Fast track course ”Schemes” Winter 2022/23

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Dies ist ein Vorlesungsskript und kein Lehrbuch.
Mit Fehlern muss gerechnet werden!
We follow the book by Hartshorne. One lecture is roughly a §.

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Chapter 1

Presheaves and Sheaves

A general reference (besides Hartshorne II.1) is the book by Godement "Théorie des faisceaux"

X a topological space.

Definition 1.1. *A* presheaf $\mathcal{F}$ *(of abelian groups) consists of*

(a) for every $U \subset X$ *open an abelian group* $\mathcal{F}(U)$*;*

(b) for every $V \subset U$ *open in* X *a homomorphism called* restriction

$$\varrho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V) \qquad s \mapsto s|_V$$

such that

(0) $\mathcal{F}(\emptyset) = 0$*;*

(1) $\varrho_{UU} = \mathrm{id}$*;*

(2) For $W \subset V \subset U$ *open* $\varrho_{VW}\varrho_{UV} = \varrho_{UW}$.

Example. Let A be an abelian group. The *constant presheaf* is given by

$$U \mapsto A \text{ for all } U \neq \emptyset\ .$$

Example. M a set

$$U \mapsto \mathrm{Maps}(U, M)$$

Remark. Equivalently, a presheaf is a contravaraint functor

$$\mathcal{F} : X \to \underline{ab}$$

where X is the category with objects $U \subset X$ open and morphisms

$$\mathrm{Mor}_X(V, U) = \begin{cases} \mathrm{incl} & V \subset U \\ \emptyset & \text{else.} \end{cases}$$

This point of view is actually very helpful and also generalises well.

Definition 1.2. *A* morphism *of presheaves is a transformation of functors. We denote* $\mathrm{PSh} = \mathrm{PSh}_X$ *the category of presheaves on* X*.*

We are actually interested in special presheaves.

Definition 1.3. *A presheaf* $\mathcal{F}$ *is called* sheaf *if:*

(3) For every open cover $U = \bigcup_{i \in I} U_i$*: if* $s \in \mathcal{F}(U)$ *such that* $s|_{U_i} = 0$ *for all* i*, then* $s = 0$*.*

(4) For every open cover $U = \bigcup_{i \in I} U_i$*: given* $s_i \in \mathcal{F}(U_i)$ *for all* $i \in I$ *such that for all* $i, j \in I$
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$
then there is $s \in \mathcal{F}(U)$ *such that*
$$s|_{U_i} = s_i \ .$$

A morphism of sheaves is a morphism of presheaves. The category of sheaves on X *is denoted* $\mathrm{Sh} = \mathrm{Sh}_X$*.*

Remark. This is equivalent to exactness of

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

where the last map is given by

$$(s_i)_{i \in I} \mapsto \left(s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}\right)_{(i,j}$$

Example.

- $U \mapsto \mathrm{Maps}(U, M)$ for any set M
- $U \mapsto C(U, \mathbb{R})$ continuous maps because "continuity is local"
- if X is a complex manifold: $U \mapsto \mathcal{O}(X)$ holomorphic maps
- the constant presheaf A is not a sheaf in general, because for $X = U \cup V$ with $U \cap V = \emptyset$
$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \times \mathcal{F}(V) \to \mathcal{F}(U) \times 0 \times 0 \times \mathcal{F}(V)$$
with the last map the zero map.

Proposition 1.4. *Let* $\mathcal{F}$ *be a presheaf. Then there is a sheaf* $\mathcal{F}^+$ *and a morphism* $\mathcal{F} \to \mathcal{F}^+$ *(unique up to unique isomorphism) with the universal property that any morphism* $\mathcal{F} \to \mathcal{G}$ *into a sheaf* $\mathcal{G}$ *factors uniquely via* $\mathcal{F}^+$*. We call* $\mathcal{F}^+$ *the* sheafification *of* $\mathcal{F}$*.*

This is an explict way of saying that there is a functor $\cdot^+$ left adjoint to the inclusion $\mathrm{Sh} \to \mathrm{PSh}$. More generally:

Definition 1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Consider two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. We call F* left adjoint *to G and G* right adjoint *to F if*

$$\mathrm{Mor}_{\mathcal{C}}(A, G(B)) = \mathrm{Mor}_{\mathcal{D}}(F(A), B)$$

for all objects A of $\mathcal{C}$ and B of $\mathcal{C}$.

In this case F is uniquely determined by G and vice versa. Hence the issue is *existence*. By general principles it follows from the existence of direct limits.

Definition 1.6. *Let I be a* directed set*, i.e, a partially ordered set such that for any two elements i, j there is an element k with $i \leq k$ and $j \leq k$.*
A direct system *indexed by I is a collection A_i for $i \in I$ of abelian groups together with homomorphisms $\phi_{ij} : A_i \to A_j$ for $i \leq j$ satisfying*

$$\phi_{ii} = \mathrm{id} \quad \textit{for all } i \in I$$
$$\phi_{jk}\phi_{ij} = \phi_{ik} \quad \textit{for all } i \leq j \leq k$$

The direct or injective limit *$A = \varinjlim A_i$ of (A_i, ϕ_{ij}) is an abelian group A together with homomorphisms $\phi_i : A_i \to A$ compatible with the ϕ_{ij} in the obvious way which is universal with this property. Given another abelian group B and homomorphisms $f_i : A_i \to B$ compatible with ϕ_{ij} then they factor uniquely via $f : A \to B$.*

For more details see for example Atiyah-Macdonald.

Idea of Proof of the Proposition. Let $\mathcal{F}$ be a presheaf. We consider the directed set of all covers $U = \bigcup_{i \in I} U_i$ with morphisms the refinement maps of covers. We define $\mathcal{F}'$ by

$$\mathcal{F}'(U) = \varinjlim_{U = \bigcup_{i \in I} U_i} \mathrm{Ker}\left(\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right)$$

(This goes also under the name of 0-the Chech cohomology. Actually, the index set is *not* directed, but it becomes direct on these kernels.) We have

$$\mathcal{F}^+ = (\mathcal{F}')' \ .$$

For more details see e.g. Tamme, Introduction to etale cohomology. The argument in Hartshorne is different. It is less conceptual but uses only partially ordered sets. □

Proposition 1.7. *The categories of presheaves and sheaves are abelian. Sheafification is exact. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then for all $U \subset X$ open*

$$\mathrm{Ker}(f)(U) = \mathrm{Ker}(f(U) \qquad \mathrm{Coker}(f)(U) = \mathrm{Coker}(f(U))$$

If, in addition, $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaf $\mathrm{Ker}(f)$ *is a sheaf and is the kernel of f in* Sh*. Moreover, the sheafification of* $\mathrm{Coker}_{\mathrm{PSh}}(f)$ *is the cokernel in the category of sheaves.*

Proof. The case of presheaves is very easy. Let now $f : \mathcal{F} \to \mathcal{G}$ in Sh. We want to verify the sheaf axiom for $\mathrm{Ker}(f)$. Let $U = \bigcup_{i \in I} U_i$ be an open cover. We consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G}(U) & \longrightarrow & \prod \mathcal{G}(U_i) & \longrightarrow & \prod \mathcal{G}(U_i \cap U_j) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \prod \mathcal{F}(U_i) & \longrightarrow & \prod \mathcal{F}(U_i \cap U_j) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathrm{Ker}\, f(U) & \longrightarrow & \prod \mathrm{Ker}\, f(U_i) & \longrightarrow & \prod \mathrm{Ker}\, f(U_i \cap U_j) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
\end{array}$$

The first two lines are exact by the sheaf axioms for $\mathcal{F}$ and $\mathcal{G}$. The vertical sequences are exact by definition and because $\prod$ is exact. We see by a diagram chase that the last line is exact.

For the statement on the cokernel, we have to verify the universal property of the cokernel. Let

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

be morphisms of sheaves such that $gf = 0$. By the universal property of $\mathrm{Coker}_{\mathrm{PSh}}$, the morphism g factors uniquely via

$$\mathcal{G} \to \mathrm{Coker}_{\mathrm{PSh}}(f) \to \mathcal{H} \ .$$

By the universal property of sheafification the second map factors uniquely via $\mathrm{Coker}_{\mathrm{PSh}}(f)^+$. □

Remark. Warning: the functor $\bigoplus$ is exact on sheaves. However, $\prod$ is not!

There is a very useful tool to study sheaves.

Definition 1.8. *Let $\mathcal{F}$ be a presheaf and $P \in X$ a point. The* stalk *of $\mathcal{F}$ in P is given by*

$$\mathcal{F}_P = \varinjlim_{P \in U} \mathcal{F}(U) \ .$$

Elements in $\mathcal{F}_P$ are called germs*. A germ is represented by an open neighbourhood U of P and a section $s \in \mathcal{F}(U)$.*

Lemma 1.9. *Sheafification preserves stalks. As sequence of sheaves is exact if and only if the induced sequences of stalks are exact for all $P \in X$.*

Note that the section functors

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$$

are exact on presheaves but only left exact on sheaves.

Finally, we want to relate sheaves for a change of topological space.

Definition 1.10. *Let $f : X \to Y$ be continuous. Let $\mathcal{F}$ be a sheaf on X. We define the* direct image $f_*\mathcal{F} \in \mathrm{Sh}_Y$ *by*

$$U \mapsto \mathcal{F}(f^{-1}(U))$$

Let the inverse image $f^{-1} : \mathrm{Sh}_Y \to \mathrm{Sh}_X$ *be left adjoint to f_*.*

Remark. Depending on the reference, the left adjoint is often called f^*. Hartshorne reserves this for the case of $\mathcal{O}_X$-modules, see later.

As with any adjoint, one needs to check existence.

Lemma 1.11.

$$f^{-1}\mathcal{G}(U) = \varinjlim_{f(U)\subset V} \mathcal{G}(V)$$

where the limit is over all open neighbourhoods V of $f(U)$. For $P \in X$

$$(f^{-1}\mathcal{G})_P = \mathcal{G}_{f(P)} \ .$$

The functor f^{-1} is exact and f_* is left exact.

Exercises

(Needs basic complex analysis) On $X = \mathbb{C}$ consider the sheaves $\mathcal{O}$ of holomorphic functions and $\mathcal{O}^*$ of non-vanishing holomorphic functions and the map

$$\exp : \mathcal{O} \to \mathcal{O}^* \qquad f \mapsto \exp \circ f$$

Check that it is not surjective as morphism of presheaves but surjective as morphism of sheaves.
Hartshorne II 1.3, 1.8, 1.17, 1.19

Chapter 2

Schemes

We follow Hartshorne Kapitel II.2.

