Profinite Sets, Extremally Disconnected Spaces and Basics of Condensed Mathematics

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von Jannek Link
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Prof. Dr. Annette Huber-Klawitter

Introduction

In this bachelor's thesis, we are going to explain the category of profinite sets, study some of its properties and in the end see the basic notions of condensed mathematics, where profinite sets are used.

The main theorems presented in this thesis are the equivalence of the category of profinite sets to the category of totally disconnected compact Hausdorff spaces (Def. 1.4.3 and Theorem 1.4.11) and the characterization of the projectives of this category as extremally disconnected spaces (Def. 3.2.1 and Theorem 3.2.10).

The text aims to coherently explain the theory of profinite sets with a view toward both category theory and topology. We will not develop the theory of Boolean Algebras another way to view this topic. Thereby we aim to lay the groundwork for an introduction to condensed mathematics.

Condensed mathematics, our main motivation for the study of profinite sets, is a very new field of mathematics only developed in the last few years by Dustin Clausen and Peter Scholze [CS19b]. It aims to give a new foundation of analytic non-archimedean [CS19a] and complex geometry [CS22], where one can apply a wide range of tools from algebraic geometry. As the notion of a topological group is not well-behaved when looking at exact sequences another kind of object was needed. This seems to be the category of condensed sets and its subcategories of condensed groups, rings, and so forth. This approach has been so far very successful, as they were able to reprove the Riemann-Roch theorem [CS22] in the theory of condensed mathematics.

Another very interesting aspect of this recently developed field is the Liquid Tensor experiment. This was a challenge, posted online by Peter Scholze, to verify the theorem of liquid tensor spaces using computer assisted proof checking software, with an emphasis on the proof assistant Lean (Liquid Tensor Experiment). He was very sure that it was true, but wanted to be absolutely sure, as he was convinced that the theory of condensed mathematics rose or fell with this theorem. In July 2022 the Liquid Tensor Experiment was completed by a group of mathematicians led by Johan Commelin with support from Peter Scholze and many contributions from Adam Topaz and many others (Completion of LTE). This marked, besides verifying the important theorem, a great success in computer-assisted proofs and proof checking.

Apart from those very recent developments most of what is done in this thesis is based on material from the middle of the twentieth century. The first appearance of profinite sets was not as such, but as totally disconnected compact Hausdorff spaces. Stone studied Boolean algebras [Sto36] which led to him proving that the categories of Boolean algebras and totally disconnected compact Hausdorff spaces are dual, which is the cause for them being called Stone Spaces [Sto37]. In the same paper, he also proved that extremally disconnected compact Hausdorff spaces, also known as Stonean Spaces, are dual to complete Boolean algebras. Additionally, simultaneously with Čech [Cec37], he discovered the Stone-Čech compactification.

Whereas Stone did it from the perspective of the underlying Boolean algebras of frames and locales the motivation of Čech was the continuous extension of real-valued functions on completely regular spaces to a compact Hausdorff space, in which the space we started with is densely embedded. The characterization of extremally disconnected spaces as projectives was mostly developed in a paper by Gleason [Gle58] and adapted in a way, more in spirit with category theory, in a paper by Rainwater [Rai59]. There he also introduces the notion to view the Stone-Čech compactification of discrete spaces as free objects.

The discoveries described in the previous paragraph mostly look at our topic from a topological point of view. The other aspect, the category theoretical one, was developed by Grothendieck in [SGA4], although there the focus lay on the ind-categories, dual to pro-categories. This also just introduced the abstract construction of those categories and did not consider the special case of profinite sets. Unsurprisingly this is a field very close to Peter Scholze's work before beginning to develop the theory of condensed mathematics.

Contents

In Chapter 1, we are going to introduce the notion of a projective limit from category theory and use this to define the pro-category. This category consists of all formal projective limits in a category, which can be thought of as a kind of completion. We will apply this to finite sets to gain the category of profinite sets. Having done this, we will first examine properties of projective limits of topological spaces and find out, among other properties, that the limits of compact Hausdorff spaces are again compact Hausdorff. Equipping finite sets with the discrete topology, they are compact Hausdorff. Thus we can view our profinite sets as a subcategory of compact Hausdorff spaces. It turns out that the profinite sets are exactly the totally disconnected compact Hausdorff spaces and even that the categories are equivalent.

Before continuing to examine the category of profinite sets, Chapter 2 introduces a tool from topology, the Stone-Čech compactification. We approach this as historically done by Čech from the angle of continuous real-valued functions and explicitly construct the Stone-Čech compactification. We also characterize it by a universal property that any continuous map to a compact Hausdorff space uniquely extends to the Stone-Čech compactification. We will mainly use the Stone-Čech compactification of discrete spaces.

Applying this tool, in Chapter 3 we again examine the categories of profinite sets and compact Hausdorff spaces. A question that originated in homological algebra is that of projective objects. It turns out that the projectives in both categories are also characterized by a topological property. They are extremally disconnected spaces. Furthermore, will we show that the category of compact Hausdorff spaces, and in particular that of profinite sets, has enough projectives. To do this we will use the

Stone-Čech compactifications of discrete spaces. They play the role of the free objects in both our categories and are in particular extremally disconnected.

We end the thesis in Chapter 4 with a brief introduction to condensed sets as this is the main motivation for the work done beforehand. We see how there does not exist a useful notion of an exact sequence for topological groups. As this is quite essential for modern algebra we go on to define condensed sets as sheaves on **CHaus**. Then, we show some of their connections to topological spaces. In the end, we show that the different definitions of condensed sets, as sheaves on **ProFin** or the extremally disconnected compact Hausdorff spaces, are equivalent.

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Jannek Link
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Contents

Terminology and Notation

In the following we introduce some terminology and notation. We will mostly use it this way, but notation may change, so additionally it will (hopefully) always be stated what kind of object we refer to at the moment.

When talking about categories, the collection of objects is not necessarily a set, but the hom-sets are always sets. By small we mean that a collection is actually a set. So if we talk about a small category we mean a category where the collection of objects forms a set.

Given objects a, b in an arbitrary category C, we write a morphism $f : a \to b$ like a map to clarify domain and codomain. We denote the hom-set of all morphisms with domain a and codomain b by $C(a, b)$. In an abelian category $\mathcal{A}$ we write $\mathrm{Hom}_{\mathcal{A}}(a, b)$ for the hom-sets.

Given a functor F from the category J to C, i, j, objects in J, and $f : i \to j$, a morphism in J, we write Fi, Fj for the objects F sends i, j to and $Ff : Fi \to Fj$ for the morphism f is sent to.

When we write diagrams they are always meant to commute. When we say something is dual we refer to the situation where all arrows are reversed. When we say a morphism $f : a \to b$ factors through d we mean that there exist morphisms $g : a \to d, h : d \to b$ such that $f = h \circ g$.

When talking about topological spaces maps are supposed to be continuous and we use the product topology for products and the subspace topology for any subspaces. A clopen set is a set that is both open and closed.

$\mathbb{N}$	natural numbers, including 0
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\subset$	any, not necessarily proper, subset
$\subsetneq$	proper subset
$A \setminus B$	$= \{a \in A \mid a \notin B\}$
$f^{-1}(A)$	for $f : X \to Y$ and $A \subset Y$, preimage of A under f
$\pi_i : X \to X_i$	usual projection to the i-th component, for $X = \prod_{i \in I} X_i$.
$\overline{A}$	for $A \subset X$ a topological space, closure of A in X
(a, b)	for $a < b \in \mathbb{R}$, open interval of the real line
$[a, b]$	for $a < b \in \mathbb{R}$, closed interval of the real line
$\hookrightarrow$	injections or monomorphisms
$\twoheadrightarrow$	surjections or epimorphisms
Set	category of sets with all set theoretic maps
Fin	full subcategory of **Set** of finite sets
Top	category of topological spaces with all continuous maps
CHaus	full subcategory of **Top** of compact Hausdorff spaces

Chapter 1

Profinite Sets

A fundamental element of condensed mathematics is the category of profinite sets. So, it is of interest what this category looks like. In fact, this category is equivalent to the category of totally disconnected compact Hausdorff spaces, often referred to as *Stone spaces*. This is due to the duality of Stone spaces to Boolean Algebras first shown by Stone [Sto37]. Further Reading can be found here [Joh86]. This chapter aims to show the equivalence of the categories of profinite sets and Stone spaces, just relying on the knowledge of basic category theory and set-theoretic topology.

1.1 Limits

The definitions and constructions of Limits, Colimits and other subjects from category theory are structured after *MacLane* [Mac98].

We begin with some well-known facts about functors and categories.

Lemma 1.1.1 ([Mac98, II.4.]). *Given a small category J and an arbitrary category C. Then the collection of all covariant functors from J to C together with natural transformations of functors forms a category.*

Proof. The composition of natural transformations is again a natural transformation and associative, this can be found in [Mac98, II.4.].

As J is a small category the collection of its objects forms a set. And the collection of all set theoretic maps from the set of objects of J into itself is again a set. Therefore, the natural transformations, a sub-collection of all set theoretic maps, form a set as well. □

The category constructed in the previous lemma is called *functor-category* and denoted by C^J. We refer to J as the *index category* and always mean a small category if we talk about an index category. If we have a functor $F : J \to C$, we say its *component-objects* are the objects Fj and if we have a natural transformation $\tau : F \to G$ of functors, we say $\tau_j : Fj \to Gj$ are its *components*.

Any category C has a canonical image in its functor category given by the *diagonal* functor $\Delta : C \to C^J$, where c is sent to the functor that has as its component-objects just c and as morphisms between just the identity. This is clearly a functor.

Example 1.1.2. Some examples for C an arbitrary category:

(i) Take $J = *$, the category consisting of one object and only the identity morphism, then a functor from J to C is equivalent to choosing an object in C, so $C^J = C$.

(ii) Take J to be any discrete category, i.e. it consists only of objects and their identities. Then C^J is the product category of C with itself, indexed by J, as any small discrete category is basically a set.

(iii) [Awo10, 1.6, 3. Arrow category] Take $J = \rightarrow$, the category with two objects and one non-identity arrow. Then a functor from J to C amounts to choosing an arrow in C. The morphisms in C^J are just the commutative diagrams. This is called the *arrow category*.

(iv) Take $J = \downarrow\downarrow$, the category with two objects and two non-identity morphisms with same domain and codomain, such arrows we call *parallel*. Then a functor from J to C equals to choosing two parallel arrows in C. So the category C^J consists of all combinations of parallel arrows $i_1, i_2 : a \to b$ and $j_1, j_2 : d \to e$ in C as objects. As morphisms it has arrows $f : a \to d, g : b \to e$ of C such that the resulting two diagrams commute:

$$\begin{tikzcd} a \arrow[r, "i_1"] \arrow[d, "f"] & b \arrow[d, "g"] \\ d \arrow[r, "j_1"] & e \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "i_2"] \arrow[d, "f"] & b \arrow[d, "g"] \\ d \arrow[r, "j_2"] & e \end{tikzcd}$$

(v) [Mac98, I.4.] For a fixed index-category J a morphism of functors F, G is a natural transformation τ. This means that for any i, j, objects in J, and $\phi : i \to j$, a morphism in J, the following diagram commutes:

$$\begin{tikzcd} i \arrow[d, "\phi"] \\ j \end{tikzcd} \qquad \begin{tikzcd} Fi \arrow[r, "\tau_i"] \arrow[d, "F\phi"] & Gi \arrow[d, "G\phi"] \\ Fj \arrow[r, "\tau_j"] & Gj \end{tikzcd}$$

Remark. As Examples 1.1.2 show, the functor category C^J can be used to define diagrams and their transformations of the form J in a category C. That is why, we also refer to these functors as *diagrams of the form of J*.

We say that a collection of maps $\theta_j : x \to Fj$ is *compatible* with F (a diagram of the form of J) if the diagrams we get for any $f : i \to j$ in J commute with those maps:

$$\begin{tikzcd} & Fi \arrow[d, "Ff"] \\ x \arrow[ur, "\theta_i"] \arrow[r, "\theta_j"'] & Fj \end{tikzcd} \qquad \begin{tikzcd} i \arrow[d, "f"] \\ j \end{tikzcd}$$

Definition 1.1.3 ([Mac98, III.4.])**.** Given a functor F from a small category J to a category C, a *limit* of F, if it exists, is an *limit object* $\varprojlim F$ together with *projection morphisms* $\phi_j : \varprojlim F \to Fj$ in C such that for any object x in C and compatible morphisms $\theta_j : x \to Fj$ there exists a unique $\psi : c \to \varprojlim F$ such that all diagrams, as seen below, commute for any $f : i \to j$ in J:

$$\begin{tikzcd}
\varprojlim F \arrow[r] \arrow[dr] & Fi \arrow[d, "Ff"] & i \arrow[d, "f"] \\
x \arrow[u, dashed, "\psi"] \arrow[r] \arrow[ur] & Fj & j
\end{tikzcd}$$

Remark. A *colimit* is the dual notion of a *limit.* It consists of the *colimit-object* $\varinjlim F$ and morphisms $\phi_j : Fj \to \varinjlim F$. It is characterized by a unique map from $\varinjlim F$ to any other object x with compatible maps $\theta_j : Fj \to x$ (where we mean compatible in a way which is dual to how we defined it above).

If the limit of F exists, we also say that $\varprojlim F$ is a limit of the *form of* J.

Example 1.1.4. Some examples of limits (and colimits) for an arbitrary category C and a functor $F : J \to C$, for more details see [Mac98, III.3,III.4]:

(i) Given an arbitrary index category J, the limit of Δc is the limit-object c with the identity as projections. We see this as any collection of morphisms $\theta_j : x \to c$ all compatible with the identity have to be the same morphism. One can see that this limit always exists.

(ii) Take J to be the discrete category of two objects. Then, the limit of two objects a, b, if it exists, is called the *product* $a \times b$ with the projections $p : a \times b \to a$, $q : a \times b \to b$. It has the universal property that given morphisms $f : x \to a$, $g : x \to b$ they factor through $a \times b$ via the same morphism denoted by (f, g):

$$\begin{tikzcd}
a & a \times b \arrow[l, "p"'] \arrow[r, "q"] & b \\
& x \arrow[ul, "f"] \arrow[u, dashed] \arrow[ur, "g"'] &
\end{tikzcd}$$

(or, in case of colimits, we get the dual version the *coproduct*)

(iii) [Mac98, III.4.] Take J to be the category that gives us the following diagram in C (we leave out the identities):

$$a \xrightarrow{f} d \xleftarrow{g} b$$

Then we denote the limit object, if it exists, by $a \times_d b$, we call it the *fibre product.* Additionally we have projections $p : a \times_d b \to a, q : a \times_d b \to b$ (We leave out the map to d as it is completely defined by p, q). And for any x with compatible maps $x \to a, x \to b$ it fulfills the universal property described by the following commutative diagram:

$$\begin{tikzcd}
x \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dashed] & & \\
& a \times_d b \arrow[r, "q"] \arrow[d, "p"] & b \arrow[d, "g"] \\
& a \arrow[r, "f"] & d
\end{tikzcd}$$

(iv) [Mac98, III.4.] Take $J = \downdownarrows$, then, if the limit $\langle d, e \rangle$ of two parallel arrows $f, g : a \to b$ exists, is called *equalizer*. So, each arrow $h : x \to a$ such that $f \circ h = g \circ h$ factors through d via $h' : x \to a$, so $h = e \circ h'$:

$$\begin{tikzcd}
x \arrow[r, "h"] \arrow[d, dashed, "h'"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
d \arrow[ur, "e"'] & &
\end{tikzcd}$$

(Or, in the case of colimits, we get the *coequalizer*.)

(v) In the case of an abelian category $\mathcal{A}$ the equalizer of an arrow $f : A \to B$ and the zero arrow $0 : A \to B$ is called the *kernel*. (Or, in the dual case, it is called the *cokernel*.)

We take a look at the *category of topological spaces* denoted by **Top**, with the continuous maps as morphisms.

Example 1.1.5. Let $C =$ **Top** and J the discrete category of two objects, as in Example 1.1.4(ii).

Then, applying the example we get that the product of two spaces A, B is a space $A \times B$ with continuous projections $p : A \times B \to A, q : A \times B \to B$ such that for any topological space X and any continuous maps $f : X \to A, g : X \to B$ there is a continuous map $(f, g) : X \to A \times B$ such that

$$p \circ (f, g) = f \text{ and } q \circ (f, g) = g.$$

As we can take X to be the one point space, the underlying set of $A \times B$ has to be the set theoretic product of A and B, because we have to have a unique preimage for each combination of $a \in A$ and $b \in B$. Now, applying the forgetful functor from **Top** to the category **Set**, we see that any set theoretic map fulfilling the universal property already has to be (f, g).

The condition that p, q are continuous defines a topology on $A \times B$ which has to be the topology of the product by its universal property. This topology is exactly the *product topology* we know from topology.

