Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 1

Each exercise gives two points for a total of eight points on this sheet.

1. (a)Let M be a manifold and let g be an Einstein metric on M, i.e.

$$\mathrm{Ric}_g = \lambda g.$$

Show that

$$g(t) = (1 - 2\lambda t)g$$

is a solution of the Ricci flow. Determine the interval on which this solution is well defined.

(b)Let M be a manifold, g a Riemannian metric on M, and $X \in \Gamma(TM)$ is a vector field. Suppose

$$\mathrm{Ric}_g = \lambda g - \frac{1}{2}\mathcal{L}_X g.$$

A metric satisfying this equation is known as a *Ricci soliton*. Let $\phi_t : M \to M$ be the family of diffeomorphisms satisfying

$$\frac{d\phi_t(x)}{dt} = \frac{1}{1 - 2\lambda t} X(\phi_t(x)).$$

Show that

$$g(t) = (1 - 2\lambda t)\phi_t^* g$$

is a solution of the Ricci flow.

2.

Let (M, g) be a two-dimensional Riemannian manifold. Denote by Rm_g the associated curvature tensor, by Ric_g the Ricci tensor and by R_g the scalar curvature. Show that

(a)$\mathrm{Rm}_g(X, Y, Z, W) = \frac{1}{2}R_g\left(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\right)$

(b)$\mathrm{Ric}_g(X, Y) = \frac{1}{2}R_g g(X, Y)$

Hint: It is sufficient to check the identities on an orthonormal basis e_1, e_2 of T_xM.

3.

Let (M, g) be a three-dimensional Riemannian manifold. Show that the Ricci tensor Ric_g determines the curvature tensor Rm_g.

Hint: At any point x, the Ricci tensor is a symmetric bilinear form $\mathrm{Ric}_g(x) : T_xM \times T_xM \to \mathbb{R}$. Use the principal axis theorem to obtain a basis e_1, e_2, e_3 of T_xM, such that $\mathrm{Ric}_g(x)(e_i, e_j) = \lambda_i \delta_{ij}$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Then proceed similarly as in the previous exercise.

4.

A function $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ solves the heat equation with initial condition $\phi : \mathbb{R}^n \to \mathbb{R}$, if

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) \text{ for all } (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \\ u(0,x) = \phi(x) \text{ for } x \in \mathbb{R}^n \end{cases}$$

On $\mathbb{R}^n$ the *heat kernel* is given by

$$k : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

$$k_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp(-|x-y|^2/(4t)).$$

(a) Show that the function $(t,x) \mapsto k_t(x,y)$ solves the heat equation for any $y \in \mathbb{R}^n$ on $\mathbb{R}_{>0} \times \mathbb{R}^n$.

(b) Given $\phi \in L^1(\mathbb{R}^n)$, consider

$$u(x,t) = \int_{\mathbb{R}^n} k_t(x,y)\phi(y)dy$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_{>0}$.

It can be shown that

$$\lim_{t \searrow 0} u(x,t) = \phi(x).$$

In this sense, u solves the heat equation with initial condition ϕ.

Show that u is infinitely differentiable the space variable for any $t > 0$.