Definition 2.1. *A* scheme *is a locally ringed space* $i(X, \mathcal{O}_X)$ *such that for every point there is a an open neighbourhood* U *and a ring* A *such that*

$$(U, \mathcal{O}_X|_U) \cong (\mathrm{Spec}A, \mathcal{O}) \ .$$

We need to explain:

Definition 2.2. *A* locally ringed space *is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings, i.e., all* $\mathcal{O}_X(U)$ *are rings, all* ϱ_{UV} *are ring homomorphisms.* $\mathcal{O}_X$ *is called* structure sheaf. *A* locally ringed space *is a ringed space such that all stalks* $\mathcal{O}_{X,P}$ *are local rings.*

All rings are commutative with 1. We allow the ring 0 with $0 = 1$.

Example. (i) Let X be a topological space, $\mathbb{Z}$ the constant sheaf $\mathbb{Z}$ on X (i.e, the sheafification of the constant presheaf $\mathbb{Z}$). Then $(X, \mathbb{Z})$ is a ringed space, but not locally ringed.

(ii) Let X be a smooth manifold, $\mathcal{C}$ the sheaf of differentiable functions. Then $(X, \mathcal{C})$ is a locally ringed space because a germ of a differentiable function is invertible if and only if its value is non-zero. (Analysis 2).

(iii) Let X be a Riemann surface, $\mathcal{O}$ the sheaf of holomorphic functions. Then $(X, \mathcal{O})$ is locally ringed for the same reason (Funktionentheorie).

(iv) Let V be an affine variety, $\mathcal{O}_V$ the sheaf of algebraic/regular functions on V. Then $(V, \mathcal{O}_V)$ is a locally ringed space.

The spectrum of a ring

Definition 2.3. *Let* A *be ring. Let*

$$\mathrm{Spec}A = \{\mathfrak{p} \subset A | \mathfrak{p} \textit{ prime ideal } \}$$

For every ideal I let

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}A | I \subset \mathfrak{p}\}$$

For $f \in A$ let

$$V(f) = V(\langle f \rangle)$$

and

$$U_f = \mathrm{Spec}A \setminus V(f)$$

The Zariski topology *on* Spec$A$ *has the sets of the form $V(I)$ as closed sets.*

Lemma 2.4. *This is a topology.*

Proof.

$$V(0) = \mathrm{Spec}A$$
$$V(1) = \emptyset$$
$$\bigcap_{j \in J} V(I_j) = V(\sum I_j)$$
$$V(I_1) \cup V(I_2) = V(I_1 I_2)$$

□

- A point of Spec$A$ is closed if and only if $\mathfrak{p}$ is maximal.
- If A is without zero divisors, then $0 \in \mathrm{Spec}A$ and this point is dense.
- The open sets of the form U_f are a basis of the topology because

  $$V(I) = \bigcap_{f \in I} V(f) \ .$$

  We have

  $$U_f \cap U_g = U_{fg} \ .$$

- Spec$A$ is quasi-compact, ie., every open cover has a finite subcover.

  Proof. Without loss of generality we consider a cover by U_{f_i} für $i \in I$. By assumption

  $$\bigcap_{i \in I} V(f_i) = V(\langle f_i | i \in I \rangle) = \emptyset$$

  If $I = \langle f_i | i \in I \rangle$ was a proper ideal, it would be contained in a maximal ideal, making $V(I)$ non-empty. Hence I is not a proper ideal, i.e, it contains 1. Then

  $$1 = \sum_{i \in I} a_i f_i$$

  where only finitely many $a_i \neq 0$. Hence only finitely many of U_{f_i} suffice. □

Proposition 2.5. *There is a unique sheaf of rings* $\mathcal{O}$ *on* $\mathrm{Spec} A$ *such that*

$$\mathcal{O}(U_f) = A_f$$

For $\mathfrak{p} \in \mathrm{Spec} A$ *we have*

$$\mathcal{O}_{\mathfrak{p}} = A_{\mathfrak{p}}$$

Proof. Uniqueness follows from the sheaf condition because every U can be coverd by U_f's. The computation of the stalk follows because

$$\mathcal{O}_{\mathfrak{p}} = \lim_{\mathfrak{p} \in U_f} A_f = \mathcal{O}_{\mathfrak{p}}$$

The essential step in the proof of existence is the verification of the sheaf condition in the special case $U_f = \bigcup_{i \in I} U_{f_i}$, see Hartshorne. □

This finishes the definition of schemes!

Morphisms

Definition 2.6. *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be ringed spaces. A morphism*

$$(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a continuous map

$$f : X \to Y$$

together with a morphism of sheaves of rings on Y

$$f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$$

A morphism of locally ringed spaces is a morphism of ringed spaces such that in addition for all $P \in X$ *the induced morphism*

$$f_P^\# : \mathcal{O}_{Y, f(P)} \to \mathcal{O}_{X,P}$$

is local, i.e., the preimage of the maximal ideal is the maximal ideal.
A morphism of schemes is a morphism of locally ringed spaces.

It is easy to see that

$$A \mapsto (\mathrm{Spec} A, \mathcal{O})$$

is a contravariant functor from the category of rings into the category of locally ringed spaces. Conversely,

$$(X, \mathcal{O}_X) \to \mathcal{O}_X(X)$$

is a functor form the category of locally ringed spaces into the category of rings. The composition

$$A \mapsto (\mathrm{Spec} A, \mathcal{O}) \to \mathcal{O}(\mathrm{Spec} A)$$

is the identity. This is not obvious for the converse composition!

Proposition 2.7. *Let A, B be rings. Every morphism of locally ringed spaces*

$$(f, f^{\#}) : (\mathrm{Spec} A, \mathcal{O}) \to (\mathrm{Spec} B, \mathcal{O})$$

is induced by a ring homomorphism $\phi : B \to A$.

In other words, the category of rings is equivalent to a full subcategory of the category of schemes.

Remark. Only when actually going through this proof, it becomes clear why the strange locality condition in the definition of morphisms is there.

Excercises

(i) Let $(X, \mathcal{O}_X)$ be a locally ringes space such that every point has a neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(U', \mathcal{C})$ where $U \subset \mathbb{R}^n$ is open and $\mathcal{C}$ the sheaf of differentiable functions on U. Show that X has a unique structure of a differentiable manifold such that $\mathcal{O}_X$ is the sheaf of differentiable functions on U.

(ii) Work out $\mathrm{Spec} A$ and its topology in the following cases: $\mathbb{Z}$, k field, $k[X]$ and $k[X, Y]$ for k algebraically closed field, $\mathbb{Z}_p$ (p-adic numbers), $\mathbb{Z}/p^n$ for p prime, $k[[X]]$ for k a field.

(iii) $\mathrm{Spec}(A \oplus B) = \mathrm{Spec} A \cup \mathrm{Spec} B$.

Hartshorne II.2.1, 2.3, 2.5, 2.7, 2.19

Chapter 3

Examples and first properties

Hartshorne II.2 2. half and II.3 first half

Varieties

Let k be an algebraically closed field, $V \subset \mathbb{A}^n$ affine variety over k. We have the functors

$$V \mapsto k[V] \mapsto \mathcal{V} = \mathrm{Spec}(k[V])$$

We want to understand $\mathcal{V}$ better. By Hilbert's Nullstellensatz the points of V correspond to the maximal ideals of $k[V]$. Hence:

Definition 3.1. *Let X a topological space. Let $|X|$ be the set of closed point of X with the induced topology.*

Proposition 3.2. *$V = |\mathcal{V}|$ as topological space.*

Proof. The closed set in V have the form

$$\{P \in V | f(P) = 0 \text{ for all } f \in I\}$$

for an ideal $I \subset k[V]$. The closed sets in $\mathcal{V}$ have the form

$$\{\mathfrak{p} \in \mathrm{Spec}(k[V] | f \in \mathfrak{p} \text{ for alle } f \in I\}$$

By the Nullstellensatz we have

$$V(I) = |V(I)|$$

□

Corollary 3.3. *The map $i : V \to \mathrm{Spec}(k[V])$ is continuous and induces a bijections between the topologies. In particular, sheaves on V are equivalent to sheaves on* $\mathrm{Spec}(k[V])$.

Corollary 3.4. *$V \to \mathrm{Spec}(k[V])$ is a morphism of locally ringed spaces.*

Proof. Let $U \subset \mathrm{Spec}(k[V])$ be open. Then

$$\mathcal{O}_{\mathcal{V}}(U) = \mathcal{O}_V(|U|)$$

It suffices to check this on sets of the form U_f for $f \in k[V]$. In this case both sides are equal to $k[V]_f$. □

How do we get $\mathcal{V}$ from V? The additional points are prime ideals. They correspond to the irreducible closed subsets of V.

Definition 3.5. *Let X be a topological space. We define*

$$\tilde{X} = \{Z \subset X | Z \textit{ irreducible, closed } \}$$

with the topology where the sets of the form $\tilde{Z}$ with $Z \subset X$ are closed.

Corollary 3.6. $\mathcal{V} = \tilde{V}$

The objects V and $\mathcal{V}$ carry the same information. It is standard to identifiy them implicitly. What about morphisms?

$$\mathrm{Mor}_{\mathrm{Var}_k}(V, W) = \mathrm{Mor}_{k-\mathrm{Alg}}(k[W], k[V])$$
$$\mathrm{Mor}_{\mathrm{Sch}}(V, W) = \mathrm{Mor}_{\mathrm{Ring}}(k[W], k[V])$$

The category of k-varieties is *not* a full subcategory of the category of schemes! This is because the scheme $\mathcal{V}$ does not know about the ground field k.

Grothendieck: Everything is relative!

Definition 3.7. *Let S be a scheme. The category of S-schemes has as objects morphisms $X \to S$ and as morphisms those scheme morphisms for which the obvious diagram commutes. We write X/S or simply X instead of $X \to S$.*

Example. Let $S = \mathrm{Spec}A$. A morphism $X \to S$ is equivalent to having $\mathcal{O}_X$ a sheaf of A-algebras (i.e., we are choosing such a structure!) For a morphism $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of S-schemes the map $f^\#$ is A-linear.

Proposition 3.8. *The category of k-varieties is a full subcategory of the category of k-schemes.*

We generalise:

Definition 3.9. *Let A be a ring. Then*

$$\mathbb{A}^n_A = \mathrm{Spec}A[X_1, \ldots, X_n]$$

is called n-dimensional affine space over A.