Remark. The product of two objects in **Top** always exist, as we can equip the set theoretic product with the product topology. Actually all limits exist because **Top** also has all equalizers, again the same as in set, but with the subspace topology. We will show this construction explicitly for projective limits in Proposition 1.3.1.

If the limit of F in C^J exists and we have a natural transformation of F, G in C^J, we can in a natural way define the limit of the natural transformation by its components.

Definition 1.1.6. Let $\langle \varprojlim F, v_j \rangle$ be the limit of a functor $F : J \to C$ and $\tau : F \to G$ a natural transformation of functors. Then, the *limit* of τ, denoted by $\varprojlim \tau$, is defined via its components $(\varprojlim \tau)_j = \tau_j \circ v_j$:

$$\begin{tikzcd}
\varprojlim F \arrow[r, "v_j"] \arrow[dr, "(\varprojlim \tau)_j"'] & Fj \arrow[d, "\tau_j"] \\
& Gj
\end{tikzcd}$$

Remark. We get that $\varprojlim \tau$ is compatible with G because τ is a natural transformation.

Here, we have defined the limit of a morphism as a morphism from $\varprojlim F$ to G, because we do not need the existence of the limit of G to do this. But if $\langle \varprojlim G, w_j \rangle$ exists, one readily sees by the universal property of the limit $\varprojlim G$ that $\varprojlim \tau$ gives rise to a morphism from $\varprojlim F$ to $\varprojlim G$ that composed with the w_j gives us $\varprojlim \tau$. By abuse of notation we call this morphism also $\varprojlim \tau$, it lets this diagram commute:

$$\begin{tikzcd}
\varprojlim F \arrow[r, "\varprojlim \tau"] \arrow[d, "v_j"] & \varprojlim G \arrow[d, "w_j"] \\
F_j \arrow[r, "\tau_j"] & G_j
\end{tikzcd}$$

So, if we have a functor category C^J and all limits the form of J exist, this leads us to an obvious functor $\varprojlim$ from C^J to C. This fact is formulated here:

Lemma 1.1.7. *Given a category C and an index category J such that all limits of the form of J exist, then $\varprojlim$ is a functor from C^J to C.*

Proof. By assumption and the discussion above we know that $\varprojlim F$ exists for all $F \in C^J$, so we know what $\varprojlim$ does on objects and morphisms (the natural transformations).

We have to check that $\varprojlim$ commutes with composition of morphisms. This is clear, if we extend the diagram from above, where each small square commutes, the whole diagram commutes as well:

$$\begin{tikzcd}
\varprojlim F \arrow[r, "\varprojlim \tau"] \arrow[d, "v_j"] & \varprojlim G \arrow[r, "\varprojlim \theta"] \arrow[d, "w_j"] & \varprojlim H \arrow[d, "u_j"] \\
F_j \arrow[r, "\tau_j"] & G_j \arrow[r, "\theta_j"] & H_j
\end{tikzcd}$$

As we have defined the limit of our morphisms by components it is now clear that $\varprojlim \theta \circ \varprojlim \tau = \varprojlim(\theta \circ \tau)$. $\square$

1.2 Cofiltered Limits

If we now do not admit arbitrary index categories J, but only certain ones, we can get a special kind of limit a *(co-)filtered limit*. The first part of this section is again oriented at *MacLane* [Mac98].

First, we consider directed posets, which form categories:

Definition 1.2.1 ([RZ10, 1.1])**.** A *directed poset* (directed partially ordered set) $(I, \preceq)$ is a set I with a binary relation $\preceq$ satisfying:

(i) $i \preceq i$ for all $i \in I$.

(ii) $i \preceq j, j \preceq k \implies i \preceq k$ for all $i, j, k \in I$.

(iii) $i \preceq j, j \preceq i \implies i = j$ for all $i, j \in I$.

(iv) for all $i, j \in I$ there exists $k \in I$ such that $i, j \preceq k$.

Remark. (i),(ii),(iii) are just the ordinary reflexivity, transitivity and anti-symmetry of a *poset*, but then (iv) gives our directed poset its direction.

And it is easy to see that a poset forms a category:

Lemma 1.2.2. *A directed poset is a category by taking its elements as objects and its relations as morphisms, this means, if $i \succeq j$, we interpret it as an arrow $i \to j$.*

Proof. (i) shows us that each element has a identity morphism, (ii) shows us that we can compose arrows and it is well defined. And, if we have $i \succeq j, j \succeq k, k \succeq l$, applying (ii) twice implies $i \succeq l$ and we get for the corresponding arrows

$$[(i \to j) \circ (j \to k)] \circ (k \to l) = (i \to l) = (i \to j) \circ [(j \to k) \circ (k \to l)],$$

because it clearly does not matter which elements we compare first. □

Remark. The direction of our arrows to make a directed poset into a category is arbitrary, $i \succeq j$ constituting an arrow $j \to i$ would work as well, it is dual and gives the opposite category. It makes a difference as soon as we start talking about limits of the form of this category.

Example 1.2.3. (i) Any totally ordered set is a directed set. e.g. $\mathbb{N}$:

$$\cdots \to 3 \to 2 \to 1 \to 0$$

(ii) The real plane $\mathbb{R}^2$ with $P \preceq Q$ if Q lies on the straight line (including endpoints) from the origin to P (and $0 \preceq 0$). This is a *directed poset* as it clearly satisfies (i),(ii) and (iii) and $P \preceq 0$ for all P.

(iii) The set of open covers of a topological space ordered by refinement (we say $\{U_i\}_I \preceq \{U_j\}_J$ if $\{U_i\}_I$ is a refinement of $\{U_j\}_J$), here the arrows are just those related to the ordering induced by refinement not the refinement maps between the covers.

We have a special case of the example (iii) above if we just take the finite disjoint open covers (then the covers are also closed, because their complement is open) of a space ordered by refinement. Then the morphisms between the covers actually correspond to the refining morphisms as there is always only one choice for a refining morphism because the covers are disjoint.

The categorical properties of categories we get from directed posets can be generalized to define (co-)filtered categories and thereby (co-)filtered limits (and colimits):

Definition 1.2.4 ([Mac98, IX.1])**.** A *filtered category* is a non-empty category J with the following properties:

(i) Given any two objects $i, j \in J$ there exists $k \in J$ and morphisms $i \to k$ and $j \to k$.

(ii) Given two parallel morphisms $u, v : i \to j$, then there exists $k \in J$ and a morphism $w : j \to k$ such that $w \circ u = w \circ v$.

A *cofiltered category* is the opposite category of a filtered category.

Example 1.2.5. Any directed poset is a cofiltered category (When seen as a category as in Lemma 1.2.2) as property (i) is derived from the existence of bigger elements (Definition 1.2.1(iv)) and there are only unique arrows so property (ii) is trivial.

Definition 1.2.6. A *(co-)filtered (co-)limit* is a (co-)limit of an object in the functor category C^J for J a (co-)filtered category.

Remark. Looking at a filtered limit, one easily sees that being filtered does not give us any relevant information about the limit, similarly for cofiltered colimits. Therefore, one only considers cofiltered limits and filtered colimits, which are often defined and referred to as "filtered" limits and colimits. But then one needs a contravariant functor from a filtered category to get a "filtered" limit. So when talking about "filtered" limits it is often implied that the functor from the filtered category is contravariant. To keep in line with our convention to only have covariant functors we will use cofiltered to stress that we use covariant functors from cofiltered categories. This is also in line with sources like [Stacks].

When proving things about general cofiltered limits it is tedious to always consider general cofiltered index categories. And in fact we can just use the subcollection of the categories of directed posets (As defined in 1.2.2), because a cofiltered limit is always isomorphic to the limit of the form of the opposite category of a directed poset. For this, see [Stacks, Lemma 0032].

If we are only using directed posets and not general cofiltered categories, there are only unique arrows (Example 1.2.5). Therefore, we will introduce some notation to simplify talking about our functors from cofiltered categories.

Definition 1.2.7. A *projective system* $\{X_i, \phi_{ij}, I\}$ in a category C is a diagram of the form of a cofiltered category I in the case our cofiltered category is a directed poset. The ϕ_{ij} denote the unique arrows in the image of our functor from I to C.

A *projective limit* $\langle \varprojlim X_i, \phi_i \rangle$ of a projective system $\{X_i, \phi_{ij}, I\}$ is its cofiltered limit.

Explicitly, for all $i \succeq j \in I$ we have a morphism $\phi_{ij} : X_i \to X_j$ such that the following diagram commutes:

$$\begin{tikzcd} \varprojlim X_i \arrow[r, "\phi_i"] \arrow[dr, "\phi_j"'] & X_i \arrow[d, "\phi_{ij}"] \\ & X_j \end{tikzcd}$$

In this text we will mostly consider cofiltered limits, as defined above. One important category in the context of this text is the category **Set** of all sets and its full subcategory **Fin** of finite sets.

Example 1.2.8. We consider the cofiltered category $J = \mathbb{N}$ and $C = \mathbf{Fin}$. We define a functor F by sending n to $\{\infty, 0, \ldots, n\}$ and the arrow $i \to j$, for $i > j$, to

$$\phi_{ij} : \{\infty, 0, \ldots, i\} \to \{\infty, 0, \ldots, j\}, x \mapsto \begin{cases} \infty & x > j \\ x & x \leq j \end{cases} .$$

At first we consider ∞ just as a symbol bigger than any other number, we will later see how it makes sense to call it infinity.

This functor does not have a limit in **Fin**, because the limit-object of such a limit would need a preimage of each natural number and can thereby not be finite.

But one sees that $\varprojlim F = \mathbb{N} \cup \{\infty\}$ with the obvious projections ϕ_i, when we consider it in the category **Set**.

Another similar but important example are the p-adic integers:

Example 1.2.9. Again, we take $J = \mathbb{N}^{op}$, $C = \mathbf{Fin}$, p a prime, and consider the functor F from $\mathbb{N}^{op}$ to **Fin**, sending n to $\mathbb{Z}/p^n\mathbb{Z}$ and $i \to j$ to the natural map

$$\phi_{ij} : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}, x \bmod p^i \mapsto x \bmod p^j.$$

Again, we see that there is no limit of this functor in **Fin**. But if we consider it as a subcategory of **Set**, we get a limit, namely

$$\mathbb{Z}_p = \{(x_n)_{n \in \mathbb{N}} | x_n \in \mathbb{Z} \text{ and } x_i \equiv x_j (\bmod p^j) \text{ for } j \leq i\},$$

the p-adic integers.

Remark. Obviously we have more structure on the p-adic integers than just those set theoretic maps, as all of those maps are also group-homomorphisms, but here we are not interested in those properties.

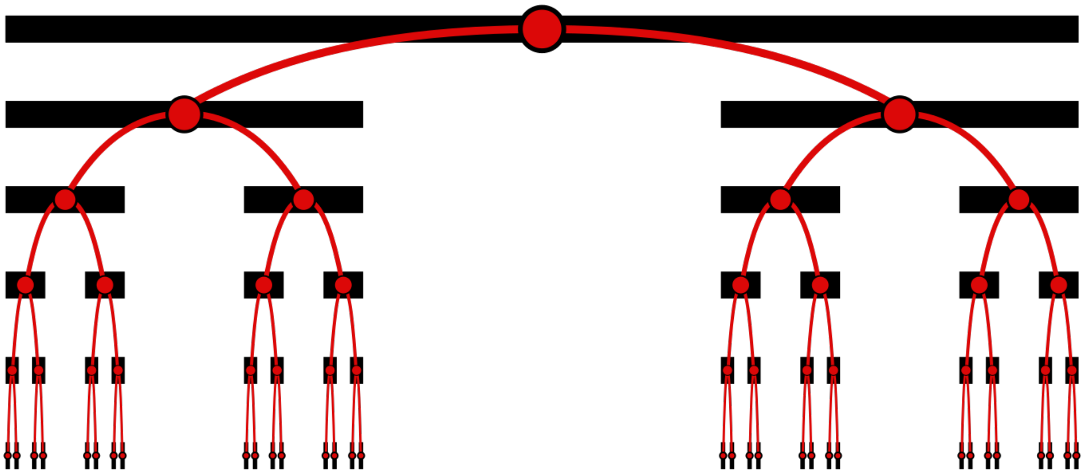

Figure 1.1: First six iterations of Cantor set construction
Source: `https://en.wikipedia.org/wiki/Cantor_set`

Example 1.2.10. A special case of Example 1.2.9 is $p = 2$. In this case we can look at $\mathbb{Z}/2^n\mathbb{Z}$ as the n-th iteration in the construction of the Cantor set and $\mathbb{Z}_2$ as the Cantor set, see Figure 1.1.

Until now, we have only looked at one index category at a time and then it is easy to compare limits if we have a component-wise map between our diagrams. But if we look at all objects that are cofiltered limits in a category C, we would like them to be a category, so we need morphisms for different index categories. Furthermore, we want to have all cofiltered limits to exists, which they often do not. As we saw in Examples 1.2.8 and 1.2.9 from above, the limits often do not exists in our original category, but only when looking at them as objects in **Set**. This method generalizes:

Definition 1.2.11 ([SGA4, 8.10.]). Given a category C then we define $Pro(C)$ to consist of *pro-objects* all formal cofiltered limits in C and define

$$Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{j\in J} Y_j) := \varprojlim_{j\in J} \varinjlim_{i\in I} C(X_i, Y_j).$$

For this to be a category we need a composition map on our hom-sets.

Proposition 1.2.12. *Given a category C, there is a composition map*

$$Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{j\in J} Y_j) \times Pro(C)(\varprojlim_{j\in J} Y_j, \varprojlim_{k\in K} Z_k) \to Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{k\in K} Z_k)$$

making $Pro(C)$ into a category, such that C is a faithful subcategory of $Pro(C)$ and the formal limit $\varprojlim X_i$ is the projective limit of the X_i in $Pro(C)$.

Proof. Based on an explanation by professor Huber. Let $(I, \preceq_I), (J, \preceq_J), (K, \preceq_K)$ be directed posets, then let the projective systems be given by

$$\{X_i, \phi_{il}, I\}, \{Y_j, \psi_{jn}, J\}, \{Z_k, \pi_{km}, K\}.$$

We first notice that for our conditions to hold, that

$$\theta \in \varprojlim_{j\in J} \varinjlim_{i\in I} C(X_i, Y_j) = Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{j\in J} Y_j)$$

has as data for each $j \in J$ an $i(j) \in I$ and a morphism $\theta_j : X_{i(j)} \to Y_j$ with the following property: for all $j' \in J$ with $j' \preceq_J j$ and there exists $i \in I$ such that $i \succeq_I i(j)$ and $i \succeq_I i(j')$ such that the diagram commutes

$$\begin{tikzcd}
 & X_{i(j)} \arrow[r, "\theta_j"] & Y_j \arrow[d, "\psi_{jj'}"] \\
X_i \arrow[ur, "\phi_{ii(j)}"] \arrow[dr, "\phi_{ii(j')}"'] & & \\
 & X_{i(j')} \arrow[r, "\theta_{j'}"] & Y_{j'}
\end{tikzcd}$$

Let

$$\theta \in Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{j\in J} Y_j) \text{ and } \tau \in Pro(C)(\varprojlim_{j\in J} Y_j, \varprojlim_{k\in K} Z_k)$$

We want to map (θ, τ) to

$$\rho = \tau \circ \theta \in Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{k\in K} Z_k).$$

Now for each $k \in K$ there exists $j(k) \in J$ for which there exists $i(k) := i(j(k)) \in I$, so let

$$\rho_k = \tau_k \circ \theta_{j(k)} : X_{i(k)} \to Z_k.$$

We need to check that it fulfills all needed properties. Let $k, k' \in K$ such that $k' \preceq_K k$. Take any $i \in I$ such that $i \succeq_I i(k)$ and $i \succeq_I i(k')$. There exists $j \in J$ such that $j \succeq_J j(k)$, $j \succeq_J j(k')$ and we can choose i big enough such that $i(j) \preceq_I i$. We get that all the inner subdiagrams from θ and τ and therefore the whole diagram commutes:

$$\begin{tikzcd}
 & X_{i(k)} \arrow[rr, "\theta_{j(k)}"] \arrow[rrr, bend left, "\rho_k"] & & Y_{j(k)} \arrow[r, "\tau_k"] & Z_k \arrow[dd, "\pi_{kk'}"] \\
X_i \arrow[ur, "\phi_{ii(k)}"] \arrow[r, "\phi_{ii(j)}"] \arrow[dr, "\phi_{ii(k')}"'] & X_{i(j)} \arrow[r, "\theta_j"] & Y_j \arrow[ur, "\psi_{jj(k)}"] \arrow[dr, "\psi_{jj(k')}"] & & \\
 & X_{i(k')} \arrow[rr, "\theta_{j(k')}"] \arrow[rrr, bend right, "\rho_{k'}"'] & & Y_{j(k')} \arrow[r, "\tau_{k'}"] & Z_{k'}
\end{tikzcd}$$

The way we have defined ρ the outmost diagram gives us the property we wanted and $\rho \in Pro(C)(\varprojlim_{i\in I} X_i, \varprojlim_{k\in K} Z_k)$. Thereby our composition map is well defined.

