ds_{\gamma_0} \leq (\partial_t\varphi)(\xi_0, t_0) + (\partial_\xi\varphi)(\xi_0, t_0)\left(2\xi_0 - 1 + \frac{1}{2\pi}\int_{\gamma_0} k_{\gamma_0} ds_{g_{t_0}}\right).}$$

Next, we consider variations in the domain. To this end, suppose γ_δ is a family of curves with γ_0 as before. Then γ_δ also bounds a domain Ω_δ. We define

$$\hat{L}(\delta) = L_{g_{t_0}}(\gamma_\delta) \text{ and } \hat{V}(\delta) = \mathrm{Vol}_{g_{t_0}}(\Omega_\delta)/(4\pi).$$

From the definition of the isoperimetric profile, we see that

$$\hat{L}(\delta) \geq h_{g_{t_0}}(\hat{V}(\delta)) \geq \varphi(\hat{V}(\delta), t_0)$$

and

$$\hat{L}(0) = h_{g_{t_0}}(\hat{V}(0)) = \varphi(\hat{V}(0), t_0).$$

This implies

$$\hat{L}'(0) = \partial_\delta|_{\delta=0}\varphi(\hat{V}(\delta), t_0) = (\partial_\xi\varphi)(\hat{V}(0), t_0)\hat{V}'(0)$$

and

$$\hat{L}''(0) \geq (\partial_\xi\partial_\xi\varphi)(\hat{V}(0), t_0)(\hat{V}'(0))^2 + (\partial_\xi\varphi)(\xi_0, t_0)\hat{V}''(0).$$

Supposing that

$$\gamma_\delta(s) = \exp_{\gamma(s)}(tf(s)\nu(s)),$$

the lemma from before gives

$$\hat{L}'(\delta) = \int_{\gamma_\delta} k_{\gamma_\delta} f ds_{g_{t_0}}$$

and

$$\hat{V}'(\delta) = \frac{1}{4\pi}\int_{\gamma_\delta} f ds_{g_{t_0}}.$$

In conclusion, we obtain

$$\int_{\gamma_0} k_{\gamma_0} f ds_{g_{t_0}} = (\partial_\xi\varphi)(\xi_0, t_0)\frac{1}{4\pi}\int_{\gamma_0} f ds_{g_{t_0}}.$$

Since this holds for every f, this relation implies

$$\boxed{k_{\gamma_0} = \frac{1}{4\pi}(\partial_\xi\varphi)(\xi_0, t_0).}$$

Now suppose $f \equiv 1$. Then $\hat{V}'(\delta) = \frac{1}{4\pi}\hat{L}(\delta)$ and so

$$\hat{V}''(0) = \hat{L}'(0) = \frac{1}{4\pi}\int_{\gamma_0} k_{\gamma_0} ds_{\gamma_0} = \left(\frac{1}{4\pi}\right)^2 (\partial_\xi\varphi)(\xi_0, t_0)\varphi(\xi_0, t_0).$$

To compute the second derivative of $\hat{L}$, we first apply Gauß–Bonnet to obtain

$$\hat{L}'(\delta) = \int_{\gamma_\delta} k_{\gamma_\delta} ds_{g_{t_0}} = 2\pi - \int_{\Omega(\delta))} K_{g_{t_0}} \operatorname{vol}_{g_{t_0}}.$$

Then by the lemma

$$\hat{L}''(0) = -\int_{\gamma_0} K_{g_{t_0}} ds_{g_{t_0}}.$$

Thus

$$-\int_{\gamma_0} K_{g_{t_0}} ds_{g_{t_0}} \geq (\partial_\xi\partial_\xi\varphi)(\xi_0, t_0)\left(\frac{\varphi(\xi_0, t_0)}{4\pi}\right)^2 + (\partial_\xi\varphi)(\xi_0, t_0)\left(\frac{1}{4\pi}\right)^2 (\partial_\xi\varphi)(\xi_0, t_0)\varphi(\xi_0, t_0)$$

or equivalently

$$\boxed{-\int_{\gamma_0} K_{g_{t_0}} ds_{g_{t_0}} \geq \left(\frac{1}{4\pi}\right)^2 \left[(\partial_\xi \partial_\xi \varphi)\varphi^2 + (\partial_\xi \varphi)^2 \varphi\right] (\xi_0, t_0).}$$

The computation of the mean curvature of γ_0 simplifies the first boxed inequality to

$$\partial_t \varphi \geq \varphi + (\partial_\xi \varphi)(1 - 2\xi_0) - \frac{1}{8\pi^2}(\partial_\xi \varphi)^2 \varphi - \int_{\gamma_0} K_{g_{t_0}} ds_{\gamma_0}.$$