Projective varieties

Let V/k be a projective variety. Then $\mathcal{V} = \tilde{V}$ with the structure sheaf

$$\mathcal{O}_{\mathcal{V}}(U) = \mathcal{O}(|U|)$$

is a scheme, because locally a projective variety is affine. We can also define $\mathcal{V}$ directly in the language of schemes.

Definition 3.10. *Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and $S_+ = \bigoplus_{d>0} S_d$. We put*

$$\mathrm{Proj}(S) = \{\mathfrak{p} \subset S| \text{ homogenous prime ideal with } S \subsetneq \mathfrak{p}\}$$

with the Zariski topology with respect to homogenous ideals.

As in the affine case there is a basis of the topology consisting of open sets U_f for $f \in S$ homogenous. The structure sheaf is uniquely determined by

$$\mathcal{O}_{\mathrm{Proj}S}(U_f) = S_{(f)} = \left\{ \frac{a}{f^d} \in S_f | a \in S_d \right\}$$

i.e., $S_{(f)}$ consists of the homogenous elements of degree 0 in the $\mathbb{Z}$-graduated ring S_f.

Lemma 3.11. *This is a scheme.*

Example. Let A be a ring, $S = A[X_0, \ldots, X_n]$. Then

$$\mathbb{P}^n_{\mathrm{Spec}A} = \mathbb{P}^n_A = \mathrm{Proj}S$$

is called *projective space over* Spec$A$.

3.0.1 First properties

A scheme $(X, \mathcal{O}_X)$ is called:

- *affine*, if it is isomorphic to $(\mathrm{Spec}A, \mathcal{O})$ (then: $A = \mathcal{O}_X(X)$)
- *connected/irreducible*, if the topological space X is connected/irreducible.
- *reduced*, if all $\mathcal{O}_X(U)$ are reduced (only 0 is nilpotent). (Equivalently: all stalks are reduced; there is an affine open cover where the sections are reduced)
- *integral*, if all $\mathcal{O}_X(U)$ are integral domains, i.e., without zero divisors (Equivalently: reduced and connected)
- *locally noetherian*, if there is an open affine cover by SpecA_i with A_i *noetherian*
- *noetherian*, if it is there is a finite open cover by SpecA_i with A_i noetherian

The *dimension* of X is its dimension as a topological space, i.e., the maximal length of a chain of irreducible closed subsets of X

Example. $\mathbb{A}_A^n$ has dimension $n + \dim A$.

A morphism $f : X \to Y$ of schemes is called

- *affine*, if $f^{-1}(U)$ affine for all affine $U \subset Y$
- *locally of finite type*, if there is an open affine cover $\{U_i\}_{i \in I}$ of Y, such that $f^{-1}U_i$ has an open cover $\{V_{ij}\}_{j \in J_i}$, such that the morphism

  $$f^{\#} : \mathcal{O}(U_i) \to \mathcal{O}(V_{ij}$$

  turns $\mathcal{O}(V_{ij})$ into a finitely generated $\mathcal{O}_{U_i}$-algebra. I.e, locally on X and Y the ring homomorphism is of finite type.
- *of finite type*, if in addition, the index sets J_i can be chosen to be finite.
- *finite*, if f is affine and for $U = \mathrm{Spec} A \subset Y$ open is $\mathcal{O}(f^{-1}(U))$ a finite A-Algebra, i.e., finitely generated as an A-module.

Excercises

Hartshorne II 2.10, 2.11, 3.5, 3.6, 3.8

Chapter 4

Immersions and points

Ha II.3

4.1 Immersions

Let $U \subset X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is a scheme and $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ a morphism of schemes.

Definition 4.1. *A morphism of this form is called* open immersion.

Closed subsets are more difficult. Let $V(I) \subset \mathrm{Spec}A$ closed. Then $V(I) \cong \mathrm{Spec}A/I$ as topological spaces. The induced map $\mathrm{Spec}A/I \to \mathrm{Spec}A$ is is a morphism of schemes.

Definition 4.2. $i : Y \to X$ *is called* closed immersion*, if there is an open affine cover* $\{U_i\}_{i \in I}$ *of* X *such that* $i : f^{-1}(U_i) \to U_i$ *is of the form* $\mathrm{Spec}A/I \to \mathrm{Spec}A$*. Then* Y *is called a* closed subscheme *of* X*.*

Hence closed immersions are affine and even finite!
The scheme structure on $V(I)$ depends on the choice of I, it is *not* unique.

Example. k field, $X = \mathbb{A}^1_k$, $x = 0 \in k$ viewed as a point of $\mathbb{A}^1_k$. The corresponding maximal ideal is generated by T. Let $I = (T)$. Then

$$V(T) = V(T^2) = \{(T)\}$$

is the point zero in $\mathbb{A}^1_k$. ABut

$$\mathrm{Spec}(k[T]/T^2) \neq \mathrm{Spec}(k[T]/T)$$

as schemes. Both are closed subschemes of the line.
Indeed,

$$\mathrm{Spec}(k[T]/T) \to \mathrm{Spec}(k[T]/T^2)$$

is also a closed immersion. In the language of schemes, we do not view these "thickened" points as a stupid accident, but rather as a nice thing to have. It is the difference between intersecting two lines or a line with a parabola in a point of tangency.

4.2 Points

Example. Let k be algebraically closed. Then

$$x \mapsto x^2$$

defines a morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$. It is a morphism, because x^2 is a polynomial in x.
How do we treat this in the scheme situation? Let R be a ring. We often write

$$f : \mathbb{A}^1_R \to \mathbb{A}^1_R \qquad x \mapsto x^2 \ .$$

What should this mean for prime ideals? One clear way is to look at the corresonding homomorphism of rings:

$$R[T] \leftarrow R[T] : f^{\#}$$

It is R-linear and maps T to T^2. This is correct, but unintuitive. It would mean to work out the effect of a map in terms of generators and relations.
We work with T-valued points instead.

Definition 4.3. *Let X/S be an S-scheme. For every T/S we call*

$$X(T) = \mathrm{Mor}_S(T, X)$$

the set of T-valued points *of X.*

Example. Let $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k[X,Y]/f)$ and K/k be an extension of fields. Then

$$\begin{aligned} X(\mathrm{Spec}K) &= \mathrm{Mor}_k(\mathrm{Spec}K, X) \\ &= \mathrm{Mor}_k(k[X,Y]/f, K) = \{(x,y) \in K^2 | f(x,y) = 0\} \end{aligned}$$

Example. Let $S = \mathrm{Spec}R$, $X = \mathbb{A}^1_R$. Then

$$X(T) = \mathrm{Mor}_S(T, \mathbb{A}^1_R) = \mathcal{O}_T(T)$$

In order to check this, we first consider affine $T = \mathrm{Spec}R'$. Then we have again

$$X(T) = \mathrm{Mor}_R(R[T], R') = R' = \mathcal{O}(\mathrm{Spec}R')$$

The general case follows by gluing morpisms.

The assignment $T \mapsto X(T)$ is a functor.

Lemma 4.4. *Let $X, Y/S$ be two S-schemes. Then a morphism $f : X \to Y$ of S-schemes is the same as a transformation of functors*

$$f(\cdot) : X(\cdot) \to Y(\cdot)$$

Proof. This is the Yoneda lemma. It is true in any category. □

It is often easier and more intuitive to specify a map on T-valued points!

Example. Let $X = \mathbb{A}^1_R$. Then $x \mapsto x^2$ is the map on T-valued points

$$X(T) = \mathcal{O}(T) \to \mathcal{O}(T) \qquad x \mapsto x^2$$

In the situation of varieties, we get back the map we had there.

The point of view of T-valued points allows us to check equalities.

Example.

$$(x \mapsto x^2) \circ (x \mapsto x^2) = (x \mapsto x^4)$$

as morphisms von schemes, because the assertion is true for T-valued points for all T.

The method requires a functorial assigment, but this is something we have anyway most of the time. This holds in particular if the object X was defined by its T-valued points in the first place. We call this a *moduli space.*
This point of view also allows us to understand how to translate a definition from the category of sets to the category of schemes. Of particular importance is the case of *fibre products.*

4.3 Fibre product

Definition 4.5. *Let X, Y, S sets and $f : X \to S$ and $g : Y \to S$ maps. Then*

$$X \times_S Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$$

is called fibre product of X and Y over S.

Fibrewise, we have the product: for $s \in S$ we have

$$(X \times_S Y)_s = X_s \times Y_s$$

The fibre product has a universal property, the property of a product in the category of sets over S. More precisely:
Let $\phi : T \to X$ and $\gamma : T \to Y$ be maps with $g\gamma = g\phi$, then there is a unique map

$$(\phi, \gamma) : T \to X \times_S Y$$

over which ϕ and γ factor.

Definition 4.6. *Let X, Y be two S-schemes. Then the* fibre product $X \times_S Y$ *is the uniquely determined scheme with*

$$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T)$$

for alle T/S. Equivalently, $X \times_S Y$ has the universal property of fibre products in the category of S-schemes.

Proposition 4.7. *The fibre product exists.*

Proof. We obtain it by gluing the fibre product in the affine case.
In the first step, we reduce to S affine, then X affine and finally Y affine.
Now let $X = \mathrm{Spec} A, Y = \mathrm{Spec} B, S = \mathrm{Spec} C$ affine. We consider the universal property for $T = \mathrm{Spec} R$ and get precisely the universal property of the tensor product. A homorphism $A \otimes_C B$ of C-algebras towards R corresponds 1:1 to a pair of C-algebra homomorphisms $A \to R$ and $B \to R$. □

Let $s \in S$ be a point of the topological space. Then we get morphism of schemes

$$s : \mathrm{Spec}(\kappa(s)) \to S$$

where $\kappa(s) = \mathcal{O}_{S,s}/m_s$ is the *residue field of s.*

Definition 4.8. *Let $f : X \to S$ be a morphism, $s \in S$ a point. Then*

$$X_s = X \times_S \mathrm{Spec}(\kappa(s))$$

is called fibre of f over s.

Remark. In this situation we really have

$$X_s = f^{-1}(s)$$

as sets. The scheme structure is described by the definition!

Definition 4.9. *Let S' be an S-scheme. Then the functor*

$$\mathrm{Sch}_S \to \mathrm{Sch}_{S'} \qquad X \mapsto X \times_S S'$$

is called base change.

