Algebraic Number Theory 2018 — Set 3

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Deadline: 12.00 on Friday, the 11th of May, 2018

Each exercise is worth 4 points. The bonus exercise is also worth 4 points. If you get more then 16 points, you can transfer the excess points to other exercise sets.

Exercise 1. Let d be a square-free integer. Show that the discriminant of the quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ is equal to

$$\begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{otherwise.} \end{cases}$$

Exercise 2. Let K be a number field. Show that the discriminant of $K/\mathbb{Q}$ is congruent to 0 or 1 modulo 4.

Hint: The definition of the discriminant involves the determinant of a certain matrix. This determinant is a sum of the form $\sum_{s \in \mathfrak{S}_n} (-1)^{\mathrm{sign}(s)}(\ldots)$. Split this sum into two pieces: $P = \sum_{s|\mathrm{sign}(s)=1}(\ldots)$, and $N = \sum_{s|\mathrm{sign}(s)=-1}(\ldots)$. Then the determinant is equal to $P - N$. Show that $P + N$ and PN are elements of $\mathbb{Z}$ (using the Galois action). Now use $(P-N)^2 = (P+N)^2 - 4PN$.

Exercise 3. Let $a_1, \ldots, a_n$ be elements of a commutative ring R. Calculate the determinant of the matrix:

$$\begin{pmatrix} a_1^0 & a_1^1 & \cdots & a_1^{n-1} \\ a_2^0 & a_2^1 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^{n-1} \end{pmatrix}$$

Exercise 4. Look up the Smith normal form: `en.wikipedia.org/wiki/Smith_normal_form`. Use the algorithm to compute the structure of $\mathbb{Z}^3/(M\mathbb{Z}^3)$, where M is the matrix:

$$\begin{pmatrix} -167 & 22 & -140 \\ -151 & -14 & -184 \\ 110 & 94 & 276 \end{pmatrix}$$