Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 5

Each exercise gives two points for a total of eight points on this sheet.

1. Let (M, g) be a Riemannian manifold. Show that for any $\alpha \in \Gamma(T^*M)$ and $X, Y, Z \in \Gamma(TM)$, the following formula holds

$$[R(X,Y)\alpha](Z) = -\alpha(R(X,Y)Z).$$

On the right hand side, R is the curvature of the induced connection, i.e.

$$R(X,Y)\alpha = \nabla^g_X \nabla^g_Y \alpha - \nabla^g_Y \nabla^g_X \alpha - \nabla^g_{[X,Y]}\alpha.$$

2. Let (M, g) be a Riemannian manifold and $f \in C^2(M)$. Let $e_1, \dots, e_n$ be an orthonormal frame of TM, i.e. local vector fields, such that $g(e_i, e_j) \equiv 1$. Then

$$\sum_{i=1}^{n} \mathrm{Hess}_g(f)(\nabla^g_X e_i, e_i) = 0$$

for any $X \in \Gamma(TM)$.

Hint: Expand

$$\nabla^g_X e_i = \sum_{j=1}^{n} g(\nabla^g_X e_i, e_j).$$

Use the symmetry of $\mathrm{Hess}_g(f)$ and that

$$0 = Xg(e_i, e_j) = g(\nabla^g_X e_i, e_j) + g(e_i, \nabla^g_X e_j).$$

3. *(Counts as two exercises.)*

Denote by $\langle \cdot, \cdot \rangle_0$ the standard inner product on $\mathbb{R}^n$. The associated volume form is

$$\mathrm{vol}_0(v_1, \dots, v_n) = \det(A),$$

where A is the matrix satisfying $Ae_i = v_i$.

Now suppose that $\langle \cdot, \cdot \rangle$ is any other standard inner product on $\mathbb{R}^n$. There exists a symmetric endomorphism $B : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\langle v, w \rangle = \langle Bv, Bw \rangle.$$

(a)Show that the volume form vol associated to $\langle \cdot, \cdot \rangle$ satisfies

$$\mathrm{vol} = \det(B)\,\mathrm{vol}_0\,.$$

(Recall that vol is uniquely specified by the condition

$$\mathrm{vol}(e_1, \dots, e_n) = 1$$

for any oriented $\langle \cdot, \cdot \rangle$-orthonormal basis.)

(b)Suppose that $(\langle\cdot,\cdot\rangle_t)_{t\in(a,b)}$ is a smooth family of inner products on $\mathbb{R}^n$. We denote the time derivative by β_t, i.e.

$$\beta_t(v,w) = \frac{d}{dt}\langle v,w\rangle_t.$$

The *trace* of β_t with respect to $\langle\cdot,\cdot\rangle_t$ is defined to be

$$\operatorname{tr}_t(\beta_t) = \sum_{i=1}^{n} \beta_t(e_i,e_i),$$

where $e_1,\ldots,e_n$ is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle_t$.
Show that

$$\frac{d}{dt}\operatorname{vol}_t = \frac{1}{2}\operatorname{tr}_t(\beta_t)\operatorname{vol}_t.$$

Hint: You may assume that

$$\langle v,w\rangle_t = \langle B_t v, B_t w\rangle$$

for a smooth family of symmetric endomorphisms $B_t : \mathbb{R}^n \to \mathbb{R}^n$. Then apply (a).