Example. For every ring R we have

$$\mathbb{A}^n_{\mathrm{Spec} R} = \mathbb{A}^n_{\mathrm{Spec}\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec} R$$

hence we define for every scheme S

$$\mathbb{A}^n_S = \mathbb{A}^n_{\mathrm{Spec}\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} S$$
$$\mathbb{P}^n_S = \mathbb{P}^n_{\mathrm{Spec}\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} S$$

Easy to check: the properties affine, finite, finite type, locally of finite type, open immersion, closed immersion are stable under base change.

Excercises

Ha II 3.9, 3.10, 3.13

Let X/S and Y/S be schemes and $f : X \to Y$ an S-morphism. We define its *Graph* $\Gamma_f \subset X \times_S Y$ as

$$\Gamma_f(T) = \{(x, f(x)) \in X(T) \times_{S(T)} Y(T)\}$$

Show that the natural map

$$\Gamma_f \to X$$

is an isomorphismus, in particular Γ_f is a scheme.

Let $g : Y \to Z$ another S-morphism. Show:

$$\Gamma_{g \circ f} = \Gamma_f \times_Y \Gamma_g$$

Chapter 5

Separated and proper morphisms

Ha II.4

We want to discuss the analogues of "Hausdorff" and "compact". We start in the topological case.

5.1 Separated

Let M be a topological space that is locally homoeorphic to an open subset of $\mathbb{R}^n$.

Definition 5.1. *M is called* hausdorff*, if any two points x, y have disjoint open neighbourhoods, i.e., $x \in U_x, y \in U_y$ such that $U_x \cap U_y = \emptyset$.*

Open subsets of $\mathbb{R}^n$ are hausdorff. With the same argument open subsets of metric spaces are hausdorff. However, this is not true for M as above. To be hausdorff is a *global* property, not a local one.

Example. Let $M = \mathbb{R} \times \{0,1\}/\sim$ with the equivalence relation $(x,0) \sim (x,1)$ for all $x \neq 0$. This is a line with the origin doubled. We call a subset of M open, if its preimage in $\mathbb{R} \times \{0,1\}$ is open. The two origins *cannot* be separated by open neighbourhoods.

In differential geometry, we always make the additional assumption that all spaces are hausdorff. We have the analogous problem in the category of algebraic varieties. This uses a different characterisation of hausdorff.

Lemma 5.2. *Let M be a topological space. M is hausdorff, if and only if the diagonal $\{(x,x) | x \in M\}$ is a closed subset.*

Definition 5.3. *A scheme X is called* separated*, if the diagonal*

$$\Delta : X \to X \times X$$

is a closed immersion. A morphism $f : X \to X$ is called separated*, if the relative diagonal*

$$X \to X \times_S X$$

is a closed immersion.

This property is local in S, but not in X.

Example. (i) Affine schemes are always separated over $\mathbb{Z}$, because the diagonal

$$\mathrm{Spec}A \to \mathrm{Spec}A \otimes_{\mathbb{Z}} A$$

corresponds to mulitplication $A \otimes A \to A$, hence it is surjektive.

(ii) Open and closed immersions are separated.

(iii) The doubled line $\mathbb{A}^1 \times \{0, 1\}$ glued along U_0 is not separated. The product with itself has 4 points 0×0 (all in the closure of $\Delta(\mathbb{A}^1 - \{0\})$, but only two of them are in the image of the diagonal.

Further properties

- stable under composition.
- stable under base change $S' \to S$.
- stable under products.
- $f : X \to Y$, $g : Y \to Z$, $g \circ f$ separated. Then f is separated.

The last property allows us to consider the category of separated S-schemes. All morphisms in this category are separated.

Definition 5.4. *Let k be a field. A variety over k is as scheme over k, that is (irreducible?) reduced, of finite type and separated.*

Quasi-projective varieties are always separated, hence the property is never mentioned in the classical theory.
The easiest method to verify the properties is the valuative criterion. We will get to this later.

5.2 Proper (eigentlich)

We return to the topological situation. A topological space is called *quasi-compact*, if every open cover has a finite subcover. It is called *compact*, if it is quasi-compact and hausdorff.

Example. (i) Closed and bounded subset of $\mathbb{R}^n$ are compact.

(ii) Projektive spaces (over $\mathbb{R}$ or $\mathbb{C}$ with the metric topology) are compact.

Lemma 5.5. *(i) Every quasi-compact subset of a topological space is closed.*

(ii) Every closed subset of a compact topological space is compact.

(iii) Images of quasi-compact subsets under continuous maps are quasi-compact.

(iv) Every continuous map of compact spaces is closed, i.e., images of closed subsets are closed.

In fact, we want to treat the relative version, because everything is always relative for schemes.

Definition 5.6. *A continuous map $f : M \to M'$ of topological manifolds is called* proper *(German: eigentlich), if preimages of compact sets are compact.*

The inclusion of a closed subset is proper, the inclusion of an open subset usually not. Proper covers are also called unbounded ("unbegrenzt").

Lemma 5.7. *Proper maps are closed and stable under base change.*

Definition 5.8. *A morphism $f : X \to Y$ of schemes is called* proper*, if it is separated, of finite type, and, in addition, universally closed, i.e., every base change is closed as a map of topological spaces.*

Example. (i) Closed immersions are proper. Open immersions usually not.

(ii) $\mathbb{P}^n_S$ is proper over S.

Further properties:

- Stable under composition.
- Stable under base change
- Stable under products
- $f : X \to Y$ and $g : Y \to X$ morphisms, $g \circ f$ proper, g separated. Then f is proper.

The last property implies that are morphisms between proper S-schemes are proper.
These properties are again verified by a valuative criterion.

5.3 Valuation rings

We return to the topological situation.

Lemma 5.9. *Let M be locally homeomorphic to open subsets $\mathbb{R}^n$. M is quasi-comact, if every sequence has a converging subsequence. It is hausdorff, if the limit of a converging sequence is unique. M is compact if the limit exits and is unique.*

We cannot work with sequences. We rewrite a bit and consider continuous maps

$$f : (-\varepsilon, 0) \to M$$

If M is quasi compact, the f extends to $(\varepsilon, 0]$. If M is hausdorff, the extension is unique.
What is the analogue of an interval in algebraic geometry? We are closer to complex analysis, hence we really want an analogue of disc

$$\Delta_r = \{z \in \mathbb{C} | |z| < r\}$$

We imagine the radius arbitrarily small

$$\lim_{r \to 0} \mathcal{O}(\Delta_r),$$

i.e. germs of holomorphic functions in $0 \in \mathbb{C}$. This the ring of converging power series (with variable radius of convergence).

Definition 5.10. *Let K be a field. A* valuation *on K is a map*

$$v : K - \{0\} \to G$$

with values in a totally ordered abelian group such that

(i) $v(xy) = v(x) + v(y)$

(ii) $v(x + y) \geq \min(v(x), v(y))$

We often put $v(0) = \infty$. The valuation ring *of v is*

$$R = \{x \in K | v(x) \geq 0\}$$

A ring R is called valuation ring*, if it is isomorphic to the valuation ring of a valuation.*

Remark. Valuation rings are local rings with maximal ideal

$$m = \{x \in K | v(x) > 0\}$$

Example. (i) Let $R = k[[t]]$ be the ring of formal power series, $K = k((t))$ the field of formal Laurent series. Its elemetns have the form $f = \sum_{i \geq n} a_i t^i$ with $n \in \mathbb{Z}$. If $a_n \neq 0$ then $v(f) = n$.

(ii) Let k be algebraically closed, C a non-singular curve over k, $P \in C$ a point. Then $\mathcal{O}_p$ is a valuation ring. We choose a local parameter (a function in $m_p \smallsetminus m_p^2$, e.g., a coordinate function). Then every non-zero element has the form $t^n u$ with a u. Its valuation is n.

(iii) The same works for Riemann surfaces and holomorphic functions. We get back the above local ring of germs of holomorphic functions.

(iv) Let $K = \mathbb{Q}$, p a prime number and $v_p(x)$ the maximal p-power in x. Then v_p is a valuation with valuation ring $\mathbb{Z}_{(p)}$, fractions with denominator prime to p.

(v) Let K be the algebraic closure of $k((t))$. The valuation of (i) extends uniquely to K. The set of values is not longer $\mathbb{Z}$ but $\mathbb{Q}$.

Definition 5.11. *A valuation ring is called* discrete*, if it takes values in* $\mathbb{Z}$.

In this case we can normalise the valuation such that it becomes surjective.
A valuation ring is discrete if and only if it is noetherian. In this case, it is a principal ideal domain with a unique maximal ideal. The maximal ideal is generated by any element with valuation 1.

Excercises

Let A be a one-dimensional local noetherian integral comain with maximal ideal m and residue field k. The following are equivalent:

(i) A is a discrete valuation ring

(ii) A is integrally closed

(iii) m is a principal ideal

(iv) $\dim_g(m/m^2) = 1$

(v) Every non-zero ideal is is a power of m

(vi) There is $x \in A$ such that every non-zero ideal has the form (x^v) with $v \geq 0$.

Ha II 4.1, 4.2, 4.3, 4.8

Chapter 6

Valuative criteria

Ha II 4, auch Görtz-Wedhorn §15.3

Let $X \to Y$ be a morphism of schemes, R a valuation ring with field of fractions K. Let $T = \mathrm{Spec}R$, $\eta = \mathrm{Spec}K$. We consider a commutatie diagram of the form

$$\begin{array}{ccc} \eta & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

We ask about lifts $T \to X$.

Theorem 6.1. *Let X locally noetherian.*

(i) A morphism $f : X \to Y$ is separated if and only if for every valuation ring R and every diagram as above a lift $T \to X$ is unique (if it exists).

(ii) A morphism $f : X \to Y$ is proper if and only if f is of finite type and for every valuation ring R and every diagram as above, a unique lift exists.

If Y is noetherian and f locally of finite type, it suffices to consider discrete valuation rings.

There is an even more general version where the condition X "locally noetherian" is removed. Then f has to be quasi-separated, i.e., for $V \subset Y$ affine, open and $U_1, U_2 \subset f^{-1}(V)$ affine, open $U_1 \cap U_2$ is quasi-compact.

We first show how to apply the criterion.