As we have defined the composition componentwise it is clear that it is associative, because it is associative in the components.

When all our systems are constant this clearly gives us our composition from C, so we can embed it into $Pro(C)$. The other property holds as well. To see this one just has to notice that the data of a compatible system of morphisms to a projective system X_i agrees with how we constructed our morphisms and clearly gives us a unique morphism to $\varprojlim X_i$ by taking the obvious morphism. □

Remark. We calculate the factorization through one object of our projective system, when we have a morphism from our projective limit later explicitly in Lemma 1.4.10.

An equivalent characterization is given by the following lemma:

Lemma 1.2.13 ([SGA4, 8.10.5.])**.** *Given a category C, then the category $Pro(C)$ is equivalent to the full subcategory of $(\mathbf{Set}^C)^{op}$ of those functors which are cofiltered limits of representable functors under the opposite Yoneda-embedding ($C \to (\mathbf{Set}^C)^{op}$).*

Remark. A whole discussion of this subject can be found in [SGA4, 8.] or [nLab, Pro-Object]. We will only use the characterization of our definition and therefore provided the lemma without proof.

With this we have arrived at the main object of study of this text the category of profinite sets:

Definition 1.2.14. We denote $Pro(\mathbf{Fin})$ by $\mathbf{ProFin}$. We call this the category of *profinite sets*.

Examples of objects in **ProFin** are all finite sets or Examples 1.2.8 and 1.2.9.

1.3 Projective Limits of Topological Spaces

Now that we have looked at a special case of our index category J, we look at C equal to **Top**, the *category of topological spaces* and their continuous maps, and full subcategories of **Top** by restricting to certain kinds of topological spaces, mainly compact Hausdorff spaces. This section is structured as in *Ribes and Zalesskii* [RZ10].

In the Appendix A some basic topological facts and tools are stated with references, in case they are not familiar to the reader. Having those facts at hand we start to examine projective systems and their limits in **Top**.

Proposition 1.3.1. *(i) Given a projective system $\{X_i, \phi_{ij}, I\}$ of topological spaces, there exists a projective limit $\langle X, \phi_i \rangle$ where*

$$X = \{(x_i) \in \prod_{i\in I} X_i \mid \phi_{ij}(x_i) = x_j \textit{ for all } j \preceq i\} \subset \prod_{i\in I} X_i$$

with the subspace topology and the $\phi_i = \pi_i|_X$ are the restrictions of the usual projection maps.

(ii) *This limit is unique in the sense that, if (X_i, ϕ_i) and (Y_i, ψ_i) are two limits, there is a unique homeomorphism between them compatible with the projections.*

Proof. (i) Extending the proof of [RZ10, Prop 1.1.1.]. We take X and ϕ_i as constructed above. By construction the ϕ_i are clearly continuous, as restriction of continuous maps, and compatible with the ϕ_{ij}.

We need to check the universal property. Given any topological space with compatible maps (Y, ψ_i) to our *projective system* $\{X_i, \phi_{ij}, I\}$ we construct a map

$$\psi : Y \to X, y \mapsto (\psi_i(y))_{i\in I}.$$

By construction $\phi_i \circ \psi = \psi_i$, so ψ is compatible with the restriction maps. We need to check that ψ is continuous. So given a basic open set U in X, $U = \prod_{i\in I} U_i \cap X$ with $U_i \subset X_i$ a basic open and all but finitely many $U_i = X_i$. Then

$$\psi^{-1}(U) = \bigcap_{i\in I} \psi_i^{-1}(U_i) = \bigcap_{i_1,\dots,i_n} \psi_i^{-1}(U_i)$$

as $\psi_i^{-1}(X_i) = Y$ and so $\psi^{-1}(U)$ is open and ψ continuous. One easily sees that ψ is unique by construction.

(ii) See [RZ10, Prop 1.1.1.].

□

Remark. This can also be viewed as a specific instance of the existence of arbitrary products and equalizers implying the existence of all limits.

By Lemma 1.1.7 and the previous Proposition 1.3.1 we have the functor $\varprojlim$ from $\mathbf{Top}^I$ to $\mathbf{Top}$ for any fixed poset I.

So we can start to look at which properties are preserved by this functor.

Definition 1.3.2 ([RZ10, 1.1])**.** A morphism θ of $\mathbf{Top}^I$ is called surjective if each of its components is surjective.

But surjectivity is not preserved by the projective limit as the following example of p-adic integers (Example 1.2.9) shows:

Example 1.3.3 ([RZ10, 1.1])**.** There are obvious surjective maps

$$\theta_i : \mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}, x \mapsto x \bmod p^i$$

which clearly commute with all the ϕ_{ij}, for $i > j$, as

$$\theta_j(x) = x \bmod p^j = (x \bmod p^i) \bmod p^j = \phi_{ij} \circ \theta_i(x)$$

and the image of $\theta = \varprojlim \theta_i$ in $\mathbb{Z}_p$ is the set of constant sequences. But

$$(x_n)_{n\in\mathbb{N}} = (1 + \cdots + p^{n-1})_{n\in\mathbb{N}}$$

is also in $\mathbb{Z}_p$ but not a constant sequence. So θ is not surjective.

This shows that we have to add some further restrictions such that surjectivity is preserved. It turns out that the spaces for which surjectivity is preserved are the compact Hausdorff spaces. But for that we first have to see how those properties behave under the projective limit:

Lemma 1.3.4. *Given a projective system $\{X_i, \phi_{ij}, I\}$ of Hausdorff spaces then its limit $\varprojlim X_i$ is a closed subspace of $\prod_{i\in I} X_i$.*

Proof. Rephrasing [RZ10, Lemma 1.1.2]. Take $(x_i) \in \prod_{i\in I} X \setminus \varprojlim X_i$. Then there are

$$j \preceq k \in I \text{ such that } \phi_{kj}(x_k) \neq x_j.$$

As X_j is Hausdorff we can choose disjoint neighborhoods $U_j, V_j \subset X_j$ of x_j and $\phi_{kj}(x_k)$ respectively. By continuity $V_k = \phi_{kj}^{-1}(V_j) \subset X_k$ is a neighborhood of x_k. Now

$$W = \prod_{i\in I} W_i \text{ such that } W_i = \begin{cases} X_i & \text{for } i \neq k \\ V_k & \text{for } i = k \\ U_j & \text{for } i = j \end{cases}$$

is a neighborhood of $(x_i)_{i\in I}$ and W is disjoint from $\varprojlim X_i$ as $\phi_{kj}(W_k) = \phi_{kj}(V_k) = V_j$ disjoint from $W_j = U_j$. So $\varprojlim X_i$ is closed as the complement of an open set. □

Remark. This can be seen as a generalization of the fact that given a Hausdorff space X the diagonal $\Delta X \subset X \times X$ is closed, as this is a special case of the above for $I = \{1, 2\}$ and $X_i = X$ for $i = 1, 2$.

For the next steps we will need a strong result from topology, the Tychonov theorem. As this theorem is by no means trivial, we will only provide references for it: [Mun00, Ch. 5] or [Stacks, Theorem 08ZU]

Theorem 1.3.5 (Tychonov)**.** *An arbitrary product of compact spaces is compact.*

Proposition 1.3.6 ([RZ10, Prop 1.1.3])**.** *Given a projective system $\{X_i, \phi_{ij}, I\}$ of compact Hausdorff spaces then its limit $\varprojlim X_i$ is a compact Hausdorff space.*

Proof. By Tychonov 1.3.5 $\prod_{i\in I} X_i$ is compact, by Lemma 1.3.4 $\varprojlim X_i$ is a closed subspace of this compact space and so it is compact by Lemma A.1.2.

By Lemma A.1.1 $\prod_{i\in I} X_i$ is Hausdorff and as a subspace of a Hausdorff space $\varprojlim X_i$ is also Hausdorff. □

We have seen that being compact Hausdorff is preserved by the projective limit and arrive at the following result. This turns out to solve the problem of preservation of surjectivity.

Proposition 1.3.7. *Given a projective system $\{X_i, \phi_{ij}, I\}$ of non-empty compact Hausdorff spaces then the projective limit $\varprojlim X_i$ is non-empty.*

Proof. Giving the proof of [RZ10, Prop 1.1.4] in more detail. For each $j \in I$ let

$$Y_j = \{(y_i)_{i\in I} \mid \phi_{jk}(y_j) = y_k \ \forall\ j \succeq k \in I\} \subset \prod_{i\in I} X_i.$$

By the same argument as in Lemma 1.3.4 Y_j is closed. It is non-empty because we can choose a $x_j \in X_j$ and take y_i to be $\phi_{jk}(x_j)$ for all $k \preceq j$ and any other value in X_i for all other i. (Here we need the X_i to be non-empty)

Let $j \preceq j'$ and $(y_i) \in Y_{j'}$, this means that $\phi_{j'k}(y_{j'}) = y_k$ for all $k \preceq j'$. Because $\phi_{j'k} = \phi_{jk} \circ \phi_{j'j}$ and in particular $j \preceq j'$, so $\phi_{j'j}(y_{j'}) = y_j$, we get that for all $k \preceq j$

$$y_k = \phi_{j'k}(y_{j'}) = \phi_{jk}(\phi_{j'j}(y_{j'})) = \phi_{jk}(y_j).$$

So $(y_i) \in Y_j$ and $Y_{j'} \subset Y_j$.

We notice that the collection $\{Y_i\}_{i\in I}$ fulfills the finite intersection property, because given any finite subset indexed by $\{i_1, \dots i_n\} \subset I$ there is a $k \in I$ such that $i_1 \dots i_n \preceq k$ and by the assertion above this means that $Y_k \subset \bigcap_{j=1,\dots,n} Y_{i_j}$ and as Y_k nonempty the intersection is non-empty as well.

By Tychonov 1.3.5 $\prod_{i\in I} X_i$ is compact, so by Lemma A.1.6 the intersection $\bigcap_{i\in I} Y_i$ is nonempty and we can see by our construction in Lemma 1.3.1, that $\bigcap_{i\in I} Y_i = \varprojlim X_i$. So $\varprojlim X_i$ is non-empty. □

This solves the problem we encountered in Example 1.3.3.

Theorem 1.3.8. *Given a surjective morphism $\theta :\{X_i, \phi_{ij}, I\} \to \{X'_i, \phi'_{ij}, I\}$ of projective systems of compact Hausdorff spaces, then the morphism $\varprojlim \theta_i : \varprojlim X_i \to \varprojlim X'_i$ is surjective.*

Proof. Restating [RZ10, Lemma 1.1.5.]. Take any $(x'_i) \in \varprojlim X'_i$. In Hausdorff spaces one point sets are closed so by continuity $\tilde{X}_i := \theta_i^{-1}(\{x'_i\}) \subset X_i$ is closed. Because X_i is compact Lemma A.1.2 tells us that $\tilde{X}_i$ is compact. It is also Hausdorff because X_i is Hausdorff. And we observe that $\phi_{ij}(\tilde{X}_i) \subset \tilde{X}_j$, because θ is compatible with the ϕ_{ij}:

$$\theta_j \circ \phi_{ij}(\tilde{X}_i) = \phi'_{ij} \circ \theta_i(\tilde{X}_i) = \phi'_{ij}(x'_i) = x'_j.$$

So $\{\tilde{X}_i, \phi_{ij}, I\}$ is a projective system of non-empty compact Hausdorff spaces, so by Proposition 1.3.7 $\varprojlim \tilde{X}_i$ is nonempty. Choose any $(x_i) \in \varprojlim \tilde{X}_i \subset \varprojlim X_i$, this is a preimage for (x'_i). □

Corollary 1.3.9 ([RZ10, Cor 1.1.6])**.** *Given a compact Hausdorff space X and surjective maps $\theta_i : X \to X_i$ to a projective system $\{X_i, \phi_{ij}, I\}$ of compact Hausdorff spaces, the induced map $\Theta : X \to \varprojlim\ X_i$ is surjective.*

Proof. Take the surjective morphism $\theta : \Delta X \to \{X_i, \phi_{ij}, I\}$ of inverse systems that is induced by the θ_i, then apply Theorem 1.3.8. So we have a surjective map (use Example 1.1.4(i)) and get equality to Θ by uniqueness of the map to the limit.

$$\Theta = \varprojlim \theta : X = \varprojlim \Delta X \to \varprojlim X_i$$

□

1.4 Profinite Sets

We have already seen the category **ProFin** of profinite sets in Example 1.2.14 and are ready to show the equivalence of categories to the category of totally disconnected compact Hausdorff spaces, the so called Stone spaces. If the reader is not familiar with connectedness, all necessary material is stated and referenced in Appendix A.2.

Definition 1.4.1. A *profinite space* is a topological space which is the projective limit of a projective system of finite spaces equipped with the discrete topology.

We will use profinite space and profinite set interchangeably as the categories of finite sets and of discrete finite spaces are clearly equivalent and thereby each profinite set is also a profinite space.

Example 1.4.2. Any finite space in the discrete topology is profinite as it is the projective limit of the constant system.

To characterize these profinite spaces we need to introduce some more topology:

Definition 1.4.3. A topological space is *totally disconnected* if every point is its own connected component.

Clearly totally disconnectedness of a space X is equivalent to saying that any subspace of X consisting of more than two points has a separation.

We formulate some useful results for totally disconnected spaces:

Lemma 1.4.4. *An arbitrary product of totally disconnected spaces is totally disconnected. A subspace of a totally disconnected space is totally disconnected.*

Proof. For all $i \in I$ let X_i be totally disconnected and let $W \subset \prod_{i \in I} X_i$ such that it contains at least two distinct points $(x_i), (y_i)$. Then there is an index $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$ and as X_{i_0} is totally disconnected there is a separation U_{i_0}, V_{i_0} of $W_{i_0} = \pi_{i_0}(W) \supset \{x_{i_0}, y_{i_0}\}$ (projection of W to X_{i_0}). Then

$$U = \prod_{i \in I} U_i \text{ with } U_i = \begin{cases} \pi_i(W) & i \neq i_0 \\ U_{i_0} & i = i_0 \end{cases}, V = \prod_{i \in I} V_i \text{ with } V_i = \begin{cases} \pi_i(W) & i \neq i_0 \\ V_{i_0} & i = i_0 \end{cases}$$

is a separation of W. This shows that any set with two or more points is not connected, so the connected components are the one point sets and $\prod_{i \in I} X_i$ is totally disconnected.

Let X be totally disconnected and $Y \subset X$ a subspace. Y is also totally disconnected as the connected components of Y are always contained in the intersection of a connected component of X with Y, so they can only become smaller. As the connected components of X are already just the points, those are the connected components of Y. □

Remark. As equalizers in **Top** are always subspaces, this lemma tells us that the full category of totally disconnected spaces has arbitrary products and equalizers and thereby all limits. We formulate this for our case:

Corollary 1.4.5. *Given a projective system $\{X_i, \phi_{ij}, I\}$ of totally disconnected spaces then $\varprojlim X_i$ is totally disconnected.*

Proof. By Lemma 1.4.4 $\varprojlim X_i$ is totally disconnected as a subspace of a totally disconnected space $\prod_{i \in I} X_i$. $\square$

Example 1.4.6. Lemma 1.4.4 yields many examples of totally disconnected spaces, in particular the p-adic integers from Example 1.2.9, $\mathbb{N} \cup \infty$ from Example 1.2.8, and the Cantor set from Example 1.2.10. When looking at the Cantor set one can see the topology it inherits as a subspace from the unit interval very well. In general all profinite sets are totally disconnected (compact Hausdorff), because finite discrete spaces are all clearly totally disconnected (compact Hausdorff) and those properties are preserved by the projective limit

The next example, important for condensed mathematics, was explained to me by Reid Barton and Johan Commelin:

Example 1.4.7. We will look at $\mathbb{N} \cup \infty$ explicitly. We remind of the construction in Example 1.2.8. As a subspace of the product its topology has a basis consisting of

$$\{\{n\} = \phi_i^{-1}(n) | n \in \mathbb{N} \text{ and } i \geq n\} \cup \{\{n+1, n+2, \ldots, \infty\} = \phi_n^{-1}(\infty) | n \in \mathbb{N}\}.$$

We can see from this topology that $\mathbb{N} \cup \infty$ is totally disconnected as any set U that contains more than one point contains an $n \in \mathbb{N}$ so $\{n\}, U \setminus \{n\}$ is a separation, because $\{n\}$ is clopen.

This is in a way the classification of a converging sequence in a topological space X. To see this we take a continuous map

$$f : \mathbb{N} \cup \infty \to X, n \mapsto x_n, \infty \mapsto x.$$

As f is continuous given any neighborhood U of x then $f^{-1}(U)$ is a neighborhood of ∞ and by the discussion above contains all $n > n_0$ for some $n_0 \in \mathbb{N}$. So, U contains all x_n for $n > n_0$. This coincides with the definition of

$$x_n \xrightarrow[n \to \infty]{} x$$

in metric spaces and even in topological spaces where the limit is well defined.