Now applying the second boxed inequality, we obtain

$$\begin{aligned}\partial_t \varphi &\geq \varphi + (\partial_\xi \varphi)(1 - 2\xi_0) - \frac{1}{8\pi^2}(\partial_\xi \varphi)^2 \varphi + \left(\frac{1}{4\pi}\right)^2 \left[(\partial_\xi \partial_\xi \varphi)\varphi^2 + (\partial_\xi \varphi)^2 \varphi\right] \\ &= \left(\frac{1}{4\pi}\right)^2 \left((\partial_\xi \partial_\xi \varphi)\varphi^2 - (\partial_\xi \varphi)^2 \varphi\right) + \varphi + (1 - 2\xi_0)(\partial_\xi \varphi)\end{aligned}$$

at (ξ_0, t_0). This is a contradiction to the assumption

$$\partial_t \varphi < \left(\frac{1}{4\pi}\right)^2 \left((\partial_\xi \partial_\xi \varphi)\varphi^2 - (\partial_\xi \varphi)^2 \varphi\right) + \varphi + (1 - 2\xi)(\partial_\xi \varphi).$$

□

Definition 5.28

A metric g on S^2 is called *rotationally symmetric*, if there is an isometric S^1 action on (S^2, g).

There are two exceptional orbits, which consist of one point each. These points are called the *poles* of (S^2, g). The regular orbits are called *latitude circles.*

Suppose (S^2, g) is rotationally symmetric. The collection of orbits of S^1 can be parametrized by the closed interval $[0, \pi]$. More explicitly, we have a map $\phi : S^2 \to [0, \pi]$, such that $\phi^{-1}(s)$ is a latitude circle if $s \in (0, 1)$ and $\phi^{-1}(0)$ and $\phi^{-1}(1)$ are the poles.

We can then consider the function

$$L : [0, \pi] \to \mathbb{R}_+$$
$$s \mapsto L_g(\phi^{-1}(s)),$$

which computes the length of the latitude circles, and the map

$$A : [0, \pi] \to \mathbb{R}_+$$
$$s \mapsto \mathrm{Vol}_g(\phi^{-1}([0, s])),$$

which computes the area of the disk consisting of the latitude circles parametrized by the interval $[0, s]$.

The function A is strictly increasing. Moreover, we may assume that $\mathrm{Vol}_g(S^2) = 4\pi$, so that A is an invertible function $[0, \pi] \to [0, 4\pi]$.

With this in mind, we define the function

$$\varphi_g : [0,1] \to \mathbb{R}_+$$
$$\varphi_g(\xi) = L(A^{-1}(4\pi\xi)).$$

Theorem 5.29

Suppose $(g_t)_{t\in[0,\infty)}$ is a rotationally symmetric solution of the normalized Ricci flow on S^2 with $\mathrm{Vol}(S^2, g_t) = 4\pi$. Consider the function $\varphi(\xi, t) = \varphi_{g_t}(\xi)$.

Then

$$\partial_t\varphi = \frac{1}{(4\pi)^2}\left(\varphi^2\partial_\xi\partial_\xi\varphi - \varphi(\partial_\xi\varphi)^2\right) + \varphi + (\partial_\xi\varphi)(1-2\xi).$$

Corollary 5.30

Suppose $(g_t)_{t\in[0,\infty)}$ is a rotationally symmetric solution of the normalized Ricci flow on S^2 with positive Gauss curvature. Consider the function $\varphi(\xi, t) = \varphi_{g_t}(\xi)$.

Suppose that $(\hat{g}_t)_{t\in[0,\infty)}$ is another solution of the normalized Ricci flow on S^2. If the isoperimetric profile of $\hat{g}_t$ satisfies $h_{g_0}(\xi) \geq \varphi(\xi, 0)$ for all $\xi \in (0,1)$, then for $\xi \in (0,1)$ and $t \in [0,\infty)$, we have

$$h_{g_t}(\xi) \geq \varphi(\xi, t).$$

Sketch of the proof: First, one shows that φ_{g_t} is actually concave for every t, because of the positivity of the Gauss curvature.

