Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 10

Each exercise gives two points for a total of eight points on this sheet.

1. Show that if $(g_t)_{t\in[0,T)}$ is a solution of the normalized Ricci flow, then

$$\partial_t \Delta_{g_t} f = 2(K_{g_t} - \bar{K})\Delta_{g_t} f$$

for any $f \in \mathcal{C}^\infty(M)$.

2. Show that for any Riemannian manifold (M, g) and $f \in \mathcal{C}^\infty(M)$ the following inequality holds

$$(\Delta_g f)^2 \leq n|\operatorname{Hess}_g(f)|^2,$$

where n is the dimension of M.

3. Let M be a manifold, g a Riemannian metric on M and $h \in \Gamma(\operatorname{Sym}^2 T^*M)$. Show that for any three vector fields $X, Y, Z \in \Gamma(TM)$ the formula

$$\partial_t|_{t=0} g(\nabla_X^{g+th} Y, Z) = \frac{1}{2}\left((\nabla_X^g h)(Y, Z) + (\nabla_Y^g h)(X, Z) - (\nabla_Z^g h)(X, Y)\right).$$

4. Let M be a manifold, g a Riemannian metric and $h \in \Gamma(\operatorname{Sym}^2 T^*M)$. Show that for any four vector fields $X, Y, Z, W \in \Gamma(TM)$ the following formula holds

$$\begin{aligned}\partial_t\big|_{t=0} \operatorname{Rm}(X, Y, Z, W) = &\frac{1}{2}\left(h(R^g(X, Y)W, Z) - h(R^g(X, Y)Z, W)\right)\\ &+ \frac{1}{2}\big((\nabla_Y^g \nabla_W^g h)(X, Z) - (\nabla_Y^g \nabla_W^g h)(X, Z)\\ &\quad + (\nabla_Y^g \nabla_W^g h)(X, Z) - (\nabla_Y^g \nabla_W^g h)(X, Z)\big)\end{aligned}$$