Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 7

Each exercise gives two points for a total of eight points on this sheet.

1. Let M be a compact manifold. Suppose that $(g_t)_{t\in[0,T]}$ is a smooth family of Riemannian metrics. Let $f : M \times [0,T] \to \mathbb{R}$ be a smooth function. Suppose that $u : M \times [0,T] \to \mathbb{R}$ is a smooth function solving the partial differential equation

$$\partial_t u(x,t) + \Delta_{g_t} u(x,t) = f(x,t).$$

Using the maximum principle, show that

$$\max_{M\times[0,T]} |u| \leq T \max_{M\times[0,T]} |f|.$$

2. Let M be a manifold and suppose that $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow. Show that

$$\partial_t \operatorname{vol}_{g_t} = -R_{g_t} \operatorname{vol}_{g_t}.$$

Hint: Use exercise 3 of sheet 6 to prove the identity

$$\partial_t \operatorname{vol}_{g_t} = \frac{1}{2} \operatorname{tr}_{g_t}(\partial_t g_t) \operatorname{vol}_{g_t}$$

for any smooth family of Riemannian metrics $(g_t)_{t\in(a,b)}$.

3. Let (M,g) be a Riemannian manifold. Let $X \in \Gamma(TM)$. The covariant derivative $\nabla^g X \in \Gamma(T^*M \otimes TM)$ is called *skew-symmetric*, if

$$g(\nabla^g_V X, W) = -g(\nabla^g_W X, V)$$

for every $V, W \in TM$. The vector field X is called a *Killing vector field*, if $\mathcal{L}_X g = 0$.

Show that X is a Killing vector field, if and only if $\nabla^g X$ is skew-symmetric.

4. Let (M,g) be a Riemannian manifold and suppose that $X \in \Gamma(TM)$ is a Killing vector field. Let $f = \frac{1}{2}|X|_g^2$. Show that

$$\operatorname{Hess}_g f(V,V) = |\nabla^g_V X|^2 - \operatorname{Rm}(V,X,X,V)$$

for every $V \in TM$ and conclude that

$$\Delta_g f = |\nabla^g X|^2 - \operatorname{Ric}(X,X).$$

Remark. If the Ricci curvature is negative, i.e. $\operatorname{Ric}(V,V) < 0$ for every nonzero $V \in TM$, then this formula implies $\Delta_g f > 0$ everywhere. Every function on a compact manifold has at least one minimum and at this point $\Delta_g f \leq 0$. Hence there are no non-trivial Killing vector fields on such a manifold.