Algebraic Number Theory 2018 — Set 1

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Deadline: Friday, the 27th of April, 2018

Each exercise is worth 4 points.

Exercise 1. Prove that the ring $\mathbb{Z}[(1+\sqrt{-3})/2]$ is Euclidean.

Exercise 2. Let p be a prime number. Show that $x^2 + 2y^2 = p$ has an integral solution if and only if p is congruent to 1, 2, or 3 $\pmod 8$.

Exercise 3. Let K be a field (= Körper), and let $H \subset K^*$ be a finite subgroup. Write down a proof that H is cyclic. (You are allowed use the Internet and/or books.)

Exercise 4. Show that for every element $x \in \mathbb{Z}[i] \setminus \{0\}$, the norm $N(x) \in \mathbb{Z}$ is the cardinality of the residue ring $\mathbb{Z}[i]/(x\mathbb{Z}[i])$.

Exercise 5 (Bonus)**.** Show that there are no integral solutions to $x^2 + 1 = y^3$. (*Hint: Factor* x^2+1 *in* $\mathbb{Z}[i]$ *and show that* $\gcd(x+i, x-i) = 1$. *Deduce that* $x+i$ *is a cube in* $\mathbb{Z}[i]$ *and derive a contradiction.*)