Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 12

Each exercise gives two points for a total of eight points on this sheet.

1. Let M be a manifold, g a Riemannian metric and $h \in \Gamma(\mathrm{Sym}^2 T^*M)$. Define $g_t = g + th$. Show that the scalar curvature R_{g_t} satisfies

$$\partial_t|_{t=0} R_{g_t} = -g(\mathrm{Ric}_g, h) + \mathrm{div}_g \nabla^{g*} h + \Delta_g(\mathrm{tr}_g h).$$

Hint. Apply exercise 4 on sheet 11.

2. Let M be an oriented compact surface, g a Riemannian metric and $h \in \Gamma(\mathrm{Sym}^2 T^*M)$. Show that

$$\partial_t \int_M R_{g_t} \,\mathrm{vol}_{g_t} = 0.$$

Hint. Use the divergence theorem and exercise 3 (b) on sheet 6.

3. *(Gauß–Bonnet)*

(a) Let M be a manifold. Show that for any two metrics Riemannian metrics g_0, g_1 on M

$$g_t = tg_0 + (1-t)g_1$$

defines a Riemannian metric on M.

(b) Let M be an oriented compact surface. Show that for any two Riemannian metrics g_0, g_1 on M

$$\int_M R_{g_0} \,\mathrm{vol}_{g_0} = \int_M R_{g_1} \,\mathrm{vol}_{g_1}.$$

(c) Let (M_0, g_0) and (M_1, g_1) be two oriented, compact Riemannian surfaces. Show that

$$\int_{M_0} R_{g_0} \,\mathrm{vol}_{g_0} = \int_{M_1} R_{g_1} \,\mathrm{vol}_{g_1}.$$

Hint. Consider a diffeomorphism $\varphi : M_0 \to M_1$ and consider the metrics g_0 and $\varphi^* g_1$ on M_0.

4. Show that if M is diffeomorphic to S^2, then for any metric g on M

$$\int_M K_g \,\mathrm{vol}_g = 4\pi$$

and if M is diffeomorphic to the torus T^2, then

$$\int_M K_g \,\mathrm{vol}_g = 0.$$