Next, for $\epsilon \in (0,1)$ the function $(1-\epsilon)\varphi$ is considered. It can be seen that this function satisfies the assumptions of theorem 5.25 and thus

$$h_{g_t}(\xi) \geq (1-\epsilon)\varphi(\xi, t)$$

holds for every $\epsilon \in (0,1)$. Passing to the limit $\epsilon \to 0$ yields the corollary. □

To apply the theorem, we need a rotationally symmetric solution of the normalized Ricci flow on S^2 with positive curvature. Such a solution is furnished by the *King–Rosenau solution* or *sausage model*. The descriptive second name stems from the fact that far in the past the solution looks like two cigar solitons glued together.

Note that since it is a solution on S^2 we already know that the solution of the Ricci flow has a finite time singularity, i.e. the solution is at best ancient. Of course, the normalized solution is in fact defined up to infinite time.

In fact the King–Rosenau solution is an ancient solution and we can describe it explicitly as the metric

$$g_t = \frac{\sinh(e^{-2t})}{2e^{-2t}\left(\cosh(z) + \cosh(e^{-2t})\right)}\left(dz^2 + d\theta^2\right)$$

on $\mathbb{R} \times S^1$, where z denotes the coordinate on $\mathbb{R}$ and θ denotes the coordinate on $S^1 = [0, 4\pi]/0 \sim 4\pi$. The metrics g_t extend smoothly to a metric on the sphere S^2 of volume 4π and solve the normalized Ricci flow equation.

The function $\varphi_{KR}(\xi, t)$ for this solution can be compute explicitly and according to Andrews and Bryan it is given by

$$\varphi_{KR}(\xi, t) = 4\pi\sqrt{\frac{\sinh(\xi e^{-2t})\sinh((1-\xi)e^{-2t})}{\sinh(e^{-2t})e^{-2t}}}.$$

Proposition 5.31

For any isoperimetric profile h_g of a metric g on S^2 with volume 4π, there exists a t_0, such that for all $\xi \in (0, 1)$ the following inequality holds

$$h_g(\xi) \geq \varphi_{KR}(\xi, t_0).$$

Proof idea: For any $\xi \in (0, 1)$ one hase

$$\varphi_{KR}(\xi, t) \to 0$$

as $t \to -\infty$. Thus for individual ξ, there always exists a $t(\xi) \in \mathbb{R}$, such that

$$h_g(\xi) \geq \varphi_{KR}(\xi, t(\xi)).$$

In fact, $t(\xi)$ can be chosen as a continuous function $(0, 1) \to \mathbb{R} \cup \{\infty\}$.

Analysing the behavior at the endpoints yields a continuous extension of $t(\xi)$ onto the interval $[0, 1]$. By compactness, this gives a uniform lower bound for $t(\xi)$. This uniform lower bound is a valid choice for t_0. □

Theorem 5.32

Let $(g_t)_{t\in[0,\infty)}$ be a solution of the normalized Ricci flow on S^2 with volume 4π. Then there exists a $t_0 \in \mathbb{R}$, such that

$$h_{g_t}(\xi) \geq \varphi_{KR}(\xi, t + t_0).$$

Theorem 5.33

Let (M, g) be a compact Riemannian surface. Then the isoperimetric profile has the expansion

$$h_g(\xi) = \sqrt{4\pi \operatorname{Vol}_g(M)}\xi^{1/2} - \left(\frac{1}{4\sqrt{\pi}} \operatorname{Vol}_g(M)^{3/2} \sup_{x\in M} K_g(x)\right) \xi^{3/2} + O(\xi^2)$$

as $\xi \searrow 0$.

Proof: This is shown by establishing $\leq$ and $\geq$ up to $O(\xi^2)$ in the expansion above. Recall that the definition of $h_g(\xi)$ is via an infimum over domains. This means that "$\leq$"

can be shown by a good choice of domain. It turns out that small metric balls do the job, because one has the following asymptotic formulas for the area of metric balls

$$\operatorname{Vol}_g(B_r(x)) = \pi r^2 \left(1 - \frac{K_g(x)}{12} r^2 + O(r^4)\right)$$

and the length of their boundary

$$L_g(\partial B_r(x)) = 2\pi r \left(1 - \frac{K_g(x)}{6} r^2 + O(r^4)\right)$$

in a Riemannian surface (M, g) for $r \searrow 0$.