We now know that profinite sets are totally disconnected compact Hausdorff spaces, for the other direction, we need one more result from topology:

Lemma 1.4.8. *In a compact Hausdorff space X the intersection of all clopen neighborhoods of a point x is the connected component containing x.*

Proof. Rephrasing [RZ10, Lemma 1.1.11]. Let C be the connected component of x and $\{W_t\}_{t \in T}$ be the collection of all clopen neighborhoods of x. Then let

$$A = \bigcap_{t \in T} W_t.$$

If C were not a subset of A, we would have a clopen neighborhood W_{t_0} of x not containing C. Then $W_{t_0} \cap C$ and $C \setminus W_{t_0}$ are two subsets of C, which are non-empty, because $C \not\subset W_{t_0}$ and open. This would be a separation, a contradiction to C being the connected component of x. Therefore, $C \subset A$.

To see that $A \subset C$, it is enough to show that A is connected. For this let $Y, Z \subset A$ be closed in A such that $Y \cap Z = \emptyset, Y \cup Z = A$. As A is an intersection of closed sets, Y, Z are closed in the compact space X and thereby compact. By Lemma A.3.5 a compact Hausdorff space is normal, so we can separate Y, Z by disjoint open sets U, V. (Explicitly $Y \subset U, Z \subset V$ and $U \cap V = \emptyset$)

We look at

$$X \setminus (U \cup V) \subset X \setminus A = X \setminus (\bigcap_{t \in T} W_t) = \bigcup_{t \in T} (X \setminus W_t)$$

where the $X \setminus W_t$ are open. So $\{X \setminus W_t\}_{t \in T}$ is an open cover of $X \setminus (U \cup V)$. As $X \setminus (U \cup V)$ is closed it is compact and we can choose a finite subset $T' \subset T$, such that $\{X \setminus W_t\}_{t \in T'}$ is still an open cover of $X \setminus (U \cup V)$.

Therefore, the set

$$B = \bigcap_{t \in T'} W_t$$

is disjoint from $X \setminus (U \cup V)$ so $B \subset U \cup V$. As T' is finite and the W_t are clopen, B is clopen. And, as all W_t contain x, we clearly have

$$x \in B = (B \cap V) \dot{\cup} (B \cap U),$$

which is a disjoint union because V, U are disjoint. Without loss of generality assume that $x \in B \cap V$. Then $B \cap V$ is open because V and B are open. $B \cap V$ is closed in B, because we have

$$B \setminus (B \cap V) = B \cap U$$

and U is open. This makes it also closed in X because B is closed. So $B \cap V$ is a clopen neighborhood of x. By the construction of A as the intersection of all clopen neighborhoods of x we have $A \subset B \cap V \subset V$. But $Y \cap V = \emptyset$ and, because $A \subset V$, $Y \cap A = \emptyset$. As $Y \subset A$ we get that $Y = \emptyset$. □

Theorem 1.4.9. *The profinite spaces are exactly the totally disconnected compact Hausdorff spaces.*

Proof. Adapted from [Stacks, Lemma 08ZY]. A topological space that is finite and discrete is clearly a totally disconnected compact Hausdorff space as it is the finite disjoint union of all its points which are clopen sets. Now, take a projective system $\{X_i, \phi_{ij}, I\}$ of finite discrete spaces X_i then by Proposition 1.3.6 and Corollary 1.4.5 its limit $\varprojlim X_i$, as a closed subspace of the product of the X_i, is also a totally disconnected compact Hausdorff space. Being such a limit, every profinite set is a totally disconnected compact Hausdorff space.

To see the other direction, let X be a totally disconnected compact Hausdorff space. We take $\mathcal{I}$ to be the set of all finite disjoint open union decompositions. This means for any $I \in \mathcal{I}$ we have

$$X = \coprod_{i \in I} U_i$$

with U_i non-empty and open for all $i \in I$. It follows that the U_i are closed because it is the complement of the union of the others.

We define a partial order on $\mathcal{I}$ by setting $J \preceq I$ if I is a refinement of J. This gives us a projective system as we can always find a common refinement of I, J by taking the $K \in \mathcal{I}$ corresponding to the finite disjoint union decomposition we get by intersecting all $\{U_i\}_{i \in I}$ with the $\{U_j\}_{j \in J}$. The decomposition K will be again finite, because we only intersect finitely many sets, disjoint, and clearly $J, I \preceq K$. We equip $I \in \mathcal{I}$ with the discrete topology which makes I into a finite discrete space. Refinement induces maps

$$\phi_{IJ} : I \to J, \; i \mapsto j \text{ if } U_i \subset U_j, \quad \forall J \preceq I \in \mathcal{I},$$

which are clearly compatible with each other. As the covers of X are disjoint those maps are unique and they are continuous because the I, J are all discrete. So we have a projective system of finite discrete spaces and so their limit $\varprojlim \mathcal{I}$ is a profinite space.

The maps

$$\theta_I : X \to I, x \mapsto i \in I \text{ if } x \in U_i$$

are continuous because the $U_i = \theta_I^{-1}(i) \subset X$ are clopen and are clearly compatible with the transition maps. By the universal property of the limit we get a continuous map

$$\Theta : X \to \varprojlim \mathcal{I}.$$

We claim this map is a homeomorphism.

As X is in particular compact and $\varprojlim \mathcal{I}$ in particular Hausdorff any bijective continuous map is a homeomorphism by Lemma A.1.5, so it is enough to show that Θ is bijective.

For surjectivity see Corollary 1.3.9 as the θ_I are surjective for all $I \in \mathcal{I}$.

For injectivity let $x, y \in X$ be distinct. We notice that by Lemma 1.4.8 the connected component C of any $x \in X$ equals the intersection of all clopen neighborhoods of x. As our space X is totally disconnected $C = \{x\}$. So there is a clopen neighborhood U of x which does not contain y (We cannot use C as it is not necessarily open). Setting $V = X \setminus U$ we get a finite union decomposition, where $X = U \coprod V$. By construction of $\mathcal{I}$ there is a $I_0 \in \mathcal{I}$ to this separation. Clearly

$$\theta_{I_0}(x) \neq \theta_{I_0}(y).$$

Therefore, x and y cannot have the same image in $\varprojlim \mathcal{I}$ under Θ because then θ_{I_0} could not factor through $\varprojlim \mathcal{I}$ via Θ. So Θ is injective.

We conclude that any totally disconnected compact Hausdorff space is homeomorphic to a profinite set. $\square$

The above Theorem 1.4.9 now tells us that we have a functor from profinite sets to the subcategory of totally disconnected compact Hausdorff spaces of **Top** and that the functor is essentially surjective.

This results in an equivalence of categories. We prove an important property of profinite sets to show this:

Lemma 1.4.10. *Given a profinite set $\varprojlim X_i$, a projective limit of the projective system $\{X_i, \phi_{ij}, I\}$ considered as a topological space, then any continuous map f from $\varprojlim X_i$ to a finite discrete space Y factors through an X_{i_0}.*

Proof. Let $Y = \{y_1, \ldots, y_n\}$. Then by continuity we have for each k a clopen set $U_k := f^{-1}(y_k)$ disjoint from U_l for all $l \neq k$. They cover $\varprojlim X_i$, so

$$\varprojlim X_i = \coprod_{k=1}^{n} U_k.$$

Now cover each U_k by basic open sets of $\varprojlim X_i$ contained in U_k, this is possible because U_k is open. As $\varprojlim X_i$ is compact so is U_k as a closed subset. So we can choose a finite cover $\{U_{k,j}\}_{1 \leq j \leq m_k}$ of U_k.

By the construction of $\varprojlim X_i$ as the subspace of the product we get that each basic open set $U_{k,j}$ is the preimage of a basic open set of an $X_{i_{k,j}}$ under the projection $\phi_{i_{k,j}}$. Because $X_{i_{k,j}}$ is discrete the basic open sets are just the points so

$$U_{k,j} = \phi_{i_{k,j}}^{-1}(x_{k,j}), x_{k,j} \in X_{i_{k,j}}.$$

As I is directed we can choose an upper bound i_0 of all those $i_{k,j}$ (there are only finitely many!). For $k \neq l$ take any $x_k \in U_{k,j} \subset U_k$ and $x_l \in U_l$ then $\phi_{i_0}(x_k) \neq \phi_{i_0}(x_l)$ because

$$x_l \in U_l \subset X \setminus U_k \subset X \setminus U_{k,j} \implies x_l \notin U_{k,j} = \phi_{i_{k,j}}^{-1}(x_{k,j}).$$

So the images of $\phi_{i_0}(x_k), \phi_{i_0}(x_l)$ under $\phi_{i_0, i_{k,j}}$ are different, so they have to differ. This we can see in the diagram below (where $\phi_0 = \phi_{i_0}$, $\phi_{k,j} = \phi_{i_{k,j}}$ and $\phi_{0kj} = \phi_{i_0 i_{k,j}}$):

$$\begin{tikzcd}
\varprojlim X_i \arrow[r, "\phi_0"] \arrow[dr, "\phi_{kj}"'] & X_{i_0} \arrow[d, "\phi_{0kj}"] & \ni \phi_0(x_k), \phi_0(x_l) \arrow[d, mapsto] \\
& X_{i_{k,j}} & \ni x_{k,j} \neq \phi_{kj}(x_l)
\end{tikzcd}$$

So X_{i_0} distinguishes points from different U_k. This gives a well defined map

$$g : X_{i_0} \to Y, x \mapsto \begin{cases} y_k & \text{for } x \in \phi_{i_0}(U_k) \\ y_1 & \text{otherwise} \end{cases}.$$

It is continuous because X_{i_0} is discrete. By construction $g \circ \phi_{i_0} = f$. So X_{i_0} was the one we were looking for. $\square$

Theorem 1.4.11. *The categories of profinite sets and totally disconnected compact Hausdorff spaces are equivalent.*

Remark. As the category of totally disconnected compact Hausdorff spaces is a full subcategory of **CHaus** and the continuous maps between discrete spaces are exactly the set theoretic maps, we can just use the hom-functor of **CHaus** for those hom-sets.

Proof. In Theorem 1.4.9 we have seen that the obvious functor from profinite sets to totally disconnected compact Hausdorff spaces is essentially surjective. We need to show that it is fully faithful.

Let $\{X_i, \phi_{ik}, I\}$ and $\{Y_j, \psi_{jl}, J\}$ be projective systems of finite sets and $\varprojlim_{i\in I} X_i$, $\varprojlim_{j\in J} Y_j$ their corresponding profinite sets. By Theorem 1.4.9 we identify $\varprojlim_{i\in I} X_i$, $\varprojlim_{j\in J} Y_j$ with their totally disconnected compact Hausdorff spaces.

We need to show that the map of hom-sets is bijective. By definition of the Pro-category (Definition 1.2.11) we need to show the first bijection

$$\mathbf{CHaus}(\varprojlim_{i\in I} X_i, \varprojlim_{j\in J} Y_j) \overset{!}{\cong} \varprojlim_{j\in J} \varinjlim_{i\in I} \mathbf{CHaus}(X_i, Y_j) = \mathbf{ProFin}(\varprojlim X_i, \varprojlim Y_j).$$

$\mathbf{CHaus}(\varprojlim X, -)$ is a covariant functor and commutes by the universal property of the limit with all limits, so we get the bijection

$$\mathbf{CHaus}(\varprojlim X_i, \varprojlim Y_j) = \varprojlim_{j\in J} \mathbf{CHaus}(\varprojlim X_i, Y_j).$$

(This is true in any category. For details see Proposition B.2.2.) $\mathbf{CHaus}(-, Y_j)$ is a contravariant functor. And by precomposition with the ϕ_i and the universal property of the colimit we get an injection

$$\varinjlim_{i\in I} \mathbf{CHaus}(X_i, Y_j) \hookrightarrow \mathbf{CHaus}(\varprojlim X_i, Y_j).$$

(This can also be done in any category. For a concrete discussion see Proposition B.2.3) Now we get to the crucial point, which is not true in a general category, the surjectivity of the above map. We have seen in the previous Lemma 1.4.10 that any continuous map from $\varprojlim X_i$ to a finite discrete space Y_j factors through an $X_{i(j)}$. Thereby it determines an element of the filtered colimit $\varinjlim_{i\in I} \mathbf{CHaus}(X_i, Y_j)$. This shows that any $f \in \mathbf{CHaus}(\varprojlim X_i, Y_j)$ has a preimage in $\varinjlim_{i\in I} \mathbf{CHaus}(X_i, Y_j)$. We get a bijection

$$\varprojlim_{j\in J} \mathbf{CHaus}(\varprojlim X_i, Y_j) \cong \varprojlim_{j\in J} \varinjlim_{i\in I} \mathbf{CHaus}(X_i, Y_j).$$

Together with the first bijection this concludes the proof. □

Remark. As noted before we also have a duality of the category of Stone spaces and Boolean algebras. This is originally due to Stone [Sto36] and can also be found in [Joh86].

Chapter 2

The Stone-Čech Compactification

In this chapter we will construct the Stone-Čech compactification of completely regular spaces and explain how one can map any topological space onto a completely regular space and retain some invariants. To do this we will introduce the ring of continuous real-valued functions, which originally is the main motivation for most of the work here. The main reference will be *Munkres* [Mun00] and we will also use *Gillman, Jerison* [GJ60] and *Walker* [Wal74]. The main results are mostly due to *Čech* [Cec37].

Most details of this chapter will not be relevant for the following chapters, but we will make use of the fundamental result of this chapter: The universal property of the Stone-Čech compactification stated in Theorem 2.3.8.

Again we remind the reader that there are some topological facts stated in the Appendix A, especially Appendix A.3 may be interesting in the following sections.

2.1 Continuous Real-Valued Functions

Definition 2.1.1 ([GJ60, 1.3,1.4])**.** Given a topological space X, we define $C(X)$ to be all continuous functions from X to $\mathbb{R}$ and $C^*(X)$ to be all bounded continuous functions to $\mathbb{R}$.

Remark. $C(X)$ has an obvious structure as a ring and $C^*(X)$ is a subring of $C(X)$, but this will not be of interest in this text.

Definition 2.1.2 ([GJ60, 1.16])**.** Given an embedding $e : X \to Y$ of topological spaces, we call it an $C(X)$*-embedding* ($C^*(X)$*-embedding*) if every $f \in C(X)$, ($f \in C^*(X)$) extends to a $g \in C(Y)$ ($g \in C^*(Y)$). See the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\exists g"] \\ & \mathbb{R} \end{tikzcd}$$

It is easy to find trivial examples of such embeddings:

Example 2.1.3. Take Y to be a manifold, X a connected component of Y and e the inclusion. Then e is a $C(X)$-embedding, as we can define continuous functions independently on all connected components of Y, so we extend each function on X to a function on Y.

On the other hand many familiar embeddings do not satisfy either property:

Example 2.1.4 ([GJ60, 1.16.]). Take $X = \mathbb{R} \setminus \{0\} \subset Y = \mathbb{R}$, then X is not even C^*-embedded, as the bounded function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

clearly cannot be extended continuously to $\mathbb{R}$.

This leads us to Urysohn's Extension Theorem a criterium for when a subset $S \subset X$ is a $C^*(S)$-embedding, we will need one more definition to formulate it.

Definition 2.1.5 ([GJ60, 1.15]). Given a topological space X, then two subsets $A, B \subset X$ are said to be *completely separated* if there exists $f \in C(X)$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Theorem 2.1.6 (Urysohn's Extension Theorem,[GJ60, 1.17]). *A subset $S \subset X$ is $C^*(S)$-embedded in X if and only if any two completely separated sets in S are completely separated in X.*

Theorem 2.1.7 (Urysohn's Lemma,[Mun00, Theorem 33.1]). *Let X be a normal space (see Definition A.3.4) then any two disjoint closed set A, B are completely separated.*

2.2 Completely Regular Spaces

As we will see in the next section the Stone-Čech compactification is only defined for completely regular spaces, so in this part we will see what a completely regular space is and how we can still apply certain results to arbitrary topological spaces.

Definition 2.2.1 ([Mun00, §33]). A space X where one point sets are closed is called *completely regular* if given any point $x_0 \in X$ and any closed subset $x_0 \notin A \subset X$, $\{x_0\}, A$ are completely separated.

Remark. Replacing $\{x_0\}$ by any closed set B disjoint from A is the definition of *completely normal*. Though this is not a very relevant characterization anymore as clearly any completely normal space is normal and by Urysohn's Lemma 2.1.7 any normal space is completely normal.

One also sees that this separating function can be chosen to be bounded, because we can set any value higher than one to one and any value lower than zero to zero and still have a continuous function fulfilling our properties.

