Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 5

Each exercise gives two points for a total of eight points on this sheet.

1. Let (M, g) be a Riemannian manifold and $\lambda \in \mathcal{C}^\infty(M)$. Then let $\tilde{g} = \lambda^2 g$. The volume form vol_g is defined to be the unique n-form $\mathrm{vol}_g \in \Omega^n(M)$, such that

$$\mathrm{vol}_g(e_1, \ldots, e_n) = 1$$

for every oriented, orthonormal basis $e_1, \ldots, e_n \in T_x M$.

Show that

$$\mathrm{vol}_{\tilde{g}} = \lambda^n \,\mathrm{vol}_g .$$

2. Let M be a oriented surface and let $(g_t)_{t\in[0,T)}$ be a solution of the Ricci flow. Show that

$$\partial_t \,\mathrm{vol}_{g_t} = -2K_{g_t} \,\mathrm{vol}_{g_t} .$$

3. Let M be a compact surface (as always, without boundary) and suppose $(g_t)_{t\in[0,T)}$. Denote the total area of (M, g_t) by

$$A(t) = \int_M \mathrm{vol}_{g_t} .$$

Show that the time derivative $A'(t)$ is constant.

Hint: Use the divergence theorem or the Gauß–Bonnet theorem.

4. Let (M, g) be a Riemannian manifold. The connection Laplacian $\nabla^{g*}\nabla^g$ acting on any tensor field defined to be

$$\nabla^{g*}\nabla^g T = -\sum_{i=1}^{n} (\nabla^g \nabla^g T)(e_i, e_i),$$

where ∇^g denotes the connection induced on tensor fields and $e_1, \ldots, e_n$ is any orthonormal basis. (See remark 0.14 in the lecture notes.)

(a)Show that for any $\alpha \in \Gamma(T^*M)$ the formula

$$\frac{1}{2}\Delta_g |\alpha|_g^2 = g(\nabla^{g*}\nabla^g \alpha, \alpha) - |\nabla^g \alpha|_g^2$$

holds.

(b)Suppose $\dim M = 2$. For any $u \in \mathcal{C}^\infty(M)$ the formula

$$d\Delta_g u = \nabla^{g*}\nabla^g du + K_g du$$

holds. Infer that

$$\frac{1}{2}\Delta_g |df|_g^2 = g(d\Delta_g f, df) - K_g |df|_g^2 - |\nabla^g df|_g^2$$

for any smooth function $f \in \mathcal{C}^\infty(M)$.