Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 2

Each exercise gives two points for a total of eight points on this sheet.

1. Let (M, g) be a Riemannian manifold.

   (a) Show that $\mathrm{Hess}_g(f) = \nabla^g df$ is symmetric, i.e.

   $$\mathrm{Hess}_g(f)(X,Y) = \mathrm{Hess}_g(f)(Y,X).$$

   (b) Show that $\mathrm{div}_g \,\mathrm{grad}_g f = -\operatorname{tr}_g \mathrm{Hess}_g(f)$.

2. Let (M, g) be a Riemannian manifold and let $\tilde{g} = e^{2u}g$ with $u \in \mathcal{C}^\infty(M)$. Show that

$$\Delta_{\tilde{g}} f = e^{-2u}(\Delta_g f - (n-2)g(du, df)^2).$$

Hint: An orthonormal frame at a point $x \in M$ are n local vector fields $e_1, \ldots, e_n$, such that $g(e_i, e_j) = \delta_{ij}$. With respect to such an orthonormal frame one gets the formula

$$\Delta_g f = -\operatorname{tr}_g \mathrm{Hess}_g(f) = -\sum_{i=1}^{n} (\nabla^g du)(e_i, e_i).$$

Moreover, by the definition of the induced connection on T^*M, we have for $\alpha \in \Gamma(T^*M)$

$$(\nabla^g_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla^g_X Y).$$

3. Let (M, g) be a Riemannian manifold and let $\tilde{g} = e^{2u}g$ with $u \in \mathcal{C}^\infty(M)$. Denote $\tilde{g} = e^{2u}g$. For $v, w \in T_pM$ with $g(v, w) = 0$ and $g(v, v) = g(w, w) = 1$, show that

$$\sec_{\tilde{g}}(v,w) = e^{-2u}\left(\sec_g(v,w) - \mathrm{Hess}_g u(v,v) - \mathrm{Hess}_g u(w,w) + du(v)^2 + du(w)^2 - |du|_g^2\right).$$

4. Using the divergence theorem and the definition $\Delta_g = \mathrm{div}_g \,\mathrm{grad}_g$, show that

$$\int_M (\Delta_g u) v \,\mathrm{vol}_g = \int_M g(\mathrm{grad}_g u, \mathrm{grad}_g v) \,\mathrm{vol}_g = \int_M u(\Delta_g v) \,\mathrm{vol}_g$$

for any $u, v \in \mathcal{C}^\infty(M)$.