Example 2.2.2. Any metric space (X, d_X) is completely regular. It is Hausdorff so one point sets are closed. As d_X is continuous so is

$$f(x) = \frac{d_X(x_0, x)}{d_X(x_0, x) + d_X(A, x)} \text{ where } d_X(A, x) = \inf_{a \in A}(d_X(a, x)).$$

One sees that f is well defined as by [Dug73, IX 4.2.] $d_X(A, x) = 0$ if and only if $x \in \overline{A} = A$:

$$\begin{aligned} x \in \overline{A} &\iff \forall \epsilon > 0 : \ B_\epsilon(x) \cap A \neq \emptyset \\ &\iff \forall \epsilon > 0 \ \exists a_\epsilon \in A : d_X(x, a_\epsilon) < \epsilon \\ &\iff 0 = \inf_{\epsilon > 0} d_X(x, a_\epsilon) \geq \inf_{a \in A} d_X(x, a) \geq 0 \\ &\iff d_X(x, A) = 0. \end{aligned}$$

As $\{x_0\}, A$ are disjoint the denominator is never zero, so f is well defined and continuous. Applying f to points in A and $\{x_0\}$ shows that it fulfills the necessary properties.

Remark. This shows actually more than that a metric space is completely regular. If we replace $\{x_0\}$ by a closed set B disjoint from A the same calculations show that metric spaces are normal.

The example shows that in particular $\mathbb{R}$ is completely regular.

Lemma 2.2.3. *Any subspace Y of a completely regular space X is completely regular.*

Proof. If points in X are closed, so are points in Y. Given a point x_0 and a closed set $A \cap Y$ in Y then A is closed in X and by complete regularity of X we get a "separating" function f. The restriction $f|_Y$ is continuous and separates $\{x_0\}$ and $A \cap Y$. □

Theorem 2.2.4 ([Mun00, §33])**.** *A compact Hausdorff space is completely regular.*

Proof. By Theorem A.3.5 every compact Hausdorff space is normal and by Urysohn's lemma 2.1.7 any disjoint closed sets in a normal space can be completely separated. Finally, as points are closed, the space is completely separated. □

We prove a lemma we need for the next proof.

Lemma 2.2.5. *Given a set X and a non-empty collection C of maps from X to $\mathbb{R}$ such that for any $f, g \in C$ and $c \in \mathbb{R}$ then $f - c, -f$ and* $\max f, g$ *are again in C.*

We have a topology on X, which has all preimages of closed sets of $\mathbb{R}$ under any $f \in C$ as a subbasis for the closed sets. This is also the weakest topology such that all functions in C are continuous.

Then the preimages of $[0, \infty)$ under any $f \in C$ form a basis for the closed sets. In particular the preimages of all closed sets under any $f \in C$ already form a basis for the closed sets of the topology.

Proof. This is restructured from [GJ60, 3.].

We have a subbasis for our topology consisting of all preimages of closed sets in $\mathbb{R}$ under any $f \in C$. As the collection of closed rays

$$\{(-\infty, a] | a \in \mathbb{R}\} \cup \{[b, \infty) | b \in \mathbb{R}\}$$

forms a subbasis of $\mathbb{R}$ it is enough to consider preimages of those rays to still have a subbasis. We can reduce the collection of sets we have to take preimages of again. We can assume a, b to be zero because $f - a, f - b$ are again in C because

$$f^{-1}((-\infty, a]) = (f - a)^{-1}((-\infty, 0]), f^{-1}([b, \infty)) = (f - b)^{-1}([b, \infty)).$$

Similarly we reduce one step further and just take rays of the form of $[0, \infty)$, because if f is in C so is $-f$. The union of any such preimages is clearly such a preimage because $\max\{f, g\}$ is in C and

$$f^{-1}([0, \infty)) \cup g^{-1}([0, \infty)) = (\max\{f, g\})^{-1}([0, \infty)).$$

So, those preimages actually form a basis for our closed set and so the preimages of closed sets in general also form a basis of the topology. $\square$

Theorem 2.2.6. *Given any topological space X there is a completely regular space ρX and a continuous surjection $\eta : X \to \rho X$ such that any $f \in C(X)$ factors through ρX.*

Proof. Oriented at [Wal74, 1.6]. Let ρX be the set of equivalence classes of points in X where $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in C(X)$ and let

$$\eta : X \to \rho X, x \mapsto \tilde{x}$$

be the natural surjective mapping of x to its equivalence class $\tilde{x}$. As all $f \in C(X)$ are constant on all members of $\tilde{x} \in \rho X$ this defines a function ρf such that $\rho f \circ \eta = f$:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & \rho X \arrow[d, dashed, "\rho f"] \\ & \mathbb{R} \end{tikzcd}$$

As in lemma 2.2.5, we equip ρX with the topology generated by the collection of ρf for $f \in C(X)$. As for any $f, g \in C(X)$ $f - c, -f, \max f, g$ are again in $C(X)$ we can apply Lemma 2.2.5.

Therefore, a set A in ρX is closed if and only if it is the intersection of preimages of closed sets A_i in $\mathbb{R}$ under ρf_i. This makes η continuous because the preimage of any closed set in ρX is the preimage of a closed set in $\mathbb{R}$ under an $f_i \in C(X)$ so it is closed and intersections of closed sets are closed.

We need to check that ρX is completely regular. One can easily see that ρX is Hausdorff. Let $\tilde{x} \neq \tilde{y} \in \rho X$ for $x, y \in X$ and $f \in C(X)$ such that $f(x) \neq f(y)$. Now choose disjoint neighborhoods U, V of $f(x), f(y)$ then $\rho f^{-1}(U), \rho f^{-1}(V)$ are disjoint and open neighborhoods of $\tilde{x}, \tilde{y}$.

Similarly given a point $\tilde{x_0} \in \rho X$ and a closed set

$$A = \bigcap \rho f_i^{-1}(A_i) \subset \rho X \text{ for } f_i \in C(X) \text{ and } A_i \subset \mathbb{R} \text{ closed}$$

not containing $\tilde{x_0}$ there is some ρf_i such that $\rho f_i(\tilde{x_0}) \notin A_i$. As $\mathbb{R}$ is completely regular (Example 2.2.2) there is g separating $\rho f_i(\tilde{x_0})$ and A_i. Then $g \circ \rho f_i$ separates $\tilde{x_0}$ and $\rho f_i^{-1}(A_i)$. As $A \subset \rho f_i^{-1}(A_i)$ then $g \circ \rho f_i$ also separates $\tilde{x_0}$ and A.

We conclude that ρX is completely regular. □

Remark. One may think that the topology on ρX coincides with the quotient topology on ρX induced by η. This is not true in general as the quotient topology is too large. We see this in the following example.

Example 2.2.7. Based on an exercise [GJ60, 3J]. Let

$$S = \mathbb{R} \times \mathbb{R} \setminus (\{(0,0)\} \cup \{(1/n, y) \mid y \neq 0 \text{ and } n \in \mathbb{N}\}) \subset \mathbb{R} \times \mathbb{R}.$$

Now consider $E = \pi(S)$, where π is the projection to the x-coordinate, endowed with the quotient topology. As a set $E = \mathbb{R}$ and $U \subset E$ is open if and only if $\pi^{-1}(U) \subset S$ is open.

In E distinct points are completely separated and in particular one point sets are closed. Let $x, y \in E$ be distinct. We claim that

$$f : E \to \mathbb{R}, z \mapsto \frac{z - x}{y - x}$$

is continuous. Then it clearly separates x, y and they are preimages of closed sets (0 and 1 respectively) and thereby closed. Let $U \subset \mathbb{R}$ be open. Then $f^{-1}(U)$ is open if

$$\pi^{-1}(f^{-1}(U)) = U \times \mathbb{R} \cap S$$

is open, which is clearly the case, because $U \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$ is open.

E is not completely regular as it is not even regular: We notice that the set $A = \{1/n\}_{n \in \mathbb{N}}$ is closed as by construction of S

$$\pi^{-1}(A) = A \times \mathbb{R} \cap S = ((A \cup \{0\}) \times \{0\}) \cap S.$$

Now this closed set cannot be separated from the point 0 by any disjoint open sets as any open set containing A also contains 0.

If we now look at our construction of ρE, we will see that, as any distinct points in E can be completely separated, the map $\eta : E \to \rho E$ will be bijective. But, if the topology on ρE were to be the quotient topology, η, being bijective, would be a homeomorphism. This cannot be the case because ρE is completely separated, but E is not.

So we see that when looking at real-valued functions it is enough to just consider completely regular spaces, as all information about the functions is already "contained" in a completely regular space.

2.3 The Stone-Čech Compactification

Now we are ready to look at the Stone-Čech compactification, this part is mostly oriented at *Munkres* [Mun00, §38].

Definition 2.3.1 ([Mun00, §38])**.** A *compactification* of a space X is a compact Hausdorff space Y in which we can embed X such that the image of X is dense in Y.

Two compactifications Y_0, Y_1 of X are *equivalent* if there exists a homeomorphism $f : Y_0 \to Y_1$ such that $f|_X = id_X$.

Theorem 2.3.2. *Given a completely regular space X there exists a space βX that is a compactification of X and X is $C^*(X)$-embedded in βX. Such a space βX we call a Stone-Čech compactification of X.*

Proof. The proof can be found in part in [Mun00, Theorem 38.2] and [Mun00, Theorem 34.2.].

Let $\{f_j\}_{j\in J}$ be an indexing of $C^*(X)$. Now consider the map

$$ev : X \to R := \prod_{j\in J}[\inf_X f_j, \sup_X f_j], x \mapsto (f_j(x))_{j\in J}.$$

As X is completely regular any two points x, y of X can be separated by a continuous function f_{j_0}. So for the projection π_{j_0} to the j_0 the component

$$\pi_{j_0}(ev(x)) \neq \pi_{j_0}(ev(y)).$$

Thereby $ev(x) \neq ev(y)$ and ev is injective. And ev is continuous because by construction $\pi_i \circ ev = f_i$ is continuous and by the universal property of the product so is ev (Example 1.1.5). The map ev is also an open mapping. To see this take an open set $U \subset X$ and take $z_0 \in ev(U)$ and $x_0 \in U$ such that $ev(x_0) = z_0$. We will find an open neighborhood of z_0 in $ev(U)$. As X is completely regular we can find an j_0 such that $f_{j_0}(X \setminus U) = 0$ and $f_{j_0}(x_0) = 1$. Consider the following open subset of $ev(X)$

$$W = ev(X) \cap \pi_{x_0}^{-1}((0, +\infty)) \subset ev(U).$$

W contains $ev(x_0) = z_0$ so it is an open neighborhood of z_0 contained in $ev(U)$. Thereby $ev(U)$ is open and ev an open mapping. Together with the previous results we get that ev is a homeomorphism of X and $ev(X)$, it is an embedding of X into R.

Now let

$$\beta X = \overline{ev(X)} \subset R.$$

It is compact Hausdorff as it is a closed subspace of a product of compact Hausdorff spaces (Lemma A.1.2), which is compact Hausdorff (Tychonov 1.3.5 + Lemma A.1.1). We need to check that X is $C^*(X)$-embedded. So given $f_{j_0} \in C^*(X)$ and $(x_j)_{j\in J} \in \beta X$, let

$$f'_{j_0}((x_j)_{j\in J}) = \pi_{j_0}((x_j)_{j\in J}) = x_{j_0},$$

this clearly defines a continuous map and by construction an extension of f_{j_0}. $\square$

Remark. Some constructions instead just use all bounded functions with image in $[0,1]$, but as all closed intervals in $\mathbb{R}$ are homeomorphic, this is equivalent to our construction.

We formulate an insight from the proof of the Theorem.

Theorem 2.3.3 ([Mun00, Theorem 34.3]). *A space is completely regular if and only if it is homeomorphic to a subspace of* $[0,1]^J$ *for some* J.

Proof. Assume X is completely regular. Use the construction of the space R from the proof of Theorem 2.3.2. All closed intervals in $\mathbb{R}$ are homeomorphic to $[0,1]$. Apply those homeomorphisms componentwise to our space R, this yields an embedding into $[0,1]^J$.

On the other hand, $[0,1]^J$ is completely regular by Theorem 2.2.4 as it is compact Hausdorff. Then by Lemma 2.2.3 any subspace X of $[0,1]^J$ is completely regular. □

Lemma 2.3.4 ([Mun00, Lemma 38.3.]). *Let* $A \subset X$ *and* $f : A \to Z$ *into a Hausdorff space, then there is at most one continuous extension of* f *to* $\overline{A}$.

Proof. Let there be two extensions of f,

$$g, h : \overline{A} \to Z \text{ continuous s.t. } g|_A = h|_A = f.$$

Assume there exists $x_0 \in \overline{A}$ such that $g(x_0) \neq h(x_0)$. Choose disjoint neighborhoods U, V of those points. As g, h are continuous $g^{-1}(U), h^{-1}(V)$ are neighborhoods of x_0. But their intersection cannot intersect A because $g|_A = h|_A = f$ and the images U, V are disjoint. This is a contradiction to the fact that $x_0 \in \overline{A}$, because this means that every neighborhood of x_0, in particular $g^{-1}(U) \cap h^{-1}(V)$ intersects A. □

Proposition 2.3.5. *Any continuous map* $f : X \to C$ *where* X *is completely regular and* C *is a compact Hausdorff space factors uniquely through* βX.

Proof. [Mun00, Theorem 38.4.] As C is compact Hausdorff it is completely regular (Theorem 2.2.4) and we can embed it into $[0,1]^J$ for some index set J (Theorem 2.3.3). We identify C with its image in $[0,1]^J$. Denote the projections by π_j. Now each component $f_j = \pi_j \circ f$ of $f : X \to C$ is a bounded real-valued function. By Theorem 2.3.2 we can extend f_j to $\psi_j : \beta X \to C$, applying this to all components defines an extension ψ of f to βX. This is well defined because

$$\psi(\beta X) = \psi(\overline{X}) \subset \overline{\psi(X)} \subset \overline{C} = C.$$

The first inclusion is due to continuity and the last equality due to the fact that C is a compact subspace of a Hausdorff space and thereby closed (see Lemma A.1.3).

By Lemma 2.3.4 are the ψ_j and thereby ψ unique. □

Corollary 2.3.6. *Two Stone-Čech compactifications of a completely regular space* X*, Hausdorff compactifications such that* X *is* $C^*(X)$*-embedded, are equivalent as compactifications. We just speak of the Stone-Čech compactification* βX.

Proof. Oriented at [Mun00, Theorem 38.5] By previous the Lemma 2.3.5 given Y_0, Y_1 both satisfying the properties of the Stone-Čech compactification identifying X with its image in Y_0, Y_1 we get the following commutative diagram:

$$\begin{tikzcd}
Y_0 \arrow[r, "f"] & Y_1 \arrow[r, "g"] & Y_0 \\
& X \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

So $(g \circ f)|_X = id_X$ and as $Y_0 = \overline{X}$ this extends uniquely to Y_0 (Lemma 2.3.4) and as id_{Y_0} is such an extension and $g \circ f$ too, $id_{Y_0} = g \circ f$. By the same argument $f \circ g = id_{Y_1}$, so Y_0, Y_1 are equivalent compactifications of X. □

Corollary 2.3.7. *If X is compact Hausdorff then βX is homeomorphic to X.*

Proof. X is $C^*(X)$-embedded into the compact Hausdorff space X, so X is homeomorphic to βX by the Corollary 2.3.6 we just proved. □

As we will often use it we will formulate the results from Proposition 2.3.5 and Corollary 2.3.6 (uniqueness of βX up to homeomorphism) as the universal property of the Stone-Čech compactification.:

Theorem 2.3.8. *Given a completely regular space X and a continuous map $h : X \to C$ where C is a compact Hausdorff space then h extends uniquely to a map $\psi : \beta X \to C$:*

$$\begin{tikzcd}
& \beta X \arrow[dr, dashed, "\exists ! \psi"] & \\
X \arrow[ur, hook] \arrow[rr, "h"] & & C.
\end{tikzcd}$$

Chapter 3

Projective Objects

As we have characterized profinite sets as the totally disconnected compact Hausdorff spaces the next step is to understand this category better. One question, with origins in homological algebra, is that of projective objects. The main reference for this chapter is the paper *Gleason* [Gle58].

3.1 Projective Objects

We first consider the "original" case of projective objects in an abelian category:

Definition 3.1.1. An object P in an abelian category $\mathcal{A}$ is called *projective* if the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.

Remark. As the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is left exact for any $A \in \mathcal{A}$ the important property is that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ preserves epimorphisms (because in an abelian category epi is equivalent to cokernel 0 and the exactness of a short exact sequence states that the last cokernel is zero).

Example 3.1.2. Given any commutative ring R with 1, then the free R-modules are projective in the category R-Mod of R-modules. (This is the case because we can choose a basis)

How can we generalize this notion? For that we write out what it means that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.