Corollary 6.2. *Let $f : X \to Y$, $g : Y \to Z$ separated/proper. then $g \circ f$ is separated/proper.*

Proof. Consider the diagram as above. The lift to Y exists (exists and is unique) because g is separated. The lift to X existits (exists uniquely), because f is separated (proper). $\square$

Definition 6.3. *A morphism $f : X \to Y$ is called* projective*, if it factors over a closed immersion into $\mathbb{P}^n_Y$. It is called* quasi-projective*, if it factors into an open immersion followed by a projective morphism.*

Corollary 6.4. *Projective morphisms are proper. Quasi-projective morphisms are separated.*

Proof. Because of the computation rules, it suffices to show that $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}\mathbb{Z}$ is proper. This is done by explicit calculation with the valuative criterion. $\square$

In order to understand the valuative criterion better, we first need to understand morphisms of U and T into some scheme. We restrict to the noetherian case. As a warm-up:

Lemma 6.5. *Let $\eta = \mathrm{Spec}(K)$ for a field K. A morphism $\eta \to X$ into a scheme X consists of the choice of a point $x \in X$ and a ring homomorphismus*

$$\kappa(x) \to K\ .$$

Proof. $\mathrm{Spec}(K)$ consists of a single point. The point x is its image. It is contained in some open affine $U = \mathrm{Spec}A \subset X$. In order to understand $\eta \to X$, it suffices to understand $\eta \to U$. Now we are in the affine situation, i.e, we are dealing with a ring homomorphism

$$A \to K$$

It factors uniquely via A/P for a prime ideal P (in fact via $x \in \mathrm{Spec}(A)$). The residue field $\kappa(x) = (A_P)/A_P P$ is the field of fractions of A/P. The ring homomorphism $A/P \to K$ factors uniquely via the field of fractions.
Conversely, the composition $A \to \kappa(x) \to K$ defines a ring homomorphism. $\square$

Now let R be a discrete valuation ring R with field of fractions K and residue field k. Its spectrum $T = \mathrm{Spec}(R)$ has two points $\eta = \mathrm{Spec}(K)$ (generic point) and $\xi = \mathrm{Spec}(k)$ (special point). The map $\eta \to T$ is an opem immersion, $\xi \to T$ a closed immersion.
Consider $f : T \to X$. Let $x_0 = f(\xi)$, $x_1 = f(\eta)$. We consider the closure

$$Z = \{x_1\}^-$$

Then $f^{-1}Z$ is a closed subset of T, containing η. Hence $f^{-1}Z = T$. In other words: $x_0 \in Z$. We say: x_0 is a *specialisation* of x_1.
Consider an open affine neighbourhood of U of x_0. By construction Z is irreducible with generic point x_1. Every open subset of Z contains x_1, in particular $U \cap Z$. From now on, we may replace X by U and are in the affine situation. Then $Z = V(\mathfrak{p}_1)$ where $\mathfrak{p}_1$ is the prime ideal for x_1.
We equip Z with the reduced scheme structure. The morphism $T \to X$ factors via Z set theoretically. We want to check that this is also true as schemes. On the level of rings we are dealing with the map

$$A = \mathcal{O}(X) \to R$$

The composition $\mathcal{O}(X) \to R \to K$ factors by the previous lemma over $\mathcal{O}(X)/\mathfrak{p}_1$, where $\mathfrak{p}_1$ is the prime ideal for x_1. Then $Z = \mathrm{Spec}A/\mathfrak{p}_1$.

The map $A/\mathfrak{p}_1 \to K$ is not arbitrary, but factors via R such that the preimage of ξ is x_0. Consider the local ring $\mathcal{O} = \mathcal{O}_{Z,x_0}$. The morphism of schemes $T \to Z$ induces a local ring homomorphism

$$\mathcal{O} \to R$$

such that the preimage of the zero ideal is the zero ideal. We say: $\mathcal{O}$ dominates R.

Conversely, let $x_0, x_1 \in X$ be given with $x_0 \in \{x_1\}^- = Z$ (with reduced structure), and a domination of local rings

$$\mathcal{O}_{Z,x_0} \to R$$

Then this defines a morphism of schemes.

The valuative criteria are about checking that certain images of morphisms are closed. The main assertion is contained in the following property.

Proposition 6.6. *Let $f : X \to Y$ be a morphism of noetherian schemes.*

(i) Suppose $f(X)$ is closed under specialisation of points. Then $f(X)$ is closed.

(ii) Let $y \in Y$ be a point, $y_0 \in \{y\}^-$ a specialisation. Then there is a chain of points $y_0, y_1, \ldots, y_n = y$ and a sequence of discrete valuation rings R_i and morphisms

$$f_i : T_i = \mathrm{Spec}R_i \to Y$$

such that $f_i(\eta_i) = y_i$, $f_i(\xi_i) = y_{i-1}$.

Proof. We start with the topological part. Without loss of generality $Y = \overline{f(X)}$. Let $y \in Y$. We replace Y by an affine neighbourhood of y, i.e, Y is is affine. X is quasi-compact, hence there is an open affine cover $U_1, \ldots, U_n$. The point y is in the closure of $f(\bigcup U_i)$, hence in the closure of some $f(U_i)$. Without loss of generality X is affine. Both can be assumed reduced.

We consider the corresponding ring homomorphism $B \to A$. Let I be the kernel. Then f factors via the closed subset $V(I)$. As $f(X)$ is dense, we have $V(I) = \mathrm{Spec}B$, hence $I = 0$ (as B is reduced). In other words $B \to A$ is injective. The point $y \in Y$ belongs to a prime ideal $\mathfrak{p}$ of B. Let $\mathfrak{p}'$ be a minimal prime ideal of B, contained in $\mathfrak{p}$. Geometrically: $\mathrm{Spec}B$ is finite union of irreducible components, $\mathfrak{p}'$ is the generic point y' of an irreducible component containing y. Hence y is a specialisation of y'. By assumption it suffices to show that y' is in the image of f.

We localise with respect ot $\mathfrak{p}'$ und obtain a map

$$B_{\mathfrak{p}'} \to A_{\mathfrak{p}'}$$

As $\mathfrak{p}'$ was minimal, $B_{\mathfrak{p}'}$ is a field. Let $\mathfrak{q}_0'$ be any prime ideal of $A_{\mathfrak{p}'}$. Its preimage in $B_{\mathfrak{p}'}$ is a prime ideal of $B_{\mathfrak{p}'}$, hence equal to the unique prime ideal $B_{\mathfrak{p}''}\mathfrak{p}'$. Let $\mathfrak{q}'$ be the preimage of $\mathfrak{q}_0'$ in A. This is the preimage of y' in X.

We turn to the second claim. Let $y_0 \in Y$. Without loss of generality we replace Y by an affine neighbourghood of y_0. It necessarily contains y. Put $Y = \mathrm{Spec} B$. The points correspond to prime ideals $\mathfrak{p}_0$ und $\mathfrak{p}$. As the ring is noetherian, there is a maximal chain of prime dieals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \mathfrak{p}_n = \mathfrak{p}$$

This is the chain of points y_i we were looking for. We construct the descrete valuation ring R_i as the integral closure of $(B/\mathfrak{p}_{i-1})_{\mathfrak{p}_i}$. The latter is a 1-dimensional noetherian integral domain. Hece R_i is in addition integrally closed, hence a discrete valuation ring. By construction we have a map $B \to R_i$. □

Excercises

Let $k = \bar{k}$ be algbraically closed, $C = V(XY) \subset \mathbb{A}_k^2$, $R = k[[t]]$. Determine all morphisms of k-schemes $\mathrm{Spec} k[[t]] \to C$. What are the images of the special and the generic point in each case?

Hartshorne II 4.4, 4.6, 3.17

Chapter 7

Sheaves of modules

Ha II 5

Let $(X, \mathcal{O}_X)$ be a ringed space.

Definition 7.1. *A* sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ *is a sheaf of abelian groups together with a homomorphism of sheaves*

$$\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

such that for every open $U \subset X$ the abelian group $\mathcal{F}(U)$ is given the structure of an $\mathcal{O}_X(U)$-module. A morphismus of sheaves of modules is a morphism of sheaves, compatible with the $\mathcal{O}_X$-operation.

The category is abelian. Kernels and cokernels carry canonical $\mathcal{O}_X$-module structures.

Example. (i) Let X be a topological space, $\mathcal{O}_X$ the constant sheaf $\mathbb{Z}$. Then a sheaf of modules is the same as a sheaf of abelian groups.

(ii) Let X be a smooth manifold, $E \to X$ a vector bundle, $\mathcal{C}$ the sheaf of smooth functions, $\mathcal{E}$ the sheaf of smooth sections of E. Then $\mathcal{E}$ is sheaf of $\mathcal{C}$-modules.

(iii) Let $X = \mathrm{Spec}A$ be an affine scheme, M an A-Modul. Then there is a unique sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ with

$$\tilde{M}(U_f) = M_f$$

(Same proof as for the structure sheaf; or by tensoring the conditions there by M). The stalk of $\tilde{M}$ in $\mathfrak{p}$ is $M_\mathfrak{p}$.

Other notions:

- Tensor product: $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

- Internal Hom: $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ Sheafication of $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathcal{G}(U))$.
- $\mathcal{F}$ is called *free*, if it is isomorphic to a direct sum of compies of $\mathcal{O}_X$. It is called *locally free* if there is an open cover where this is true.
- $\mathcal{F}$ is called *invertible*, if is locally free of rank 1.
- A *sheaf of ideals* is a submodule sheaf of $\mathcal{O}_X$.
- *coherent*, if locally on X it is of the shape

$$\mathcal{O}_X^n \to \mathcal{O}_X^m \to \mathcal{F} \to 0$$

- Let X be scheme. $\mathcal{F}$ is called *quasi-coherent*, if there is an open affine cover such that $\mathcal{F}|_U$ is of the form $\tilde{M}$.

Hartshorne calls a sheaf coherent, if $\mathcal{F}$ is quasi-coherent with M finitely generated. For noetherian schemes, the two notion agree. The above is the good definition in the general case.

Lemma 7.2. *Every quasi-coherent sheaf on* Spec$A$ *is of the form* $\tilde{M}$. *The functor* $M \mapsto \tilde{M}$ *is an equivalence of categories.*

Corollary 7.3. *The category of quasi-coherent sheaves is abelian. If X is noetherian, the same is true for the category of coherent sheaves.*

Definition 7.4. *Let $f : X \to Y$ be a morphismus ringed spaces, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules on X. Then $f_*\mathcal{F}$ has a natural structure of a sheaf of $\mathcal{O}_Y$-modules. It is called* direct image *of $\mathcal{F}$.*
Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules on Y. Then $f^{-1}\mathcal{G}$ has a natural structure of a sheaf of $f^{-1}\mathcal{O}_Y$-modules on X. We call

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

the inverse image *von $\mathcal{G}$.*

The two functors are adjoint. The functor f_* is left exact, f^* is right exact. On schemes they respect the categories of quasi-coherent sheaves. If the schemes are noetherian, the inverse image of a oherent sheaf is coherent. This fails for direct images!

