Algebraic Number Theory 2018 — Set 4

Tutor: Vivien Vogelmann, `vivienvogelmann[at]web.de`
Deadline: 12.00 on Thursday, the 17th of May, 2018

Each exercise is worth 4 points. The bonus exercise is also worth 4 points. If you get more then 16 points, you can transfer the excess points to other exercise sets.

Exercise 1. Let $\alpha \in \mathbb{C}$ be a root of $X^3 - X^2 + 1$, and let $\beta \in \mathbb{C}$ be a root of $X^3 + X + 1$. Compute the ring of integers $\mathcal{O}_K$ for $K = \mathbb{Q}(\alpha)$ and $K = \mathbb{Q}(\beta)$.

Exercise 2. Let K be the number field $\mathbb{Q}(\sqrt{-23})$. Let

$$I_0 = (1), \qquad I_1 = \left(2, \frac{\sqrt{-23}-1}{2}\right), \qquad \text{and} \qquad I_2 = \left(2, \frac{\sqrt{-23}+1}{2}\right)$$

be three ideals of $\mathcal{O}_K$. Draw a picture of the lattices I_0, I_1, and I_2. Show that these three ideals define three different classes in the class group of K.

Exercise 3. Let R be a Dedekind domain. Show that R is a unique factorisation domain if and only if R is a principal ideal domain.

Exercise 4. Let R be a Dedekind domain. Prove that every ideal of R can be generated by 2 elements.

Hint: Let $I \subset R$ be a non-trivial ideal. (1) It suffices to show that every ideal in R/I is principal. (2) Use unique ideal factorisation in R: Decompose $I = P_1^{e_1} \cdots P_n^{e_n}$ into a product of powers of prime ideals. (3) Apply the Chinese remainder theorem to R/I. (4) It suffices to show that every ideal in $R/(P^e)$ is principal, where $P \subset R$ is a prime ideal, and $e \in \mathbb{Z}_{>0}$. (5) If $e = 1$, show that you are done. (6) If $e > 1$, take an element $x \in P \setminus P^2$. Show that every proper ideal of $R/(P^e)$ is generated by a power of $x \pmod{P}^e$.

Exercise 5 (Bonus). For all the following three polynomials:

$$f = X^2 - X + 174, \qquad X^2 - X + 190, \qquad \text{and} \qquad X^2 + 546$$

perform the following computation in Sage. Let $\alpha \in \mathbb{C}$ be a root of f. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\alpha)$. Use Sage to compute the following data about $\mathcal{O}$:

— the discriminant;
— the class number;
— generators of the class group;
— the order of every generator in your list.