Introduction to the Ricci Flow

WS 2019/2020

Exercise Sheet 9

Each exercise gives two points for a total of eight points on this sheet.

1. Let M be a compact surface. A smooth family of metrics $(g_t)_{t\in[0,T]}$ is a solution of the *normalized Ricci flow*, if

$$\partial_t g_t = -\left(K_{g_t} - \left(\int_M \mathrm{vol}_{g_t} \right)^{-1} \int_M K_{g_t} \, \mathrm{vol}_{g_t} \right) g_t.$$

Show that the volume is constant in time along the normalized Ricci flow, i.e.

$$\frac{d}{dt} \int_M \mathrm{vol}_{g_t} = 0.$$

Remark. By the Gauß–Bonnet theorem

$$\int_M K_{g_t} \, \mathrm{vol}_{g_t}$$

is also independent of t, so that

$$\left(\int_M \mathrm{vol}_{g_t} \right)^{-1} \int_M K_{g_t} \, \mathrm{vol}_{g_t}$$

is actually constant in time.

2. Let M be a compact surface and suppose that $(g_t)_{t\in[0,T]}$ is a solution of the *normalized* Ricci flow. Compute $\partial_t K_{g_t}$ and $\partial_t |dK_{g_t}|^2_{g_t}$.

3. Let M be a compact surface and suppose that $(g_t)_{t\in[0,T]}$ is a solution of the normalized Ricci flow. Suppose moreover that $K_{g_t} > 0$. Now define $L_t = \log K_{g_t}$.

Show that

$$\partial_t L_t + \Delta_{g_t} L_t = |dL_t|^2_{g_t} + 2K_{g_t} - 2 \left(\int_M \mathrm{vol}_{g_t} \right)^{-1} \int_M K_{g_t} \, \mathrm{vol}_{g_t} \, .$$

4. Let M be a compact surface and suppose that $(g_t)_{t\in[0,T]}$ is a solution of the Ricci flow. Show that

$$\frac{d}{dt} \frac{1}{2} \int_M K_{g_t}^2 \, \mathrm{vol}_{g_t} = -\int_M |dK_{g_t}|^2 \, \mathrm{vol}_{g_t} + \int_M K_{g_t}^3 \, \mathrm{vol}_{g_t} \, .$$