Example. Let $A = k[t]$. We consider $f : \mathrm{Spec}A \to \mathrm{Spec}k$ and $\mathcal{F} = \mathcal{O}_X$. Then $f_*\mathcal{F} = A$. This module is *not* finitely generated over k.

Consider a closed immersion $i : Z \to X$. By definition, there is a morphism

$$\mathcal{O}_X \to i_*\mathcal{O}_Z \ .$$

Let $\mathcal{I}_Z$ be the kernel. We want to understand the situation locally. Let $X = \mathrm{Spec}A$. By assumption $Z = \mathrm{Spec}B$ and the induced map $A \to B$ is surjective

with kernel I. On Spec$A$, we have $\mathcal{I}_Z = \tilde{I}$. Note that $i_*\mathcal{O}_Z$ has support on Z, i.e, $(i_*\mathcal{O}_Z)|_{X-Z} = 0$.
Conversely, let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of modules. Then $\mathcal{O}_X/\mathcal{I}$ has support on the closed set and this defines a closed subscheme.
Locally: $\mathrm{Spec}(A/I) \subset \mathrm{Spec}A$ has support in $V(I) = \{\mathfrak{p} | I \subset \mathfrak{p}\}$ und $\mathrm{Spec}(A/I) \to \mathrm{Spec}A$ is the closed immersion.

Projective schemes

Recall: For a graded ring S the set $\mathrm{Proj}(S)$ consists of the homogenous prime ideals of S different from $S_+ = \bigoplus_{i\geq 1} S_i$. The closed sets are the $V(I)$ for homogenous ideals I. A basis of the topology are the sets of the form $U_f = \mathrm{Proj}(S) \smallsetminus V(f)$ for a homogenous element $f \in S$. We have

$$U_f \cong \mathrm{Spec}((A_f)_0)$$

as schemes.

Definition 7.5. *Let S be a graded ring, M a graded S-module. The associated sheaf $\tilde{M}$ is uniquely characterised by*

$$U_f \mapsto (M_f)_0$$

In $\mathfrak{p}$ it has the stalk $(M_\mathfrak{p})_0$.

The sheaf quasi-coherent, because on U_f it is equal to $((M_f)_0)^\sim$.
Translation of the graded turns a graded module M into a new graded module $M(n)$. More precisely

$$M(n)_d = M_{n+d}$$

Definition 7.6. *For $n \in \mathbb{Z}$ let*

$$\mathcal{O}_X(n) = (S(n))^\sim$$

For $n = 1$ it is called $\mathcal{O}_X(1)$ (Serre) twist sheaf.

We are going to see that the definition of $\mathcal{O}_X(1)$ does not only depend on X, but also on the choice of S.

Lemma 7.7. *Let S be a graded ring, such that S is generated by S_1 as an S_0-algebra.*

(i) *$\mathcal{O}_X(n)$ is invertible.*

(ii) *$\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$*

(iii) *$\mathcal{O}_X(n)(X) = S_n$.*

(iv) *For $m > 0$ the sheaf $\mathcal{O}_X(m)$ is generated by global sections i.e, the map $\mathcal{O}_X(m)(X) \otimes_{\mathcal{O}(X)} \mathcal{O}_{X,x} \to \mathcal{O}_X(m)_x$ is surjective for all points $x \in X$.*

(v) For $n < 0$ we have $\mathcal{O}_X(m)(X) = 0$.

(vi) $\mathcal{O}_X(X) = S_0$.

An example of such a graded ring is

$$S = k[X_0, \ldots, X_n]/I$$

for a homogenous ideal I. We do the argument for $I = 0$, $X = \mathbb{P}^n_k$. The general case is done with the same arguments.

Proof. We cover $\mathbb{P}^n_k$ by the standard affines

$$U_i = U_{X_i} = \mathrm{Spec} A_i$$

with $A_i = (k[X_0, \ldots, X_n]_{X_i})_0 = k[Y_0, \ldots, \hat{Y}_i, \ldots, Y_n]$, $Y_j = X_j/X_i$. The sheaf $\mathcal{O}_X(n)$ is associated to the module

$$M_i = (k[X_0, \ldots, X_n]_{X_i})_n$$

It is isomorphic to A_i (as A_i-module) via $P \mapsto X_i^n P$. In other words: X_i^n is a basis vector for M_i. (This is also defined for negetative n because X_i is invertible.)
More generally for all graded S-modules:

$$(M \otimes N)^\sim \cong \tilde{M} \otimes \tilde{N}$$

Note: $S(a) \otimes_S S(b) = S(a + b)$.
We now consider global sections $\mathcal{O}_X(m)(\mathbb{P}^n) = S_m$. They vanish for $m < 0$ and are equal to k for $m = 0$. For $m = 1$ we have sections $X_1, \ldots, X_n$. The image of X_i is a basis vector on U_i. Hence the sheaf is generated by global sections. $\square$

The elements X_i are not *functions* on $\mathbb{P}^n_k$, but rather *sections* of an invertible sheaf.

Excercises

Ha II 5.1, 5.6, 5.7 (Tipp: Nakayama's Lemma)

Chapter 8

Projective morphisms

Ha II.5 and II.7

Let A be a noetherian ring, $S = \bigoplus_{d \geq 0} S_d$ with $S_0 = A$ and S finitely generated by S_1, i.e., S is a quotient of $A[X_0, \dots, X_n]$. We consider the scheme $\mathrm{Proj}(S)$. By construction it is projective, i.e., embedded into $\mathbb{P}^n_A$.

Recall that we can reconstruct S from $X = \mathrm{Proj}(S)$ and $\mathcal{O}_X(1)$, because

$$S = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$$

Definition 8.1. *An invertible sheaf $\mathcal{L}$ on an A-scheme X is called* very ample*, if $X \cong \mathrm{Proj}(S)$ with $S = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes n})$*

The choice of a very ample sheaf and generators $x_1, \dots, x_n \in \mathcal{L}(X)$ of S defines an embedding into projective space. How do we know if $\mathcal{L}$ is very ample?

We want to understand how we can embed schemes into projective spaces. This requires understanding quasi-coherent sheaves on $\mathbb{P}^n_A$.

Lemma 8.2. *Let $X = \mathrm{Proj} S$ as above, $\mathcal{F}$ quasi-coherent on X. Then $\mathcal{F} \cong \tilde{M}$ with*

$$M = \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

Here $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$.

Proof. We consider $\mathbb{P}^n$. Wir want to construct $\beta : \tilde{M} \to \mathcal{F}$. We restrict both sheaves to $U_i = \mathrm{Spec}(A[X_0, \dots, X_n]_{X_i})_0$. There it is enough to consider global sections. A section s of $\tilde{M}$ on U_i is of the form $m/(X_i)^d$ with m homogenous of degree d, i.e., $m \in \Gamma(X, \mathcal{F}(d))$. We view X_i^{-d} as an element of $\Gamma(U_i, \mathcal{O}_X(-d))$. Then s is a section of $\mathcal{F}(d) \otimes \mathcal{O}_X(-d) \cong \mathcal{F}$. This procedure is compatible with the change of charts. $\square$

Corollary 8.3. *Let $Y \subset \mathbb{P}^n$ be a closed subscheme. Then $Y = \mathrm{Proj}(S)$ with $S = A[X_0, \dots, X_n]/I$ for a homogenous ideal I.*

Proof. In general Y is defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n_A}$. By the lemma $\mathcal{I} = \tilde{I}$ for a homogenous ideal I. $\square$

In order to understand quasi-coherent sheaves on Y, it is now enough to understand quasi-coherent sheaves on $\mathbb{P}^n_A$.

Definition 8.4. *Let X be noetherian. An invertible sheaf $\mathcal{L}$ auf X is called* ample, *if for every coherent sheaf $\mathcal{F}$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.*

Example. Let X be affine. Then every invertible sheaf is ample, because every coherent sheaf is generated by global sections.
Let $X = \mathbb{P}^n_A$. Then $\mathcal{O}_X(-1)$ is not ample, because for $\mathcal{F} = \mathcal{O}_X$ the condition cannot be satisfied.

Theorem 8.5 (Serre)**.** *Let X be projective over A. Then $\mathcal{O}_X(1)$ is ample.*

Proof. It is enough to consider $X = \mathbb{P}^n_A$. Let $\mathcal{F}$ be coherent. Then $\mathcal{F} = \tilde{M}$. Every $\mathcal{F}|_{U_i}$ is generated by finitely many global sections. Their denominator is a power of X_i. We expand the fractions such that all denominators have the same degree. Then their nominators are in the same $M_d = \Gamma(X, \mathcal{F}(d))$. $\square$

A global section $f \in \Gamma(X, \mathcal{F}(d))$ is the same as a morphismus

$$\mathcal{O}_X \to \mathcal{F}(d)$$

because we only need to specify the image of 1. By twisting this is a morphism

$$\mathcal{O}_X(-d) \to \mathcal{F}$$

Corollary 8.6. *Let X be projective over A, $\mathcal{F}$ coherent. Then $\mathcal{F}$ is a quotient of a locally free sheaf. More precisely, there is a surjective morphism*

$$\mathcal{O}_X(d)^n \to \mathcal{F}$$

Theorem 8.7. *Let X be projective over A, $\mathcal{F}$ coherent. Then is $\mathcal{F}(X)$ a finitely generated A-module.*

Proof. There is an elementary proof in Ha 5.19. There is a better, cohomological proof in Ha III Thm 5.2. It uses the above surjection in order to reduce to the case $\mathcal{F} = \mathcal{O}_X(d)$. However, we have to compute the whole cohomology ring, not only the global sections. $\square$

Corollary 8.8. *Let $f : X \to Y$ be a projective morphism of noetherian schemes. If $\mathcal{F}$ is coherent, so is $f_*\mathcal{F}$.*

Proof. The assertion is local on Y, hence wlog $Y = \mathrm{Spec} A$. Then $f_*\mathcal{F}$ is a finitely generated A-module by the theorem. $\square$

We continue to study ample sheaves.

