Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 8

Each exercise gives two points for a total of eight points on this sheet.

1. Let M be a surface and let $\mu \in \Gamma(T^*M^{\otimes k})$. Suppose $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow. Show that

$$\partial_t |\mu|^2_{g_t} = 2kK_{g_t}|\mu|^2_{g_t}.$$

2. Let (M, g) be a Riemannian manifold. Show that for any $\mu \in \Gamma(T^*M^{\otimes k})$, the following identity holds

$$\Delta_g |\mu|^2_g = 2g(\nabla^{g*}\nabla^g \mu, \mu) - |\nabla^g \mu|^2.$$

3. Let $(M, g_t)_{t\in[0,T)}$ be a family of Riemannian metrics and suppose that

$$\sup_{t\in[0,T)} |\partial_t g_t|_{g_t} < \infty.$$

Show that there exists a constant $C > 0$, such that

$$\frac{1}{C}|v|^2_{g_{t_1}} \le |v|^2_{g_{t_2}} \le C|v|^2_{g_{t_1}}$$

for all $t_1, t_2 \in [0, T)$ and all $v \in TM$.

4. Let (M, g) be a Riemannian surface and suppose $u_t \in \mathcal{C}^\infty(M)$. Let $g_t = e^{2u_t} g$. Show that if

$$\sup_{t\in[0,T)} |\partial_t g_t|_{g_t} < \infty$$

and

$$\sup_{t\in[0,T)} |d\partial_t u_t|_{g_t} < \infty,$$

then there exists $C > 0$, such that

$$\frac{1}{C}|\nabla^{g_{t_1}}_X Y|^2_{g_{t_1}} \le |\nabla^{g_{t_2}}_X Y|^2_{g_{t_2}} \le C|\nabla^{g_{t_1}}_X Y|^2_{g_{t_1}}$$

for all $t_1, t_2 \in [0, T)$, $X, Y \in \Gamma(TM)$.