Algebraic Number Theory 2018 — Set 10

Tutor: Vivien Vogelmann, `vivienvogelmann[at]web.de`
Deadline: 12.00 on Thursday, the 5th of July, 2018

Remark. You are allowed to use exercise i in your solution of exercise j if $i < j$, even if you do not hand in exercise i.

Notation. (i) Recall that if G is a group and $\sigma \in G$, then we denote with $\langle \sigma \rangle$ the conjugacy class of σ.

(ii) Let L/K be an extension of number fields. We denote with $P(L/K)$ the set of primes $\mathfrak{p}$ of K such $\mathfrak{p}$ is unramified in L/K and such that there exists a prime $\mathfrak{P}|\mathfrak{p}$ of L that has degree 1 (that is, $\mathcal{O}_L/\mathfrak{P} \cong \mathcal{O}_K/\mathfrak{p}$).

(iii) If S and T are two sets, we write $S \mathrel{\dot{\subset}} T$ if $T \setminus S$ is finite; and we write $S \doteq T$ if $S \mathrel{\dot{\subset}} T$ and $T \mathrel{\dot{\subset}} S$.

Exercise 1. Let N/K be a Galois extension of number fields, and let L be an intermediate number field: $K \subset L \subset N$. Write $G = \mathrm{Gal}(N/K)$ and $H = \mathrm{Gal}(N/L)$. (We do not assume that L/K is Galois, so H does not have to be a normal subgroup of G!) Show that

$$P(L/K) \doteq \bigsqcup_{\langle \sigma \rangle : \langle \sigma \rangle \cap H \neq \varnothing} X_{N/K}(\sigma).$$

Exercise 2. Let L/K be an extension of number fields of degree n.

(i) Show that $P(L/K)$ has density $\geq 1/n$.

(ii) Show that L/K is Galois if and only if $P(L/K)$ has density $1/n$.

Exercise 3. Let L/K be an extension of number fields. Show that L/K is Galois if and only if every prime ideal in $P(L/K)$ splits completely in L/K.

Hint: Let N be the normal closure of L/K. Show that $P(N/K) \doteq P(L/K)$.

Exercise 4. Let L/K be a Galois extension of number fields, and let M/K be an arbitrary finite extension. Show that

$$P(M/K) \mathrel{\dot{\subset}} P(L/K) \iff L \subset M.$$

Conclude that a Galois extension of number fields L/K is uniquely determined by the set $P(L/K)$ of primes that split completely in L/K.