Lemma 8.9. *Let X be a noetherian scheme, $\mathcal{L}$ an invertible sheaf. The following are equivalent:*

(i) *$\mathcal{L}$ ample*

(ii) *$\mathcal{L}^{\otimes m}$ ample for all $m > 0$*

(iii) *$\mathcal{L}^{\otimes m}$ ample for some $m > 0$.*

Proof. (i) to (ii) is easy, (ii) to (iii) is trivial. We consider (iii) to (i). Let $\mathcal{F}$ be coherent. We apply the definition of ample to the sheaves

$$\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \dots, \mathcal{F} \otimes \mathcal{L}^{\otimes m-1}$$

By assumption there is n_0, such that $\mathcal{F} \otimes \mathcal{L}^{\otimes i} \otimes \mathcal{L}^{\otimes mn}$ is generated by global sections for all $n \geq n_0$, $i = 0, \dots, m-1$. □

Theorem 8.10. *Let X be of finite type over a noetherian ring A, $\mathcal{L}$ an invertible sheaf on X. Then: $\mathcal{L}$ is ample if and only if there is $n \geq 0$ such that $\mathcal{L}^{\otimes m}$ is very ample.*

The proof needs some preparation. Let $\phi : X \to \mathbb{P}^n_A$, $\mathcal{L} = \phi^*\mathcal{O}(1)$. Then there is a natural map

$$\mathcal{O}(1)(\mathbb{P}^n) \to \mathcal{L}(X) = \lim_{f(X) \subset U} \mathcal{L}(U)$$

Let $s_i = \phi^*(X_i)$. These are global generators of $\mathcal{L}$. We now show the converse.

Theorem 8.11. *Let $\mathcal{L}$ be an invertible sheaf on X, and $s_0, \dots, s_n$ global generators. Then there is a unique*

$$\phi : X \to \mathbb{P}^n_A$$

with $\mathcal{L}$ and s_i as above.

Proof. Roughly:

$$P \mapsto [s_0(P), \dots, s_n(P)] \in \mathbb{P}(\mathcal{L}(X))$$

The values cannot vanish simultanuously because they generate $\mathcal{L}_P \cong \mathcal{O}_P$.
Let $X_i = \{P \in X | s_i(P) \neq 0\}$ (meaning: $(s_i)_P \notin m_p\mathcal{L}_P$). This is an open subset. We define

$$X_i \to U_i \subset \mathbb{P}^n_A$$

by the ring homomorphismus

$$(A[X_0, \dots, X_n]_{X_i})_0 \to \Gamma(X_i, \mathcal{O}_{X_i})$$

by mapping X_j/X_i to s_j/s_i. The latter is a section of von $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}$. □

This morphismus is a closed immersion, if

(i) X_i affine

(ii) The above morphism $\mathcal{O}(U_i) \to \mathcal{O}(X_i)$ is surjective.

We get back to the proof of Theorem 8.10.

Proof. Let $\mathcal{L}^{\otimes m}$ be very ample, i.e., $\mathcal{L}^{\otimes m} = \phi^*\mathcal{O}(1)$ for a closed immersion ϕ. Then $\mathcal{L}^{\otimes m}$ is ample, hence also $\mathcal{L}$.

Conversely, let $\mathcal{L}$ be ample. For $P \in X$ let U be an affine open neighbourhood, on which $\mathcal{L}$ is free. Then $Y = X \smallsetminus U$ with sheaf of ideals $\mathcal{I}_Y$. This sheaf is coherent, hence there is n, such that $\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. We consider one such with $s(P) \neq 0$. We may understand s as a section of $\mathcal{L}^{\otimes n}$. We consider

$$X_s = \{Q \in U | s(Q) \neq 0\}$$

By construction, $P \in X_s$ and $X_s \subset U$. The open set U is affine and s corresponds to $f \in \mathcal{O}(U)$ under the trivialisation of $\mathcal{L}$. Hence $X_s = U_f$ is itself affine. The section s is a generator of $\mathcal{L}_Q$ for all $Q \in X_s$.

The X_s for all s, P cover X. By quasi-compactness finitely many of these X_s suffice; we may choose a single n for all of them because twisting does not change X_s. Hence we have found our global sections s of $\mathcal{L}^{\otimes n}$. What was called X_i in the last theorem is X_s, so affine. They define an affine morphism. For surjectivity, we choose finitely many algebra generators b_{ij} of $\mathcal{O}(X_i)$. After further twisting, the $b_{ij}s_i$ extend to all of X. □

Excercises

Ha II 5.13, 5.16, 5.18, 7.5

Chapter 9

Differential forms

Ha II 8, Matsumura: Commutative algebra

Recall the basics of calculus. Let $U \subset \mathbb{R}$ be open and $f : U \to \mathbb{R}$ a smooth function. In every point $x \in U$ the function f has a derivative $f'(x)$. This defines a new function $f' : U \to \mathbb{R}$. The assignment $f \mapsto f'$ satisfies some algebraic conditions.

(i) $\mathbb{R}$-Llinearity

(ii) Leibniz-rule $(fg)' = fg' + gf'$

In addition, there is the chain rule:

$$f(y(x))' = y'(x)f'(y(x))$$

This means that the assignment in *not* invariant under a change of coordinates $y(x)$ on U. However we would need this invariance in order to generalise to manifolds!

Better point of view: : $f \mapsto df = f'dx$ assigns to a function on U a *differential form.* Then

$$\frac{\partial f}{\partial x}dx = \frac{\partial f}{\partial y}dy$$

because $dy = \frac{\partial y}{\partial x}dx$.

In higher dimension, i.e., for $U \subset \mathbb{R}^n$ with coordinates $x_1, \dots, x_n$ we have

$$f \mapsto df = \sum \frac{\partial f}{\partial x_i}dx_i$$

In order to understand what happens for varieties, we should also consider the case of embedded manifolds. We consider the simplest case. Let $U \subset \mathbb{R}^n$, $F : U \to \mathbb{R}$ a smooth map and X the set of zeroes of F. By the implicit function theorem, the set X is a manifiold of dimension $n-1$, if the vector $(\partial_1 F, \dots, \partial_n F)$ is non-zero on all points of X. Suppose e.g., that $\partial_n F(x) \neq 0$ on all of X. Then $x_1, \dots, x_{n-1}$ is a system of coordinates and (locally) there is a smooth function $x_n = \phi(x_1, \dots, x_{n-1})$ such

$$F(x_1, \dots, x_n, \phi(x_1, \dots, x_n)) = 0$$

The differential forms on X are linear combinations of the dx_i and there is a relation

$$dx_n = d\phi(x_1, \dots, x_{n-1}) = \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial x_i} dx_i$$

We can rewrite the same relation without mentioning ϕ. We put

$$G(x_1, \dots, x_{n-1}) = F(x_1, \dots, x_{n-1}, \phi).$$

The function G vanishes on X and hence

$$0 = \frac{\partial G}{\partial x_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial x_n} \frac{\partial \phi}{\partial x_i}$$

This gives

$$\frac{\partial \phi}{\partial x_i} = -\left(\frac{\partial F}{\partial x_n} \right)^{-1} \frac{\partial F}{\partial x_i} \Rightarrow dx_n = -\left(\frac{\partial F}{\partial x_n} \right)^{-1} \sum_{i=1}^{n-1} \frac{\partial F}{\partial x_i} dx_i$$

and by adding up

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i = 0$$

Conversely, $dF = 0$ can be used to deduce the formula for dx_n.

Definition 9.1. *Let k be a field and $X = \operatorname{Spec}(k[X_1, \dots, X_n]/(f_1, \dots, f_m))$. We define the module of* Kähler differentials *on X by*

$$\Omega_{X/k} = \langle dX_1, \dots, dX_n \rangle_{\mathcal{O}(X)} / \langle df_1, \dots df_m \rangle$$

where the total differential $d : \mathcal{O}(X) \to \Omega_{X/k}$ is given by

$$f \mapsto \sum \frac{\partial f}{\partial X_i} dX_i$$

X is called smooth over k, *if $\Omega_{X/k}$ is locally free of rank* $\dim X$.

We will discuss the difference between the notions regular/non-singular/smooth later.

Example. (i) Let $X = V(x^2 - y) \subset \mathbb{A}^2$. Then Ω_X is generated by dx and dy. We have

$$0 = d(x^2 - y) = 2xdx + dy$$

i.e., Ω_X is free of rank 1. The variety is smooth.

(ii) $X = V(xy)$. The relation is

$$0 = d(xy) = xdy + ydx$$

On the set where $x \neq 0$ or $y \neq 0$, the module is free of rank 1. If it was locally free everywhere, it would have to be of rank 1. However, there is no neighbourhood of 0 where one generator would suffice because the relation vanishes modulo $\mathfrak{m}_0$.

(iii) $X = V(y^2 - x(x-1)(x-2)$, $\mathrm{char}(k) \neq 2$. The relation reads

$$2ydy = [(x-1)(x-2) + x(x-1) + x(x-2)]dx$$

On $y \neq 0$ the generator dx is a basis vector. For $y = 0$ we have $x = 0, 1, 2$. But then the factor in front of dx is non-zero and dy is a basis vector close to these points.

(iv) $X = V(x^2) \subset \mathbb{A}^1$ has

$$\Omega = (k[x]/x^2)dx/\langle 2xdx\rangle$$

The module is *not* locally free. (In characteristic 2 it is actually free, but of the wrong dimension: $\dim X = 0$.)

For the well-definedness of Ω_X, we compute as in calculus with the chain rule. Better: use a universal property!

Definition 9.2. *Let A be ring, B an A-algebra. An A-derivation on B is an A-linear map*

$$D : B \to M$$

into a B-module M such that the Leibniz-rule is satisfied. The module of A-differentials $\Omega_{B/A}$ is the universal A-derivation on B, i.e., every A-derivation factors via a unique B-module homomorphism $\Omega_{B/A} \to M$.

As always, uniqueness follows from the universal property. Obviously, $\Omega_{B/A}$ is generated as a B-module by the db with $b \in B$. The module with these generators and all relations necessary for a derivation satisfies the universal property. (Note $da = 0$ for a in the image of A.)

Lemma 9.3. *For $A = k$ and $B = k[X_1, \dots, X_n]/(f_1, \dots, f_m)$ we get back the first definition.*

Proof. We get a map from the universal property. The converse map can be written explicitly in terms of generators and relations. $\square$

Proposition 9.4. *$\Omega_{B/A}$ exists and has the description I/I^2 where I is the kernel of $B \otimes_A B \to B$ and the derivation*

$$B \to I/I^2 \qquad b \mapsto b \otimes 1 - 1 \otimes b$$

Proof. Matsumura p. 182. □

There are many computation rules.

(i) (Base change) Let A' be an A-Algebra, $B' = B \otimes A'$. Then

$$\Omega_{B'/A'} = \Omega_{B/A} \otimes_B B'$$

(ii) (Localisation) Let $S \subset B$ be a multiplicative system. Then

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}$$

The second rule implies that $U \mapsto \Omega_{\mathcal{O}(U)/A}$ (for affine $U \subset \mathrm{Spec}B$) defines a quasi-coherent sheaf.