As noted in the remark the real condition such that P is projective is that given any epimorphism $f : B \to X$ in an abelian category,

$$f^* : \mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, X), \psi \mapsto f \circ \psi$$

is also an epimorphism, or as we are now working with Hom-sets that f^* is surjective. Therefore, it is also equivalent to the existence of a preimage: P is projective if given an epimorphism $f : B \to X$ and a morphism $\phi : P \to X$, then there exists $\psi : P \to B$ such that $\phi = f \circ \psi$.

This is now a notion which does not depend on an abelian category, but is equivalent to the notion in the abelian case, so we take it to be our new definition.

Definition 3.1.3 ([Mac98, V.4.]). An object P of a category C is called *projective* if given an epimorphism $f : B \to X$ and a morphism $\phi : P \to X$, then there exists $\psi : P \to B$ such that $\phi = f \circ \psi$, its lifting property, see also the commutative diagram below:

$$\begin{tikzcd} & B \arrow[dr, two heads, "f"] & \\ P \arrow[ur, dashed, "\psi"] \arrow[rr, "\phi"] & & X \end{tikzcd}$$

Lemma 3.1.4. *An object P of a category C is projective if and only if $C(P, -)$ preserves epimorphisms.*

Proof. Assume P to be projective. Then given an epimorphism $f : B \to X$ and two morphisms $g, h : X \to D$ (denote the morphism $C(P, f)$ by f^*) such that we have $g^* \circ f^* = h^* \circ f^*$. Now given any $\phi \in C(P, X)$ there is a $\psi \in C(P, B)$ such that $f \circ \psi = \phi$, as P is projective. So we get :

$$g^*(\phi) = g \circ \phi = g \circ f \circ \psi = (g^* \circ f^*)(\psi) = (h^* \circ f^*)(\psi) = h \circ f \circ \psi = h \circ \phi = h^*(\phi),$$

for ϕ arbitrary so f^* cancels on the right and is therefor an epimorphism.

On the other hand if $C(P, -)$ preserves epimorphisms then given an epimorphism $f : B \to X$ the pullback $f^* : C(P, B) \to C(P, A)$ is surjective so given any $\phi \in C(P, A)$ there is a ψ such that $f \circ \psi = f^*(\psi) = \phi$, so the lifting property is fulfilled. □

Definition 3.1.5. A category is said to have *enough projectives* if for every A there is a projective object P and an epimorphism $P \to A$.

Example 3.1.6. In the category R-Mod for each module M there is a surjective homomorphism from a free module: We can just take the free module over elements of M, this is clearly surjective. So R-Mod has enough projectives.

3.2 Extremally Disconnected Spaces

Extremally disconnected spaces, also known as Stonean Spaces, different from Stone Spaces, are exactly the projective objects of the category of compact Hausdorff spaces, **CHaus**, and thereby also of our category of profinite sets **ProFin**, which are the totally disconnected compact Hausdorff spaces. The goal of this section is to show this characterization of the projective objects.

Definition 3.2.1 ([Gle58, Def 1.1]). A topological space X is called *extremally disconnected* if for each open set $U \subset X$ its closure $\overline{U}$ is again open.

Remark. It is indeed called "extremally" with an "a".

First we study some properties of extremally disconnected spaces. One clearly sees that open subspaces and disjoint unions of extremally disconnected spaces are also extremally disconnected.

Lemma 3.2.2 ([Gle58, Lemma 2.2]). *Given two disjoint open subsets U, V of an extremally disconnected space X, then $\overline{U}, \overline{V}$ are also disjoint.*

Proof. As V is open and $U \subset X \setminus V$, which is closed, $\overline{U} \subset X \setminus V$. Now $\overline{U}$ is open and $V \subset X \setminus \overline{U}$, which is closed, so $\overline{V} \subset X \setminus \overline{U}$ and $\overline{U}, \overline{V}$ are disjoint. □

Lemma 3.2.3. *An extremally disconnected Hausdorff space X is totally disconnected.*

Proof. Take any non-empty subset $A \subset X$ containing at least two distinct points x, y. As X is Hausdorff we can separate x, y by disjoint open neighborhoods U, V. By previous Lemma 3.2.2 are $\overline{U}, \overline{V}$ disjoint and by definition of extremally disconnected spaces clopen. Now $\overline{U} \cap A, A \setminus \overline{U} \supset \overline{V} \cap A$ constitutes a separation of A, because they are both disjoint, open, their union is A, and they are both non-empty, because they contain x, y respectively. □

Remark. The Hausdorff condition is necessary, as one can easily see when looking at the space $\{a, b\}$ with the trivial topology, which is extremally disconnected but not totally disconnected.

Theorem 3.2.4. *Any projective object P in **CHaus** is extremally disconnected.*

Proof. Adaptation of [Gle58, Theorem 1.2]. Let P be a projective in **CHaus** and $U \subset P$ open, we show that $\overline{U}$ is open.

Let $\{a, b\}$ be the two point space with the discrete topology. We know that $P \times \{a, b\}$ is in **CHaus**, so is the projection $\pi : P \times \{a, b\} \to P$, and the product of two objects. Therefore, we can consider

$$Y = ((P \setminus U) \times \{a\}) \cup (\overline{U} \times \{b\}) \subset P \times \{a, b\}$$

which also is in **CHaus** as it is a closed subspace of $P \times \{a, b\}$.

Now take a look at the restriction of π to Y, this is a surjective map to P because clearly $(P \setminus U) \cup \overline{U} = P$, so we get the following diagram with an induced ψ, such that $id_P = \pi \circ \psi$ from the projectivity of P:

$$\begin{tikzcd} & Y \arrow[dr, two heads, "\pi"] & \\ P \arrow[ur, dashed, "\psi"] \arrow[rr, "id_P"] & & P \end{tikzcd}$$

Now we show that $\psi^{-1}(\overline{U} \times \{b\}) = \overline{U}$ by which $\overline{U}$ is open, because $\overline{U} \times \{b\} = Y \cap P \times \{b\}$ is clearly open. We see that π maps $(u, b) \in U \times \{b\}$ to $u \in U$ one-to-one. As $\pi \circ \psi = id_P$ we get $\psi(u) = (u, b)$. Now ψ is continuous so

$$\psi(\overline{U}) \subset \overline{\psi(U)} \subset \overline{U} \times \{b\},$$

but because again $\pi \circ \psi = id_P$ we have to get equality. On the other hand if $u \notin \overline{U}$ we have $\psi(u) = (u, a)$, because the image of u cannot lie in $\overline{U} \times \{b\}$. We conclude that $\psi^{-1}(\overline{U} \times \{b\}) = \overline{U}$. □

Remark. As one may note, we did not need all the strength of the compact Hausdorff condition to proof this for a category of spaces. One only needed that

(i) the morphisms of the category are continuous,

(ii) $A \times \{a, b\}$, where $\{a, b\}$ has the discrete topology, is an object and the projection to A is in the category,

(iii) and given a closed subset of a space then it and its inclusion are in the category.

As (i),(ii) and (iii) also hold for **ProFin** we see that the projective objects in **ProFin** are also extremally disconnected. This is how it is done in more generality in [Gle58, 1.].

This now shows that in a right subcategory of **Top**, in particular **CHaus**, the projective spaces are extremally disconnected.

This leads us to some examples. Obviously discrete spaces are extremally disconnected, but their Stone-Čech compactifications are extremally disconnected as well:

Lemma 3.2.5. *Given a discrete space D, then its Stone-Čech compactification βD is projective and in particular extremally disconnected.*

Proof. Taken from [Rai59, Lemma 3]. Consider a continuous map $\phi : \beta D \to Z$ to a compact Hausdorff space and a continuous surjective map $f : Y \to Z$ from a topological space Y. Because D is a discrete space we can construct a continuous map

$$h : D \to Y, d \mapsto y \in f^{-1}(\phi(d)).$$

Here we choose which y we map d to and we can apply ϕ because D is embedded in βD. As Y is compact Hausdorff there exists by the universal property of the Stone-Čech compactification a continuous extension of h to βD. We denote this map by $\psi : \beta D \to Y$. Our construction can now be seen in this commutative diagram:

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[dr, hook] & Y \arrow[dr, two heads, "f"] & \\
& \beta D \arrow[u, dashed, "\psi"] \arrow[r, "\phi"] & Z
\end{tikzcd}$$

This makes βD projective and by Theorem 3.2.4 in particular extremally disconnected. □

Remark. It might seem as if our construction gives us a unique ψ, but this is not true as ψ depends on h for which there are many possibilities.

So if we show that in **CHaus** all extremally disconnected spaces are projective they are also the projective objects in **ProFin**.

But to show this we first need some other results about (extremally disconnected) compact Hausdorff spaces.

Lemma 3.2.6. *Given A, E in* **Top** *and $\rho : E \to A$ an epimorphism such that for any proper closed subset $E_0 \subsetneq E$ we have $\rho(E_0) \subsetneq A$, then for any open set $U \subset E$, $\rho(U) \subset \overline{A \setminus \rho(E \setminus U)}$.*

Proof. Extended the proof [Gle58, Lemma 2.1]. W.l.o.g. is U not empty.

Now let $a \in \rho(U)$, by definition of the closure $a \in \overline{A \setminus \rho(E \setminus U)}$ if for all open neighborhoods V of a:

$$V \cap (A \setminus \rho(E \setminus U)) \neq \emptyset.$$

So choose any open neighborhood V of a. By continuity $U \cap \rho^{-1}(V)$ is open. It is a nonempty open subset of E, therefore (using that ρ is surjective):

$$\begin{aligned} & E \setminus (U \cap \rho^{-1}(V)) \subsetneq E \\ \Longrightarrow\ & \rho(E \setminus (U \cap \rho^{-1}(V))) \subsetneq A \\ \Longrightarrow\ & A \setminus \rho(E \setminus (U \cap \rho^{-1}(V))) \neq \emptyset. \end{aligned}$$

Now choose $y \in A \setminus \rho(E \setminus (U \cap \rho^{-1}(V)))$, because ρ is surjective there is x such that $\rho(x) = y$. By construction

$$\begin{aligned} & y \in A \setminus \rho(E \setminus (U \cap \rho^{-1}(V))) \\ \Longrightarrow\ & y \notin \rho(E \setminus (U \cap \rho^{-1}(V))) \\ \Longrightarrow\ & x \notin E \setminus (U \cap \rho^{-1}(V)) \\ \Longrightarrow\ & x \in U \cap \rho^{-1}(V) \\ \Longrightarrow\ & x \in \rho^{-1}(V) \\ \Longrightarrow\ & y \in \rho(\rho^{-1}(V)) = V. \end{aligned}$$

This gives us $y \in V \cap (A \setminus \rho(E \setminus (U \cap \rho^{-1}(V))))$. And with

$$\begin{aligned} & E \setminus U \subset E \setminus (U \cap \rho^{-1}(V)) \\ \Longrightarrow\ & \rho(E \setminus U) \subset \rho(E \setminus (U \cap \rho^{-1}(V))) \\ \Longrightarrow\ & A \setminus \rho(E \setminus U) \supset A \setminus \rho(E \setminus (U \cap \rho^{-1}(V))). \end{aligned}$$

we get that $y \in V \cap (A \setminus \rho(E \setminus U))$. As we have seen that any neighborhood V of a intersects $A \setminus \rho(E \setminus U)$, a must lie in $\overline{A \setminus \rho(E \setminus U)}$, this concludes the proof. □

Lemma 3.2.7 ([Gle58, Lemma 2.3])**.** *Given E, A in* **CHaus***, A extremally disconnected, and $\rho : E \to A$ an epimorphism such that for any proper closed subset $E_0 \subsetneq E$ we have $\rho(E_0) \subsetneq A$, then ρ is a homeomorphism.*

Proof. A continuous map from a compact space to a Hausdorff space is already a homeomorphism if it is bijective by Lemma A.1.5. So we only need to show that ρ is injective:

Let $x_1, x_2 \in E$. We show that their images are different. By the Hausdorff condition we can find disjoint open neighborhoods U, V of x_1, x_2 respectively. $E \setminus U, E \setminus V$ are closed subsets of E and thereby compact. So $\rho(E \setminus U)$, $\rho(E \setminus V)$ are also compact and as compact subsets of the Hausdorff space A they are closed. So $A \setminus \rho(E \setminus U)$, $A \setminus \rho(E \setminus V)$ are open and they are disjoint as (with $\rho(x) = y$):

$$\begin{aligned} & y \in A \setminus \rho(E \setminus U) \\ \Longrightarrow\ & y \notin \rho(E \setminus U) \\ \Longrightarrow\ & x \notin E \setminus U \\ \Longrightarrow\ & x \in U \subset E \setminus V \\ \Longrightarrow\ & y \in \rho(E \setminus V) \\ \Longrightarrow\ & y \notin A \setminus \rho(E \setminus V). \end{aligned}$$

By Lemma 3.2.2 the closures of disjoint open sets in an extremally disconnected space are again disjoint. So $\overline{A \setminus \rho(E \setminus U)}$, $\overline{A \setminus \rho(E \setminus V)}$ are disjoint. By Lemma 3.2.6 this implies that $\rho(U)$ and $\rho(V)$, as subsets of the sets above, are disjoint. Therefore $\rho(x_1) \neq \rho(x_2)$. So ρ is injective. $\square$

Lemma 3.2.8 ([Gle58, Lemma 2.4])**.** *Given X, Y in* **CHaus** *and a surjective map $\pi : X \to Y$ then there is a compact subset $E \subset X$ with $\pi(E) = Y$ such that for any proper closed subset $E_0 \subsetneq E$ we have $\pi(E_0) \subsetneq Y$.*

Proof. We take $\mathcal{C}$ to be the collection of compact subsets of X such that for any $A \in \mathcal{C}$ $\pi(A) = Y$ and order them by inclusion. Clearly $\mathcal{C}$ is not empty as it contains X.

Now let $\Gamma = \{A_t\}_{t \in T}$ be a chain in $\mathcal{C}$ (This means that Γ with the order of $\mathcal{C}$ is totally ordered). Then we claim that the intersection

$$D = \bigcap_{t \in T} A_t$$

is a lower bound for Γ in $\mathcal{C}$. It is clear that is a lower bound. We check that it is in fact an element of $\mathcal{C}$.

By Lemma A.1.3 the elements of Γ are closed, because they are compact subsets of X Hausdorff. Their intersection D is also closed and compact by Lemma A.1.2. Now for any $y \in Y$ the collection

$$\{\pi^{-1}(\{y\}) \cap A_t\}_{t \in T}$$

fulfills the finite intersection property, because it is totally ordered by inclusion and all $\pi^{-1}(\{y\}) \cap A_t$ are nonempty. And as we are in a compact space by Lemma A.1.6

$$D \cap \pi^{-1}(\{y\}) = \bigcap_{t \in T} (\pi^{-1}(\{y\}) \cap A_t) \neq \emptyset$$

So we get that $\rho(D) = Y$ and we conclude $D \in \mathcal{C}$.

By the Lemma of Zorn this gives us a minimal element $E \in \mathcal{C}$. This subset fulfills the first two properties by construction. And given any proper closed subset E_0 of E it is compact, as a closed subset of a compact space. So $\pi(E_0) \subsetneq Y$ or we would have a contradiction to the minimality of E as E_0 would also be in $\mathcal{C}$. $\square$

Theorem 3.2.9. *In* **CHaus** *the extremally disconnected spaces are projective.*

Proof. Taken from [Gle58, Theorem 2.5]. Let A, B, C be in **CHaus**, A extremally disconnected, $f : B \to C$ an epimorphism and $\phi : A \to C$ a morphism. We need to construct a morphism $\psi : A \to B$ such that $f \circ \psi = \phi$.

