Exercises for the lecture
“Commutative Algebra and Algebraic Geometry”
SS 2019 Sheet 3,
Submission Date: 21.05.2019

Recall that for a ring A, an A-module M is said to be free if $M \cong \oplus_I A$, where I is an indexing set (may be infinite).

Exercise 1.

1. Is the $\mathbb{Z}$-module $M = \{(x, y, z) \in \mathbb{Z}^3 : 3x + 4y + 5z = 0\}$ free? If so, find a basis of the same.

2. Let k be a field. Is the ideal $\langle x \rangle \in k[x]$ free as an $k[x]$-module? Write a justification.

3. Is the ideal $\langle x, y \rangle \in k[x, y]$ free as an $k[x, y]$-module? Write a justification.

See Exercise 5 for a general statement.

(6 Points)

Exercise 2.

1. Let N denote the infinite product of $\mathbb{Z}$ as an $\mathbb{Z}$-module, i.e.,

$$N = \prod_{m=1}^{\infty} \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \dots$$

   and for each $i \in \mathbb{N}$, let $e_i \in N$ denote the element whose only non-zero entry is 1 at the i-th position. Prove that $\{e_i\}_i$ does not form a basis of N.

2. What is the $\mathbb{Z}$-submodule of N generated by the set $\{e_i\}_{i \in \mathbb{N}}$? Justify your claim.

3. Let M_1 denote a free $\mathbb{Z}$-module and let $v \in M_1$ be a non-zero element. Prove that there are only finitely many integers $n \in \mathbb{Z}$ such that the equation $v = nx$ admits a solution $x \in M_1$.

See Exercise 6.

(6 Points)

Exercise 3.
let $M \subset \mathbb{Z}^3$ be as in Exercise 1. Using the homomorphism theorem, prove that $\mathbb{Z}^3/M \cong \mathbb{Z}$.

(2 Points)

Exercise 4.
Let A be a ring and I an ideal contained in the Jacobson radical of A. Let M be an A-module, N a finitely generated A-module and let $\phi : M \to N$ be an A-module homomorphism. Using Nakayama's lemma prove that if the induced homomorphism $M/IM \to N/IN$ is surjective, then so is ϕ.

(4 Points)

Class Exercises(no points)

Remark 1 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.
Let A be a principal ideal domain. Then show that every submodule of a free finitely generated A-module is free.
(hint: prove it by induction on the rank of the free module).

Exercise 6.
Use Exercise 2, to prove that $N = \prod_{m=1}^{\infty} \mathbb{Z}$ is not a free $\mathbb{Z}$-module.