Algebraic Number Theory 2018 — Set 9

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Deadline: 12.00 on Thursday, the 28st of June, 2018

Exercise 1. Compute the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for all squarefree integers $2 < d \leq 15$.

Exercise 2. Let $K_1 = \mathbb{Q}(\sqrt{d_1})$ and $K_2 = \mathbb{Q}(\sqrt{d_2})$ be distinct real quadratic fields. Define $K_3 = \mathbb{Q}(\sqrt{d_1 d_2})$ and $K = K_1 K_2$.

(i) Show that $\mathcal{O}_{K_1}^* \mathcal{O}_{K_2}^* \mathcal{O}_{K_3}^*$ has finite index in $\mathcal{O}_K^*$.

(ii) Take $d_1 = 2$ and $d_2 = 3$. Is $\sqrt{2} + \sqrt{3}$ in $\mathcal{O}_{K_1}^* \mathcal{O}_{K_2}^* \mathcal{O}_{K_3}^*$?

Exercise 3. Let m and $n \geq 0$ be integers, and denote with ϕ the Euler function. Prove that there is a number field K such that $\mathcal{O}_K^* \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}^n$ if and only if m is even and $\phi(m)$ divides $2(n+1)$.

Exercise 4. Let $K \subset L$ be an extension of number fields. Show that $[\mathcal{O}_L^* : \mathcal{O}_K^*]$ is finite if and only if K is a totally real field and L is a totally complex quadratic extension of K. (A number field is called *totally real* (resp. *complex*) if the image of every complex embedding lies dense in $\mathbb{R}$ (resp. $\mathbb{C}$).)

Exercise 5 (Sage)**.** Implement an algorithm in Sage that takes as input a squarefree positive integer d and computes the fundamental unit of $\mathbb{Q}(\sqrt{d})$.

(Of course you are not allowed to use the existing implementation in Sage.)