Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 13

Each exercise gives two points for a total of eight points on this sheet.

1. Let g be the round metric on the sphere S^2. For $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$, we define spherical coordinates on S^2 via

$$x = \sin(\theta)\cos(\varphi),$$

$$y = \sin(\theta \sin(\varphi),$$

$$z = \cos(\theta).$$

(a)Show that $g = d\theta^2 + \sin(\theta)^2 d\varphi^2$.

(b)Show that $\Delta_g f = \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin(\theta)^2}\frac{\partial^2 f}{\partial \varphi^2} + \cot(\theta)\frac{\partial f}{\partial \theta}$.

2. Let g be the round metric on S^2 and let u_t be a family of smooth functions on S^2, each depending only on $\theta = \cos(z)$.

Show that $g_t = e^{2u_t} g$ satisfies the Ricci flow equation, if

$$\partial_t u_t = e^{-2u_t}\left(\frac{\partial u_t}{\partial \theta} + \cot(\theta)\frac{\partial u_t}{\partial \theta} - 1\right).$$

3. Let (M, g) be a compact Riemannian surface. The isoperimetric profile of (M, g) is the function

$$h(\xi) = \inf\{L_g(\partial\Omega) : \Omega \subset M, \partial\Omega \in C^1, \mathrm{Vol}_g(\Omega) = \xi \,\mathrm{Vol}_g(M)\}$$

defined for $\xi \in (0, 1)$. Now consider the standard round sphere (S^2, g). Assume without proof that the infimum is attained by a connected domain, whose boundary has constant geodesic curvature. Show that

$$h(\xi) = 4\pi\sqrt{\xi(1-\xi)}.$$

4. *(Calculations related to the isoperimetric profile of the King–Rosenau solution)*

(a)Let $\varphi_{KR} : (0, 1) \times [0, \infty) \to \mathbb{R}_+$ be the function

$$\varphi_{KR}(\xi, t) = 4\pi\sqrt{\frac{\sinh(\xi e^{-2t})\sinh((1-\xi)e^{-2t})}{\sinh(e^{-2t})e^{-2t}}}.$$

Show that

$$\varphi_{KR}(\xi, t) = 4\pi(\xi)^{1/2}\left(1 - \frac{1}{2}\exp(-2t)\coth(e^{-2t})\xi + O(\xi^2)\right)$$

as $\xi \to 0$.

(b)Let $f(t) = \coth(e^{-t})e^{-t}$. Show that

$$f(t) \leq 1 + \frac{1}{2}e^{-2t}.$$