Symmetry and Regularity of Solutions for Nonlinear Integral and PDE Systems.

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Abstract

In this talk, we will introduce the integral form of the method of moving planes and its applications, mainly in establishing symmetry for solutions of integral equations and systems as well as PDEs due to the equivalences between the two. This method is quite different from the ones for PDEs. Instead of using maximum principles, some global norms are estimated. It can be applied to obtain radial symmetry for positive solutions of the fully nonlinear integral systems involving Wolff potentials:

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x), & x \in R^n; \\ v(x) = W_{\beta,\gamma}(u^p)(x), & x \in R^n; \end{cases} \tag{1}$$

where

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

In a special case when $\beta = \frac{\alpha}{2}$ and $\gamma = 2$, system (1) reduces to

$$\begin{cases} u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} v(y)^q dy, & x \in R^n, \\ v(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^p dy, & x \in R^n. \end{cases} \tag{2}$$

The solutions (u, v) of (2) are critical points of the functional associated with the well-known Hardy-Littlewood-Sobolev inequality. The classification of solutions would provide the best constant in the HLS inequality.

We can also prove that the integral system (2) is equivalent to the system of partial differential equations

$$\left\{ \begin{array}{l} (-\Delta)^{\alpha/2} u = v^q, u > 0, \text{ in } R^n, \\ (-\Delta)^{\alpha/2} v = u^p, v > 0, \text{ in } R^n. \end{array} \right. \tag{3}$$

And in particular when $\alpha = 2$, it reduces to the well-known Lane-Emden system. And even more particularly, when $p = q = \frac{n+2}{n-2}$, it becomes the Yamabe equation.

We will also mention two convenient ways to lift regularity for solutions: one by contracting operators and the other by the combined use of contracting and shrinking operators. The latter is a new idea which has just been applied in our recent paper to establish Lipschitz continuity of positive solutions for system (1), and we believe that this idea will become a useful tool in nonlinear analysis.