For this consider

$$D = \{(a, b) \mid \phi(a) = f(b)\} \subset A \times B.$$

Because f is surjective, for any $a \in A$ we can find a preimage for $\phi(a) \in C$ under f. This means, for all $a \in A$ we have $b \in B$ such that $\phi(a) = f(b)$. Therefore if $\pi_1 : A \times B \to A$ is the projection we get that $\pi_1(D) = A$. Now we apply the previous Lemma 3.2.8 to get $E \subset D$ compact such that for any proper closed subset $E_0 \subsetneq E$ we have $\pi_1(E_0) \neq A$. Denote the resulting map by $\rho = \pi_1|_E$. Now we know by Lemma 3.2.7 that ρ is a homeomorphism. We get a morphism $\rho^{-1} : A \to E \subset D$. Given the second projection $\pi_2 : A \times B \to B$ then $\psi = \pi_2 \circ \rho^{-1}$ is the morphism we are looking for. To see this take any $a \in A$ then

$$f \circ \psi(a) = f \circ \pi_2 \circ \rho^{-1}(a) = \phi(a).$$

The last equality is due to the construction of D. As $a \in A$ was arbitrary $f \circ \psi = \phi$. □

This leads us to the characterization we were looking for:

Corollary 3.2.10. *The projective objects in* **ProFin** *are exactly the extremally disconnected spaces.*

Proof. We have seen that the projective objects in **CHaus** are exactly the extremally disconnected spaces, by Theorems 3.2.4 and 3.2.9. It is clear that Theorem 3.2.4 telling us that projectives are extremally disconnected still holds and as **ProFin** is a full subcategory of **CHaus** Theorem 3.2.9 holds too. □

Lemma 3.2.11. CHaus *has enough projectives.*

Proof. [Stacks, Lemma 090D] Let X be in **CHaus**. We need to find an extremally disconnected compact Hausdorff space X' that covers X. Let D be the discrete space of all points of X. Then there is an obvious surjective map $h : D \to X$ mapping each point to itself. Because D is discrete h is continuous and by the universal property of the Stone-Čech compactification it extends to βD. By Lemma 3.2.5 and D being discrete βD is extremally disconnected. So we can choose $X' = \beta D$. We see this below:

$$\begin{tikzcd}
& \beta D = X' \arrow[dr, dashed, two heads, "\psi"] & \\
D \arrow[ur, hook] \arrow[rr, two heads, "h"] & & X
\end{tikzcd}$$

□

Corollary 3.2.12. *The category* ***ProFin*** *has enough projectives.*

Proof. As by Lemma 3.2.3 any extremally disconnected space in **CHaus** is totally disconnected we can just apply the previous Lemma. □

Chapter 4

Condensed Sets

The main source of this chapter is a talk by Reid Barton in a series of talks titled *Crash-course on Condensed Mathematics* by Reid Barton and Johan Commelin at Freiburg University in March 2023 and the corresponding lecture notes [BC23]. The original source are the lecture notes by Clausen and Scholze[CS19b].

4.1 Condensed Mathematics

The motivation for doing condensed mathematics is that of investigating algebraic structures with a topology. We have already seen a lot of those, for example the real numbers $\mathbb{R}$, the p-adic integers $\mathbb{Z}_p$, or the real-valued functions $C(X)$ on a topological space X. This topology, together with completeness, allows us to approximate elements of those structures as we are used to from analysis. To do this the topology of our algebraic objects is of utmost importance.

The goal of condensed mathematics is now to extend the methods that are known from algebra to those objects (for example cohomology or coherent sheaves). At first glance one may try to use the category of topological groups. One could try to extend one basic notion of algebra, that of a short exact sequence, to this category. However, this is not well-behaved. To see this we turn to the example of the real numbers. On the one hand $\mathbb{R}$ with the standard topology and on the other hand $\mathbb{R}^\delta$ as the real numbers with the discrete topology, we get the following sequence we want to be exact ($i : x \mapsto x$):

$$0 \longrightarrow \mathbb{R}^\delta \xrightarrow{i} \mathbb{R} \longrightarrow C \longrightarrow 0$$

But what should C be?

The cokernel C should satisfy the category theoretical condition that for any topological group D we get

$$\mathrm{Hom}(C, D) = \{\phi : \mathbb{R} \to D \text{ continuous group homomorphism} \mid \phi|_{\mathbb{R}^\delta} = 0\}$$

As any map from $\mathbb{R}$ is completely determined by what it does on each point and $i : \mathbb{R}^\delta \to \mathbb{R}$ is surjective $\mathrm{Hom}(C, D) = 0$.

But if it were a short exact sequence $\mathbb{R}$ and $\mathbb{R}^\delta$ would need to be isomorphic, which is not the case, because their topologies do not agree. What we would want to happen is to have a cokernel that somehow shows the difference in the topologies of $\mathbb{R}$ and $\mathbb{R}^\delta$.

What is the problem here?

The problem is that i is a bijective map, of the underlying sets, but it is not an isomorphism of topological groups because it is not a homeomorphism.

A category we do not have that problem in is **CHaus** because there any bijective continuous map is a homeomorphism (Lemma A.1.5). But most of the object we want to include in our theory are not compact, not even $\mathbb{Z}$, the free abelian group with one generator, or $\mathbb{R}$.

The solution is to somehow expand **CHaus** so it contains all the objects we want it to contain and preserve its good properties.

4.2 Condensed Sets

We are now ready to define condensed sets as in [BC23].

Definition 4.2.1 ([BC23, Definition 1.2.5.])**.** A *condensed set* is a functor

$$X : \mathbf{CHaus}^{op} \to \mathbf{Set}$$

such that

(i) $X(K_1 \sqcup \cdots \sqcup K_n) = X(K_1) \times \cdots \times X(K_n)$,

(ii) if $q : K \twoheadrightarrow L$ is surjective, then

$$X(L) \longrightarrow X(K) \rightrightarrows X(K \times_L K)$$

is an equalizer,

and moreover X is the colimit of a small diagram of K_α. The category **Cond** is the category of those functors together with their natural transformations.

Remark. The last condition that X is a colimit is to resolve the set-theoretic problems of **CHaus** being a large category: The objects do not form a set. Those issues are addressed concretely in [CS19b, Lectures I, II].

Remark. One can also formulate this in the language of sheaves by defining a Grothendieck topology. Then (i),(ii) are the sheaf properties. This approach can be found in [Ásg21, Chapter 1].

We say that the *underlying set* of a condensed set is $X(*)$ the evaluation of the condensed set X at $*$, the one point space. This makes sense as this can be seen as the set of all maps from the one point set $*$ to X, which is just the set of points, when dealing with sets.

Example 4.2.2 ([CS19b, Example 1.5.]). Let Y be a T_1-space, then we have an associated condensed set

$$\underline{Y} : \mathbf{CHaus}^{op} \to \mathbf{Set},\ K \mapsto C(K, Y),$$

where $C(K, Y)$ denotes the continuous maps from K to Y.

This clearly satisfies Definition 4.2.1(i). To see that (ii) is satisfied, notice that a surjection $e : K \twoheadrightarrow L$ of compact Hausdorff spaces is a quotient map, as it is closed by Lemma A.1.4 and Lemma A.1.3. Therefore, given a map $f : L \to Y$ such that $g = f \circ e$ is continuous then f is already continuous. We have a commutative diagram:

$$\begin{tikzcd}
K \times_L K \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & K \arrow[r, two heads, "e"] \arrow[dr, "g"'] & L \arrow[d, "f"] \\
& & Y
\end{tikzcd}$$

Because in **Set** the coequalizer of the above map is just L, by our previous discussion it is also the coequalizer in **Top**. And applying the contravariant hom-functor $C(-, Y) = \mathbf{Top}(-, Y)$ turns colimits into limits (Dual argument to Proposition B.2.2). So the coequalizer is turned into an equalizer, this is condition (ii).

It is also clear that this functor works on our continuous maps by composition A continuous map $f : X \to Y$ between topological spaces is sent to a morphism $\underline{f}$. So evaluation of $\underline{f}$ at the compact Hausdorff space K is given by:

$$\underline{f}(K) : \underline{X}(K) \to \underline{Y}(K), (h : K \to X) \mapsto (f \circ h : K \to Y)$$

Remark ([BC23, 1.2.6.]). We require the T_1-condition for it to be actually a small colimit. So when talking about this functor in the future we will assume that our topological spaces are T_1.

One easily sees that $X \mapsto \underline{X}$ is a faithful functor.

Proposition 4.2.3 ([CS19b, Proposition 1.7.]). *The functor $X \mapsto \underline{X}$ is faithful.*

Proof. Let X, Y be topological spaces and $f, g : X \to Y$ continuous maps, such that the morphisms of condensed sets $\underline{f} = \underline{g} : \underline{X} \to \underline{Y}$ are the same. We look at the evaluation at $*$. This gives us a set-theoretic maps

$$\underline{f}(*) = \underline{g}(*) : \underline{X}(*) \to \underline{Y}(*).$$

As $X(*)$ is exactly the underlying set of the topological space X and by how we constructed $\underline{f}, \underline{g}$ we know that f, g agree on all points. However any continuous maps agreeing on all points are the same. □

The next part is modelled after an exercise given in the Crash course on Condensed Mathematics.

Definition 4.2.4. A topological space X is called *sequential* if a map $f : X \to Y$ is continuous if and only if for all converging sequences $x_n \to x$ in X we get a converging sequence $f(x_n) \to f(x)$.

As we have seen in Example 1.4.7 a converging sequence in X is equivalent to a continuous map from $\mathbb{N} \cup \infty$. This yields the following lemma:

Lemma 4.2.5. *Let X be a topological space, then the following are equivalent:*

(i) X is sequential.

(ii) $X \to Y$ is continuous if and only if $\mathbb{N} \cup \infty \to X \to Y$ is continuous for any continuous map $\mathbb{N} \cup \infty \to X$.

Proof. As stated in example 1.4.7 a converging sequence $x_n \to x$ in X is equivalent to a continuous map

$$\mathbb{N} \cup \infty, n \mapsto x_n, \infty \mapsto x$$

So clearly X is sequential if and only if for any converging sequence $x_n \to x$ the map

$$f \circ (\mathbb{N} \cup \infty \to X) : \mathbb{N} \cup \infty \to Y, n \mapsto f(x_n), \infty \mapsto f(x)$$

is continuous, because this states just the convergence $f(x_n) \to f(x)$. $\square$

Together with the functor from condensed sets to topological spaces this leads us to the following intermediate result.

Lemma 4.2.6. *The functor $X \mapsto \underline{X}$ restricted to sequential topological spaces is fully faithful.*

Proof. We know from Proposition 4.2.3, that the functor is faithful.

Given sequential spaces X, Y and a morphism of condensed sets $f : \underline{X} \to \underline{Y}$. We need to check that f is the image of a continuous map $X \to Y$. It is enough to see that $f(*) : \underline{X}(*) \to \underline{Y}(*)$ is continuous, when we equip the spaces with the topology of X and Y.

By Lemma 4.2.5 we know that $f(*)$ is continuous if and only if $f(*)$ composed with any continuous map $\mathbb{N} \cup \infty \to X$ is continuous. But to see this we can just look at the evaluations at $\mathbb{N} \cup \infty \in$ **CHaus**:

$$f(\mathbb{N} \cup \infty) : \underline{X}(\mathbb{N} \cup \infty) \to \underline{Y}(\mathbb{N} \cup \infty)$$

This tells us that every continuous map $\mathbb{N} \cup \infty \to X$ is sent to a continuous map $\mathbb{N} \cup \infty \to Y$. Which by definition tells us that if $x_n \to x$ then $f(*)(x_n) \to f(*)(x)$. So $f(*)$ is continuous. $\square$

We can generalize the notion of sequential in line with Lemma 4.2.5.

Definition 4.2.7. A topological space X is said to be *compactly generated* if a map $X \to Y$ is continuous if and only if for all K in **CHaus** and continuous maps $K \to X$ the composition $K \to X \to Y$ is continuous.

And we can generalize Lemma 4.2.6.

Proposition 4.2.8 ([CS19b, Proposition 1.7.])**.** *The functor $X \mapsto \underline{X}$ restricted to compactly generated topological spaces is fully faithful.*

Proof. We know by Proposition 4.2.3, that the functor is faithful.

We adapt the proof of Lemma 4.2.6. Given any compactly generated spaces X, Y and a morphism of condensed sets $f : \underline{X} \to \underline{Y}$. We need to check that f is the image of a continuous map $X \to Y$. So we just need to show that $f(*) : \underline{X}(*) \to \underline{Y}(*)$ is continuous.

By definition $f(*)$ is continuous if and only if $f(*)$ composed with any map $K \to X$ from a compact Hausdorff space K is continuous. To see this we notice that f is a natural transformation, so the following diagram commutes for any morphism $* \to K$, which is basically choosing a point in K (See also Example 1.1.2(v), we have a different direction because our functor is contravariant):

$$\begin{tikzcd} K & \underline{X}(K) \arrow[r, "f(K)"] \arrow[d] & \underline{Y}(K) \arrow[d] \\ * \arrow[u] & \underline{X}(*) \arrow[r, "f(*)"] & \underline{Y}(*) \end{tikzcd}$$

So $f(K)$ is pointwise defined and therefore completely defined:

$$f(K) : \ \underline{X}(K) \to \underline{Y}(K), \ h \mapsto f(*) \circ h$$

As K was an arbitrary compact Hausdorff space and $\underline{X}(K) = C(K, X)$ we see that $f(*)$ is continuous, by definition of compactly generated spaces. □

Corollary 4.2.9 ([CS19b, Proposition 1.7.])**.** *The functor $X \mapsto \underline{X}$ restricted to compactly generated topological spaces admits a left adjoint functor.*

We get this left adjoint functor from **Cond** to topological spaces explicitly by taking the underlying set $X(*)$ and equipping it with the quotient topology we get from the map

$$\bigsqcup_{\alpha} K_\alpha \to X(*).$$

Here the K_α are those that X is a colimit of, as in definition 4.2.1. By this we gain a topological space $X(*)_{top}$.

4.3 Equivalence of Definitions

There are different definitions of condensed sets. In [CS19b] the category **CHaus** in Definition 4.2.1 is replaced by **ProFin** and one can go even as far as to just consider extremally disconnected spaces. Those different definitions just amount to restricting our functors to those subcategories. Due to the covering property (ii) and **CHaus** having enough projectives (exactly the extremally disconnected spaces) those definitions are all equivalent.

Remark. There are also some differences in how the set-theoretic issues are solved but we are not going to mention those here.

Proposition 4.3.1. *The category* **Cond** *is equivalent to the category we gain when restricting each condensed set to the extremally disconnected spaces.*

Proof. Expanded upon [CS19b, Proposition 2.7.] Let K be in **CHaus** and X in **Cond**. Then by Lemma 3.2.11 there exists a projective object P and a surjection $P \twoheadrightarrow K$. By Definition 4.2.1 (ii) we have an equalizer:

$$X(K) \xrightarrow{\ e\ } X(P) \underset{p_2}{\overset{p_1}{\rightrightarrows}} X(P \times_K P).$$

We can also find another projective object P' covering $P \times_K P$ so we get again by Definition 4.2.1 (ii) the equalizer

$$X(P \times_K P) \xrightarrow{\ e'\ } X(P') \underset{p_2'}{\overset{p_1'}{\rightrightarrows}} X(P' \times_{P \times_K P} P').$$

We want to show that $X(K)$ is the equalizer of $p_1 \circ e, p_2 \circ e$. So, taking both diagrams together, given the set Y, and any map $f : Y \to X(P)$ such that

$$(e' \circ p_1) \circ f = (e' \circ p_2) \circ f =: g : Y \to X(P')$$

we get the commutative diagram

$$\begin{tikzcd}
X(K) \arrow[r, "e"] & X(P) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X(P \times_K P) \arrow[r, "e'"] & X(P') \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] & X(P' \times_{(P \times_K P)} P') \\
& Y \arrow[ul, dashed, "d"] \arrow[u, "f"] \arrow[ur, dashed, "h"] \arrow[urr, "g"'] & & &
\end{tikzcd}$$

As $p_1' \circ e' = p_2' \circ e'$ by construction $p_1' \circ g = p_2' \circ g$. As e' is the equalizer of p_1', p_2' this induces a map

$$h : Y \to X(P \times_K P)$$

in the commutative diagram above. So by uniqueness of this map, as an equalizer is a limit, we get $p_1 \circ f = h = p_2 \circ f$. Now by the universal property of the equalizer e this induces a map

$$d : Y \to X(K)$$

commuting with the others. This shows that e is also the equalizer for $e' \circ p_1$ and $e' \circ p_2$. So $X(K)$ is already completely determined by those two maps $p_1, p_2 : X(P) \to X(P')$.

We already know that the projectives in **CHaus** are exactly the extremally disconnected spaces (Corollary 3.2.10), so P, P' are extremally disconnected. As the maps $X(P) \to X(P')$ are images of morphisms in **CHaus** and the extremally disconnected spaces are a full subcategory of **CHaus**, $X(P) \to X(P')$ are images of morphisms of the extremally disconnected spaces.

Thereby an X in **Cond** is already completely determined by its values on extremally disconnected spaces and **Cond** is equivalent to the category valued on extremally disconnected spaces. $\square$

Remark. The construction of $h = p_1 \circ f = p_2 \circ f$ is just a proof that every equalizer is a monomorphism and applying it by canceling e' in $(e' \circ p_1) \circ f = (e' \circ p_2) \circ f$.

Corollary 4.3.2. *The category* **Cond** *is equivalent to the category we gain when restricting each condensed set to* **ProFin**.

Proof. We apply the same proof as in Proposition 4.3.1. As any extremally disconnected compact Hausdorff space is profinite by Lemma 3.2.3 and **ProFin** is subcategory of **CHaus** we are done. $\square$

Remark. Those results allow one to calculate results for condensed sets on the S-valued points for S just profinite or extremally disconnected.

Appendix A

Topology

A.1 Facts and Definitons

Some topological defiitions and facts supplied without proof.

