Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 4

Each exercise gives two points for a total of eight points on this sheet.

1. Let M be a compact manifold. Let g be a Riemannian metric. A diffeomorphism $f : M \to M$ is called an isometry of (M, g), if $f^*g = g$, or more explicitly

$$g_{f(x)}(df(x)v, df(x)w) = g_x(v, w) \text{ for all } x \in M, v, w \in T_xM.$$

Suppose that f is an isometry. Denote by $(g_t)_{t\in[0,T_{\max})}$ the maximal solution of the Ricci flow with initial condition g. Show that f is an isometry of g_t for every $t \in [0, T_{\max})$.

Hint: Use the uniqueness of solutions and the diffeomorphism equivariance $f^* \operatorname{Ric}_g = \operatorname{Ric}_{f^*g}$ of the Ricci tensor.

2. *(Parabolic rescaling)* Show that if $(g_t)_{t\in[0,T_{\max})}$ is a solution of the Ricci flow, then

$$\hat{g}_t = \lambda^{-1} g_{\lambda t}$$

is a solution of the Ricci flow on $[0, \lambda^{-1}T_{\max})$.

3. *(Cigar soliton)*

(a) Show that the metric

$$g_{cig} = \frac{1}{1 + x^2 + y^2} \left(dx^2 + dy^2\right)$$

on $\mathbb{R}^2$ is a Ricci soliton (cf. exercise 1, sheet 1) with respect to the vector field $V = -2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$.

(b) Show that $(\mathbb{R}^2, g_{cig})$ is complete.

Hint: Check that Cauchy sequences in $(\mathbb{R}^2, g_{cig})$ are Cauchy sequences in $\mathbb{R}^2$.

4. *(Liouville energy)* Suppose that M is a surface and suppose that $g_t = e^{2u_t}g$ is a solution of the Ricci flow. The Liouville energy of u is

$$E_L(u) = \frac{1}{2} \int_M |grad_g u|_g^2 + 2K_g u \operatorname{vol}_g .$$

Using the fact

$$\operatorname{vol}_{g_t} = e^{2u_t} \operatorname{vol}_g,$$

show that

$$\frac{d}{dt} E_L(u_t) = -\int_M K_{g_t}^2 \operatorname{vol}_{g_t} .$$