Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 3

Each exercise gives two points for a total of eight points on this sheet.

1. Prove the following theorem from the lecture: Let M be a closed manifold and suppose g is a Riemannian metric on M. Then there exists a unique, maximal solution $(g_t)_{t\in[0,T_{\max})}$ of the Ricci flow with initial value g.

Hint: Consider the set of all solutions of the Ricci flow with initial value g. This set is non-empty by the short time existence theorem. This set can be given a partial order by comparing the length of the interval, on which the flow exists. More precisely, let $(g_t^1)_{t\in[0,T_1)}$, $(g_t^2)_{t\in[0,T_2)}$ be two solutions with initial value g. Then we define a partial order $\preceq$ by

$$(g_t^1)_{t\in[0,T_1)} \preceq (g_t^2)_{t\in[0,T_2)} :\Leftrightarrow T_1 \leq T_2.$$

The lemma of Zorn can then be applied to show that a maximal solution exists. Finally, conclude it must be unique.

2. *(Conformal Laplacian)* Let (M,g) be a n-dimensional Riemannian manifold.

(a)Let $u \in \mathcal{C}^\infty(M)$ and $\tilde{g} = e^{2u}g$. Show that

$$R_g = e^{-2u}\left(2(n-1)\Delta_g u + (2-n)(n-1)|du|_g^2 + R_g\right)$$

Hint: Use the formula for the sectional curvature from last sheet. Note that

$$R_g = \sum_{i=1}^n \sum_{\substack{j=1,\\ j\neq i}}^n \sec_g(e_i, e_j)$$

for any orthogonal basis $e_1, \dots, e_n$.

(b)Suppose $n \geq 3$. The conformal Laplacian L_g is the differential operator

$$L_g f = 4\frac{n-1}{n-2}\Delta_g f + R_g f.$$

Let $v \in \mathcal{C}^\infty(M)$ and $\tilde{g} = v^{\frac{4}{n-2}}g$. Show that

$$R_{\tilde{g}} = v^{-\frac{n+2}{n-2}} L_g v.$$

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\alpha \in (0,1)$.

(a)Show that if $u, v \in C^\alpha(\Omega)$ then $uv \in C^\alpha(\Omega)$ and

$$\|uv\|_{C^\alpha(\Omega)} \leq \|u\|_{C^\alpha(\Omega)}\|v\|_{C^\alpha(\Omega)}.$$

(b)Show that if $u \in C^\alpha(\Omega)$, then $\exp \circ u \in C^\alpha(\Omega)$ and

$$\|\exp \circ u\|_{C^0(\Omega)} \leq \exp(\|u\|_{C^0(\Omega)})$$

and

$$[\exp \circ u]_\alpha \leq \exp(\|u\|_{C^0(\Omega)})[u]_\alpha.$$

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\alpha \in (0,1)$. Furthermore, suppose $K \in C^\alpha(\Omega)$. Define

$$N : C^{2,\alpha}(\Omega) \to C^\alpha(\Omega)$$
$$u \mapsto (1 - e^{-2u})\Delta_g u + Ke^{-2u}.$$

(a)Show that N is well-defined.

(b)Show that there exists a $C > 0$, such that

$$\|N(u)\|_{C^\alpha(\Omega)} \leq C \exp(2\|u\|_{C^0(\Omega)})(1 + \|u\|_{C^\alpha(\Omega)})(1 + \|u\|_{C^{2,\alpha}(\Omega)})$$

for all $u \in C^{2,\alpha}(\Omega)$.

(c)Show that there exists a $C > 0$ and a continuous function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$\|N(u) - N(v)\|_{C^\alpha(\Omega)} \leq CF(\|u\|_{C^\alpha(\Omega)} + \|v\|_{C^\alpha(\Omega)})(\|u - v\|_{C^{2,\alpha}})$$

for all $u, v \in C^{2,\alpha}(\Omega)$.

Hint: Use the formula

$$e^u - e^v = e^v(e^{u-v} - 1) = e^v \frac{e^{u-v} - 1}{u - v}(u - v).$$