Not surprisngly, for $\geq$ one needs deeper results. The Bol-Fiala inequality says that if (M, g) is simply connected and has curvature bounded above by κ, then for any smooth domain $\Omega \subset M$ one has

$$L_g(\partial\Omega)^2 \geq 4\pi \operatorname{Vol}_g(\Omega) - \kappa \operatorname{Vol}_g(\Omega)^2.$$

Now let ξ be sufficiently small, so that $h_g(\xi)$ is smaller than the injectivity radius. This implies that the optimal domain Ω lies wholly in a geodesic ball, which is diffeomorphic to a disk and hence simply connected. Thus the Bol-Fiala inequality applies and we get

$$(h_g(\xi))^2 = L_g(\partial\Omega)^2 \geq 4\pi\xi \operatorname{Vol}_g(M) - \sup_M K(\xi \operatorname{Vol}_g(M))^2.$$

Now use the expansion

$$\sqrt{\alpha x - \beta x^2} = \sqrt{\alpha} x^{1/2} - \frac{\beta}{2\sqrt{\alpha}} x^{3/2} + O(x^2)$$

to obtain

$$h_g(\xi) \geq \sqrt{4\pi \operatorname{Vol}_g(M)} \xi^{1/2} - \frac{\sup_M K\xi}{4\sqrt{\pi \operatorname{Vol}_g(M)}} \xi^{3/2} + O(\xi^2).$$

□

Corollary 5.34

Let $(g_t)_{t\in[0,\infty)}$ be a solution of the normalized Ricci flow on S^2 and assume that the volume $\operatorname{Vol}(S^2, g_t)$ is 4π.

Then there exists $t_0 \in \mathbb{R}$, such that

$$K_{g_t}(x) \leq \coth(e^{-2(t+t_0)}) e^{-2(t+t_0)} \leq 1 + \frac{1}{2} e^{-4(t+t_0)}.$$

Proof: Let $t_0 \in \mathbb{R}$ be as in theorem 5.32, so that

$$h_{g_t}(\xi) \geq \varphi_{KR}(\xi, t + t_0)$$

for every $t \geq 0$.

By an exercise, we have the asymptotic expansion

$$\varphi_{KR}(\xi, t) = 4\pi\xi^{1/2}\left(1 - \frac{1}{2}\exp(-2t)\coth(e^{-2t})\xi + O(\xi^2)\right)$$

as $\xi \to 0$ and by theorem 5.33 we have

$$h_{g_t}(\xi) = 4\pi\xi^{1/2} - \left(\frac{1}{4\sqrt{\pi}}(4\pi)^{3/2}\sup_{x\in M} K_{g_t}(x)\right)\xi^{3/2} + O(\xi^2).$$

Thus the difference $h_{g_t}(\xi) - \varphi_{KR}(\xi, t + t_0)$ has the expansion

$$\left(2\pi\exp(-2(t+t_0))\coth(e^{-2(t+t_0)}) - \frac{1}{4\sqrt{\pi}}(4\pi)^{3/2}\sup_{x\in M} K_{g_t}(x)\right)\xi^{3/2} + O(\xi^2).$$

Since $h_{g_t}(\xi) \geq \varphi_{KR}(\xi, t + t_0)$, this implies

$$\sup_{x\in M} K_{g_t}(x) \leq e^{-2(t+t_0)}\coth(e^{-2(t+t_0)}).$$

By an exercise

$$\coth(e^{-t})e^{-t} \leq 1 + \frac{1}{2}e^{-2t}$$

and so

$$\sup_{x\in M} K_{g_t}(x) \leq 1 + \frac{1}{2}e^{-4(t+t_0)}.$$

□

Applying Gauß–Bonnet gives

$$\begin{aligned} 0 &= \int_M K_{g_t} - 1 \operatorname{vol}_{g_t} \\ &= \int_{K_{g_t}\geq 1} K_{g_t} - 1 \operatorname{vol}_{g_t} + \int_{K_{g_t}<1} K_{g_t} - 1 \operatorname{vol}_{g_t} \\ &\leq \frac{1}{2}e^{-4(t+t_0)} + \int_{K_{g_t}<1} K_{g_t} - 1 \operatorname{vol}_{g_t}, \end{aligned}$$

or equivalently

$$\int_{K_{g_t}<1} |K_{g_t} - 1| \operatorname{vol}_{g_t} \leq \frac{1}{2}e^{-4(t+t_0)}.$$

Hence

$$\int_M |K_{g_t} - 1| \operatorname{vol}_{g_t} \leq e^{-4(t+t_0)}.$$

Using that the derivatives of K_{g_t} remain uniformly bounded along the flow and the Gagliardo–Nirenberg inequality

$$\|\nabla^k K\|_{L^\infty(M)} \leq C\|K_{g_t} - 1\|_{L^1(M)}^{(k+2)/(m+2)}\|\nabla^m K_{g_t}\|^{(k+2)/(m+2)}$$

one obtains exponential decay. (Note that the constant C in this inequality depends on isoperimetric constant, which is controlled by the isoperimetric profile.)

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