Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 10

Each exercise gives two points for a total of eight points on this sheet.

1. Show that if $(g_t)_{t\in[0,T_{\max})}$ is the solution of the normalized Ricci flow with initial condition g, then

$$\hat{g}_t = (1 - 2\bar{K}t)g_{\psi(t)}$$

with $\psi(t) = -\log(1 - 2\bar{K}t)/(2\bar{K})$ is a solution of the Ricci flow on the interval $[0, \hat{T}_{\max})$ where $\hat{T}_{\max} = (1 - \exp(-2\bar{K}T_{\max}))/(2\bar{K})$.

2. Let M be a compact surface and let g be a Riemannian metric on M. Show that there exists a unique maximal solution $(g_t)_{t\in[0,T_{\max})}$ of the normalized Ricci flow with initial condition $g_0 = g$.

3. Let (M, g) be a Riemannian manifold and let $f \in \mathcal{C}^\infty(M)$. Show that

$$\mathcal{L}_{(df)^\sharp} g = \mathrm{Hess}_g(f) = \nabla^g df.$$

4. *(Uniformization for the torus)* Let (M, g) be a compact Riemannian surface. It is a fact that for any $f \in \mathcal{C}^\infty(M)$ with $\int_M f \,\mathrm{vol}_g = 0$ there exists a $u \in \mathcal{C}^\infty(M)$ with $\Delta_g u = f$.

Now suppose that $\int_M K_g \,\mathrm{vol}_g$. Show that there exists a flat conformal metric $\tilde{g}$, i.e. $K_{\tilde{g}} \equiv 0$.