Algebraic Number Theory 2018 — Set 7

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Deadline: 12.00 on Thursday, the 14th of June, 2018

Exercise 1 (8 points). A map $Q\colon \mathbb{Z}^2 \to \mathbb{Z}$ of the form $(x, y) \mapsto ax^2 + bxy + cy^2$ is called a (binary) *quadratic form*. An integer $n \in \mathbb{Z}$ is called *representable* by $\mathbb{Q}$ if there exist $x, y \in \mathbb{Z}$ such that $Q(x, y) = n$. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension of $\mathbb{Q}$ with discriminant Δ. Show that

(1) The norm map $N_{K/\mathbb{Q}}\colon \mathcal{O}_K \to \mathbb{Z}$ is a quadratic form with respect to every choice of basis.

(2) If K has class number $h_K = 1$, and p is a prime number unramified in K, then $\pm p$ is representable by $N_{K/\mathbb{Q}}$ if and only if $(\frac{\Delta}{p}) = 1$.

(3) Generalise (2) for class number $h_K > 1$: Let $\mathfrak{a}_1, \ldots \mathfrak{a}_{h_K}$ be ideals representing the ideal classes of K. To $\mathfrak{a}_i$, associate the function $Q_i\colon \mathfrak{a}_i^{-1} \to \mathbb{Z}$ given by $Q_i(\alpha) = \frac{N(\alpha)}{N(\mathfrak{a}_i^{-1})}$. Show that Q_i is a quadratic form. Show that $\pm p$ is representably by one of these quadratic forms if and only if $(\frac{\Delta}{p}) = 1$.

(4) Illustrate (3): Choose a quadratic extension $K/\mathbb{Q}$ such that $h_K > 1$, and explicitly calculate the quadratic forms Q_i. Explictly represent a prime number by one of these quadratic forms.

Hints: $(\frac{\Delta}{p}) = 1$ if and only if there is an ideal $\mathfrak{p}$ with $N(\mathfrak{p}) = p$. Write $\mathfrak{p} = a \cdot \mathfrak{a}_i$, and note that $a \in \mathfrak{a}_i^{-1}$.

Exercise 2. (See definition 3.2.2 and the lines following it, in the lecture notes.) Let V be an n-dimensional real vector space, and let $\Gamma \subset V$ be a lattice. Prove that the following are equivalent.

(0) Γ is complete.

(1) $V = \bigcup_{\gamma \in \Gamma} (\Phi + \gamma)$, where Φ is a principal mesh (Grundmasche) for Γ.

(2) The natural map $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism.

(3) V/Γ is compact.

Exercise 3.

(i) Let $A \to B$ be a ring homomorphism between two commutative rings. Let M be a free A-module. Show that $M \otimes_A B$ is a free B-module.

(ii) Let V be a finite-dimensional vector space over a field K. Show that there is a natural isomorphism $\phi\colon V \otimes V^* \to \mathrm{End}(V)$. Show that there is a natural map $\mathrm{ev}\colon V \otimes V^* \to K$. Show that $\mathrm{ev} \circ \phi^{-1}$ is the trace map $\mathrm{End}(V) \to K$.