Lemma A.1.1. *[Mun00, Theorem 19.4.] The arbitrary product of Hausdorff spaces is Hausdorff.*

Lemma A.1.2. *[Mun00, Theorem 26.2.] A closed subspace of a compact space is compact.*

Lemma A.1.3. *[Mun00, Theorem 26.3.] A compact subspace of a Hausdorff space is closed.*

Lemma A.1.4. *[Mun00, Theorem 26.5.] Given a continuous map $f : X \to Y$ of topological spaces and X compact, then $f(X) \subset Y$ is compact.*

Corollary A.1.5. *[Mun00, Theorem 26.6.] Given a bijective continuous map $f : X \to Y$, where X is a compact space and Y a Hausdorff space, then f is a homeomorphism.*

Lemma A.1.6. *[Mun00, p. 26.9.] A space X is compact if and only if given a family of closed sets $\{V_j\}_{j\in J}$ in X such that $\bigcap_{j\in J'} V_j \neq \emptyset$ for any finite $J' \subset J$ then $\bigcap_{j\in J} V_j \neq \emptyset$.*

We call the property that any finite intersection is nonempty the *Finite Intersection Property.*

Definition A.1.7. [Mun00, §18] A continuous map $e : X \to Y$ is called an *embedding* if it is injective and a homeomorphism between X and its image $e(X) \subset Y$.

Definition A.1.8. Given a subspace $Y \subset X$ of a topological space X and a continuous map $f : Y \to Z$. Then we say a continuous map $g : X \to Z$ is a continuous extension of f if $g|_Y = f$.

A.2 Connectedness

Definition A.2.1. [Mun00, §23] A *separation* of a space X are two non-empty open sets $U, V \subset X$ such that $U \cap V = \emptyset$ and $U \cup V = X$.

A space X is said to be *connected* if there is no separation.

Remark. As U, V are complements of each other U, V are always clopen.

The idea behind connectedness is that we can not divide a space into two disjoint spaces. We cannot separate it. This gives rise to a relation on the set of points of a space X:

$$x \sim y \iff \exists\, U \subset X \text{ connected such that } x, y \in U.$$

Lemma A.2.2. *[Mun00, §25] The relation defined above is an equivalence relation and the equivalence classes are the biggest connected subspaces of X.*

We call those equivalence classes the *connected components* of X.

A.3 Separation Axioms

Definition A.3.1. [Mun00, §31] A topological space is said to be T_1 if one point sets are closed.

Definition A.3.2. [Mun00, §17] A topological space X is said to be Hausdorff if for any pair of distinct points x_1, x_2 there exist disjoint neighborhoods U_1, U_2 of x_1, x_2 respectively.

Definition A.3.3. [Mun00, §31] A topological space X that is T_1 is said to be regular if for any closed set A and any $x \notin A$ there are disjoint open sets U, V containing A, x respectively ($A \subset U, x \in V$).

Definition A.3.4. [Mun00, §31] A topological space X that is T_1 is said to be *normal* if for any two disjoint closed subset $A, B \subset X$ there exist disjoint open sets U, V containing A, B respectively ($A \subset U, B \subset V$).

Together with the definition of completely regular (Definition 2.2.1) and Urysohn's Lemma 2.1.7 we get the following implications of properties of a topological space [Mun00, §31]:

$$T_1 \Longleftarrow \text{Hausdorff} \Longleftarrow \text{regular} \Longleftarrow \text{completely regular} \Longleftarrow \text{normal}$$

And we have the following result without proof.

Theorem A.3.5. *[Mun00, Theorem 32.3] Any compact Hausdorff space is normal.*

Appendix B

Category Theory

B.1 Final functors and subcategories

Definition B.1.1. [Stacks, Definition 04E6] A functor $H : I \to J$ is called *final* if for any object k in J there exists an object x in I, a morphism $k \to Hx$ and if for any two morphisms $k \to Hx, k \to Hx'$ there exist

$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \dots x_{2n} = x',$$

such that the following diagram commutes for all $r < n$:

$$\begin{tikzcd} & k \arrow[dl] \arrow[d] \arrow[dr] & \\ Hx_{2r} & Hx_{2r+1} \arrow[l] \arrow[r] & Hx_{2r+2} \end{tikzcd}$$

Definition B.1.2. [Mac98, p. IX.3.] A subcategory J' of J is called *final* if the inclusion functor is final.

Definition B.1.3. A subset J' of a directed poset $(J, \preceq)$ is called *final* if for any $j \in J$ there exists $j' \in J'$ such that $j \preceq j'$

Lemma B.1.4. *Both notions of being final agree for a directed poset.*

Proof. Clear by applying the definitions and the "directed" property. □

Theorem B.1.5. *Given a final functor $L : I \to J$, a functor $F : J \to X$ such that the colimit $\varinjlim FL$ exists then $\varinjlim F$ exists and is canonically isomorphic to $\varinjlim FL$.*

Proof. See [Mac98, p. IX.3.].

□

B.2 Preservation of Limits

Definition B.2.1. [Mac98, V.4.] A functor $H : C \to D$ is said to *preserve limits* if given a functor $F : J \to C$, for J a small category, with a limit consisting of the limit-object $\varprojlim F$ and projections ϕ_j, then the limit of HF exists and consists of the limit object $\varprojlim HF \cong H \varprojlim F$ and projections $H\phi_j$.

Proposition B.2.2. *Given a category C and an object c of C, then*

$$C(c, -) : C \to \mathbf{Set}$$

preserves all limits.

Proof. Adapted from [Mac98, V.4.]. Let $F : J \to C$ be a functor from a small category J to C and its limit be given by the limit-object $\varprojlim F$ and projections ϕ_j. Then applying the covariant functor $C(c, -)$ yields the following diagram, where $\phi_{*j} = C(c, \phi_j)$ and X in $\mathbf{Set}$ is arbitrary:

$$\begin{tikzcd}
C(c, \varprojlim F) \arrow[r, "\phi_{*j}"] & C(c, Fj) \arrow[d, equal] \\
X \arrow[u, dashed, "k"] \arrow[r, "\theta_j"] & C(c, Fj)
\end{tikzcd}$$

The question now is how k is induced. For this take any element $x \in X$. Applying the $\theta_j(x)$ gives us a collection of maps from c to the diagram FJ and by the universal property of the limit this induces a morphism $h_x : c \to \varprojlim F$ such that

$$\phi_{*j}(h_x) = \phi_j \circ h_x = \theta_j(x)$$

This justifies setting $k(x) = h_x$ to construct the map $k : X \to C(c, \varprojlim F)$. As we have seen this is done in a unique way for each x. Thereby $C(c, \varprojlim F)$ satisfies the universal property of the limit of the functor $C(c, F) : J \to \mathbf{Set}$. So $C(c, \varprojlim F) \cong \varprojlim C(c, F)$. ☐

Remark. This can actually be used to define the limit.

Proposition B.2.3. *Given a category C, an object c of C, a small cofiltered index category J, and the cofiltered limit of a functor $F : J \to C$, with limit object $\varprojlim F$, we get an injective map from the filtered colimit:*

$$\varinjlim C(F, c) \to C(\varprojlim F, c)$$

It is surjective if for any $f \in C(\varprojlim F, c)$ there exist an object j_0 of J such that f factors through Fj_0.

Proof. W.l.o.g. J^{op} a directed poset. Consider the following diagramm of our limit of F under the contravariant functor $C(-, c)$:

$$\begin{tikzcd}
C(\varprojlim F, c) & C(Fi, c) \arrow[l] \arrow[dl] \arrow[d, "i_k"] \\
\varinjlim C(F, c) \arrow[u, dashed] & \coprod_{j \in J} C(Fj, c) \arrow[l, two heads]
\end{tikzcd}$$

This we get from the universal property of the colimit, which exists in **Set** as all (co-)limits do.

We need to show that the map is injective. We first notice that any element in the coproduct $\coprod_{j \in J} C(Fj, c)$ determines an element in the filtered colimit $\varinjlim C(F, c)$, as seen in the diagram above. And we notice that two elements $g \in C(Fi_1, c)$, $h \in C(Fi_2, c)$ of the coproduct, indexed by J, determine the same object in the filtered colimit if

$$\forall j \succeq i_1, i_2 : \ C(\phi_{ji_1}, c)(g) = C(\phi_{ji_2}, c)(h)$$

Now take $g', h' \in \varinjlim C(F, c)$ with the same image f in $C(\varprojlim F, c)$ and choose any preimages $g \in C(Fi_1, c), h \in C(Fi_2, c)$ in the coproduct. For any $j \succeq i_1, i_2$ we get the following diagram:

$$\begin{tikzcd}
 & Fi_1 \arrow[dr, "g"] & \\
Fj \arrow[ur, "\phi_{ji_1}"] \arrow[dr, "\phi_{ji_2}"'] & \varprojlim F \arrow[l, "\phi_j"'] \arrow[u, "\phi_{i_1}"'] \arrow[d, "\phi_{i_2}"] \arrow[r, "f"] & c \\
 & Fi_2 \arrow[ur, "h"'] &
\end{tikzcd}$$

and it has to commute because of the g, h being mapped to f. By what we noticed about the filtered colimit g, h determine the same element in $\varinjlim C(F, c)$ so our map is injective.

Surjectivity in the case any $f \in C(\varprojlim F, c)$ factors through some Fj_0 should be clear as well because a preimage is just the image of the morphism $Fj_0 \to c$ in the filtered colimit. $\square$

Zusammenfassung

Diese Arbeit befasst sich mit der Konstruktion der Kategorie der proendlichen Mengen, ihren Eigenschaften und ihrer Rolle in dem Feld der Condensed Mathematics.

Dabei sind die wichtigsten Ergebnisse die Äquivalenz der Kategorie der proendlichen Mengen zu der Kategorie der total unzusammenhängenden Hausdorff Räumen (Theorem 1.4.11) und die Charakterisierung der projektiven Objekte in der Kategorie der kompakten Hausdorff Räume, und damit auch in der Kategorie der proendlichen Mengen, als die extremal unzusammenhängende Räume (Theorem 3.2.9).

Das Ziel dieser Arbeit ist die Einführung und das Verständniss proendlicher Mengen, um damit in das Feld der condensed mathematics einsteigen zu können.

Wir beginnen in Kapitel 1 mit dem Konzept eines projektiven Limes aus der Kategorientheorie und führen damit für jede Kategorie C die Pro-Kategorie $Pro(C)$ als die Kategorie aller formalen projektiven Systeme in C ein; dies kann als Vervollständigung von C nach projektiven limiten gesehen werden. Daraufhin betrachten wir den Spezialfall der proendlichen Mengen, die Pro-Kategorie der endlichen Mengen. Zuerst versehen wir dazu endliche Mengen mit der diskretene Topologie, das macht sie kompakt Hausdorff, und betrachten generell projektive Systeme und Limiten von kompakten Hausdorff Räumen. Dieses Wissen nutzen wir dann, um zu zeigen, dass alle proendlichen Mengen total unzusammenhängende kompakte Hausdorff Räume sind und kommen zu dem Schluss, dass sie sogar äquivalent sind.

In Kapitel 2 verlassen wir zunächst die Kategorie der proendlichen Mengen, um ein topologisches Werkzeug einzufüren, die Stone-Čech Kompaktifizierung. Wir führen diese, wie historisch durch Čech, als den kompakten Hausdorff Raum βX ein in die sich der komplett reguläre topologische Raum X dicht einbetten lässt, sodass sich alle beschränkten reellwertigen Funktionen von X stetg auf βX fortsetzen lassen. Wir werden die Stpne-Čech Kompaktifizierung auch über eine andere universelle Eigenschaft charakterisieren. So lässt sich nämlich auch jede stetige Abbildung in einen kompakten Hausdorff Raum eindeutig stetig fortsetzen.

Mit diesem Werkzeug in der Schublade wenden wir uns in Kapitel 3 wieder den Kategorien der proendlichen Mengen und kompakten Hausdorff Räumen zu. Eine Fragestellung aus der homologischen Algebra ist die der projektiven Objekte. So verallgemeinern wir den Begriff des projektiven Objektes auf allgemeine Kategorien und zeigen, dass die projektiven Objekte der Kategorie der kompakten Hausdorff Räume auch über einen topologische Eigenschaft bestimmt sind, sie sind alle extremal unzusammenhängend. Des weiteren hat die Kategorie der kompakten Hausdorff Räume genug Projektive, da wir jeden Raum mit der Stone-Čech Kompatifizierung seiner zugrundeliegenden Menge als diskreter Raum überdekt werden kann. Außerdem sehen wir, dass alle extremal unzusammenhängenden kompakten Hausdorff Räume proendlich sind und sich somit die Charakterisierung und die Existenz von genügend Projektiven auf die Kategorie der proendlichen Mengen überträgt.

In Kapitel 4 sehen wir schlussendlich die Motivation für Condensed Mathematics, eine Theorie von algebraischen Objekten mit Topologie, die wohlartig ist, sodass wir Methoden der algebraischen Geometrie anwenden können. Wir werden lediglich die Konstruktion von kondensierten Mengen und einen Teil ihrer Verbindung zu topologischen Räumen sehen. Schließlich werden wir noch zeigen, dass teilweise verschiedene Definitionen übereinstimmen.

References

[Ásg21] Dagur Ásgeirsson. “Foundations of Condensed Mathematics”. English. Unpublished Master-Thesis at Université de Paris. 2021. URL: `https://dagur.sites.ku.dk/files/2022/01/condensed-foundations.pdf`.

[Awo10] Steve Awodey. *Category theory.* English. 2nd ed. Vol. 52. Oxf. Logic Guides. Oxford: Oxford University Press, 2010. ISBN: 978-0-19-958736-0.

[Cec37] Eduard Cech. “On Bicompact Spaces”. In: *Annals of Mathematics* 38.4 (1937), pp. 823–844. ISSN: 0003486X. URL: `http://www.jstor.org/stable/1968839` (visited on 02/10/2023).

[Dug73] James Dugundji. *Topology. 8th printing.* English. 1973.

[GJ60] Leonard Gillman and Meyer Jerison. *Rings of continuous functions.* English. Princeton-Toronto-London-New York: D. Van Nostrand Company, Inc., 1960.

[Gle58] Andrew M. Gleason. “Projective topological spaces”. In: *Illinois Journal of Mathematics* 2.4A (1958), pp. 482–489. DOI: `10.1215/ijm/1255454110`.

[Joh86] Peter T. Johnstone. *Stone spaces.* English. Vol. 3. Camb. Stud. Adv. Math. Cambridge University Press, Cambridge, 1986.

[Mac98] Saunders Mac Lane. *Categories for the working mathematician.* English. 2nd ed. Vol. 5. Grad. Texts Math. New York, NY: Springer, 1998. ISBN: 0-387-98403-8.

[Mun00] James R. Munkres. *Topology.* English. 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2000. ISBN: 0-13-181629-2.

[Rai59] John Rainwater. “A Note on Projective Resolutions”. In: *Proceedings of the American Mathematical Society* 10.5 (1959), pp. 734–735. DOI: `10.2307/2033466`. URL: `http://www.jstor.org/stable/2033466`.

[RZ10] Luis Ribes and Pavel Zalesskii. *Profinite groups.* English. 2nd ed. Vol. 40. Ergeb. Math. Grenzgeb., 3. Folge. Berlin: Springer, 2010. ISBN: 978-3-642-01641-7.

[SGA4] Michael Artin et al., eds. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 1: Théorie des topos. Exposés I à IV. 2e éd.* French. Vol. 269. Lect. Notes Math. Springer, Cham, 1972. DOI: `10.1007/BFb0081551`.

[Sto36] M. H. Stone. "The theory of representations for Boolean algebras". English. In: *Trans. Am. Math. Soc.* 40 (1936), pp. 37–111. ISSN: 0002-9947. DOI: `10.2307/1989664`.

[Sto37] M. H. Stone. "Applications of the theory of Boolean rings to general topology". English. In: *Trans. Am. Math. Soc.* 41 (1937), pp. 375–481. ISSN: 0002-9947. DOI: `10.2307/1989788`.

[Wal74] Russell C. Walker. *The Stone-Cech compactification.* English. Vol. 83. Ergeb. Math. Grenzgeb. Springer-Verlag, Berlin, 1974.

[BC23] Reid Barton and Johan Commelin. *Condensed sets.* English. 2023. URL: `https://math.commelin.net/2023/condensed/`.

[CS19a] Dustin Clausen and Peter Scholze. *Lectures in analytic Geometry.* English. 2019. URL: `https://people.mpim-bonn.mpg.de/scholze/Analytic.pdf`.

[CS19b] Dustin Clausen and Peter Scholze. *Lectures on Condensed Mathematics.* English. 2019. URL: `https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf`.

[CS22] Dustin Clausen and Peter Scholze. *Condensed Mathematics and Complex Geometry.* English. 2022. URL: `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`.

[nLab] The nLab Authors. *nLab.* 2023. URL: `https://ncatlab.org`.

[Stacks] The Stacks Project Authors. *Stacks Project.* 2023. URL: `https://stacks.math.columbia.edu`.