Proposition 9.5. *Let $X \to Y$ be a morphism of schemes. Then there is a uinqiue quasi-coherent sheaf $\Omega_{X/Y}$ on X such that for* $\mathrm{Spec}B \subset X$ *mappint to* $\mathrm{Spec}A \subset Y$

$$\Omega_{X/Y}(\mathrm{Spec}B) = \Omega_{B/A}$$

It is given by $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the sheaf of ideals for the immersion $\Delta : X \to X \times_Y X$.

Example. Let $X = \mathbb{P}^1_k$, $Y = \mathrm{Spec}k$. Then $\Omega_{X/Y} = \mathcal{O}(-2)$, because on U_0 the sheaf is generated by dy_1 where $y_1 = X_1/X_0$, on U_1 by dy_0 where $y_0 = X_0/X_1$. The transition map $y_0 = y_1^{-1}$ induces $dy_0 = -1/y_1^2 dy_1$. This is the transition map of $\mathcal{O}(-2)$.

Theorem 9.6 (1. fundamental exact sequence)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is a an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

This is often applied in the case $Z = \mathrm{Spec}(k)$ for some ground field. We often write Ω_X instead of $\Omega_{X/k}$. If X/k is smooth, then $\Omega_{X/k}$ is locally free of rank $\dim X$. The sequence allows to compute relative differentials from absolute differentials.

Theorem 9.7 (2. fundamental exact sequence)**.** *Let $f : X \to Y$ be a morphism, $Z \subset X$ a closed subscheme with sheaf of ideals $\mathcal{I}$. Then there is an exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0$$

It it the same sequence as before, but in addition the term $\Omega_{Z/X} = 0$ vanishes. Instead we can describe the kernel on the left.

Example. Let $Y = \operatorname{Spec} k$, $X = \mathbb{A}^n$, $\mathcal{I} = (f_1, \ldots, f_m)$. The affine version of the sequence is

$$\mathcal{I}/\mathcal{I}^2 \to \langle dX_1, \ldots, X_n \rangle_{\mathcal{O}(Z)} \to \Omega_Z \to 0$$

This is quite similar to our previous description. The new insight is the vanishing of d on $\mathcal{I}^2$. The sequence is *not* left exact in general!

Übungen

(i) Ha II 8.3 (a)

(ii) Compute the rank of $\Omega_{E/k}$ for the projective curve $E = V(y^2 - x(x-1)(x-2))$, where $\operatorname{char}(k) \neq 2$

(iii) Ha Theorem 8.13

(iv) Read up on the proof of the assertions from commutative algebra.

Chapter 10

Regularity and smoothness

Ha I 5, II 8, III 9 und III 10, Goertz/Wedhorn 6.8
From now on all schemes and rings are noetherian.

Definition 10.1. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k. Then A is called* regular*, if*

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$$

A scheme X is called regular*, if all local rings are regular.*

We always have $\geq$. Hence the condition is equivalent to asking that $\mathfrak{m}$ is generated by $\dim A$ many elements. They are called *local parameters.*

Example. (i) $k[[t_1, \dots, t_n]]$ is regular with parameter system $t_1, \dots, t_n$.

(ii) In dimension 0 a local ring is regular if and only it is a field.

(iii) In dimension 1 a local ring is regular if and only if it is a discrete valuation ring.

(iv) If A is a Dekekind domain (noetherian, dimension 1, integrally closed), then A is regular.

(v) If k is a field, then $\mathbb{A}^n_k$ is regular.

(vi) $A = k[[x, y]]/xy$ has $\mathfrak{m} = \langle x, y\rangle$, $\mathfrak{m}^2 = \langle x^2, xy = 0, y^2\rangle$. Hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$ (basis $\overline{x}, \overline{y}$). The ring is not regular.

(vii) The variety $V(XY) \subset \mathbb{A}^2$ is not regular.

Remark. For $x \in \mathrm{Spec} A$ we call $\mathfrak{m}_x/\mathfrak{m}_x^2$ the *cotantent space* of A. If you try to develop the germ of a function in x into a Taylor series, then $\mathfrak{m}_x/\mathfrak{m}_x^2$ captures the linear part.

We want to understand, when an affine variety $\mathrm{Spec} k[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ over a field k is regular. We consider the Taylor series but restrict attention to the linear term. In other words: we consider derivatives.

Proposition 10.2. *Let k be an algebraically closed field $X = \mathrm{Spec} k[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$. Then X is regular in $P \in X$, if and only if the matrix*

$$\mathrm{Jac}(X)(P) = \left(\frac{\partial f_i}{\partial X_j}(P) \right)$$

has rank $n - \dim X$.

Such varieties are also called *non-singular* in P.

Proof. As k is algebraically closed, it suffices to consider $P = 0$. The maximal ideal is generated by $X_1, \ldots, X_n$. As $0 \in X$, the equations f_i vanish in 0. In $\mathfrak{m}/\mathfrak{m}^2$ we have the relations

$$0 = \sum \frac{\partial f_i}{\partial X_j}(0) X_j \mod \mathfrak{m}^2$$

Hence the assumption on the rank allows us to eliminate generators. □

There is an alternative point of view. We consider the 2nd fundamental sequence for $P \in X$:

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,P}/\mathfrak{m}_P$$

is an isomorphism. This is what we spell out in generators and relations.

Proposition 10.3. *Let k be algebraically closed. The X/k is smooth, if and only if X is regular.*

Proof. If X smooth, then Ω_X is locally free of rank $r = \dim X$. By the above isomorphism we compute $\dim \mathfrak{m}_P/\mathfrak{m}_P^2$. If conversely X is regular, then all $\Omega_{X/k} \otimes \kappa(P)$ have dimension r. I.e., the Jacobi-matrix

$$\mathrm{Jac}(X) = \left(\frac{\partial f_i}{\partial X_j}(P) \right)$$

has rank $n - r$. Then the same is true in a neighbourhood of P. In this neighbourhood, we obtain a free $\mathcal{O}_X$-module. The basis is a subset of the X_j. □

What happens, if k is *not* algbraically closed?

Example. Let k be field, $K = k[X]/F$ a field extension. Obviously $X = \mathrm{Spec} K$ is regular. On the other hand,

$$\Omega_{K/k} = \langle dX \rangle_K / F' dX$$

This module has K-dimension 0, if $F' \neq 0$. This happens precisely if the extension is separable.

The notion regular and smooth do *not* agree in general. Regularity is absolute, whereas smoothness is relative. It depends on the choice of base. Smoothness is stable under base change.

Lemma 10.4. *Let X/k of finite type and K/k a field extension. Then X/k is smooth if and only if X_K/K is smooth.*

Proof. $\Omega_{X_K/K} = \Omega_{X/k} \otimes_k K$. Torsion can be detected after the extension to K. □

Regularity is *not* stable under base change.

Example. Let K/k be a purely inseparable extension of charakteristic p, e.g. $K = k[X]/F$ with $F = X^p - a$ irreducible. Then

$$K \otimes_k K = k[X]/F \otimes_k K = K[X]/F$$

Over K the polynomial F has zero of multiplity p. Hence $K[X]/F$ is not reduced, hence not regular.

General base schemes

We now want to consider X/Y.

Definition 10.5. *Let $f : X \to Y$ be a morphism of schemes, $r \geq 1$. We say f is* smooth of relative dimension r, *if for every point $x \in X$ there are affine neighbourhoods $V = \mathrm{Spec}R$ of $f(x)$ and $U = \mathrm{Spec}R[X_1, \dots, X_n]/(f_1, \dots, f_{n-r})$ of x such that the Jacobi-matrix*

$$\left(\frac{\partial f_i}{\partial X_j}(x)\right) \in M_{n,n-r}(\kappa(x)$$

has rank $n - r$.
If f is smooth of relative dimension 0, we also call f etale.

The condition is equivalent to the condition that $\Omega_{X/Y}$ is local free of rank equal to the relative dimension. For $Y = \mathrm{Spec}k$ we get back our earlier definition. The condition is stable under base change. In particular:

Lemma 10.6. *Let $f : X \to Y$ be smooth, $y \in Y$. Then $X_y/\kappa(y)$ is smooth and $X_{\overline{y}}/\overline{\kappa(y)}$ non-singular where $\overline{\kappa(y)}$ is an algebraic closure of $\kappa(y)$. We say: The geometric fibres are non-singular.*

Example.

Open immersions are etale.

Closed immersionen are etale only if it the inclusion of a connected component.

If X/k is smooth, then $X \times Y \to Y$ is smooth.

If f is an etale morphism of varieties over $\mathbb{C}$, then the condition of the definition is the Jacobi-criterium: f is locally invertible (in the analytic topology).

Proposition 10.7. *$f : X \to Y$ is smooth of relative dimension d if and only if f is flat of finite type and all geometric fibres are non-singular.*

Hartshorne uses this as the definition.

In order to understand this characterisation, we need to understand what "flat" means. The notion does not have a good analogon on differential geometry. It guarantees that the different fibres of f "are related" E.g., they all have the same dimension.

Definition 10.8. *Let $\phi : A \to B$ be a ring homomorphismus. ϕ is called* flat*, if B is flat as an A-module, i.e., $- \otimes_A B$ is exact on the category of A-modules.* *Let $f : X \to Y$ be a morphism of schemes. f is called* flat*, if all $f_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ are.*

Example. All algebras over a field are flat. If A is a discrete valuation ring, then B is flat if and only if B is without A-torsion.

For a complete understanding of flatness, we would need to get into homological algebra.

Excercises

Ha I 5.10, II 8.6, III 10.1

Chapter 11

Line bundles and Chech-cohomology

Ha II §6, III §4

We want to understand invertible sheaves, i.e., locally free $\mathcal{O}_X$-modules of rank 1

Definition 11.1. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. The group* Pic(X) *of isomorphism classes of invertible sheaves is called* Picard group of X.

Example. (i) $\mathrm{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$ for a field k

(ii) If C is a smooth projective curve of genus g over $\mathbb{C}$, then

$$\mathrm{Pic}(C) = \mathbb{Z} \times \mathbb{C}^g/\Lambda,$$

where Λ is a lattice (discrete subgroup of rank $2g$). In fact $\mathbb{C}^g/\Lambda = \mathrm{Jac}(X)$ is a projektive algebraic variety.

(iii) $\mathrm{Pic}(\mathrm{Spec}A) = \mathrm{Cl}(A)$ (class group) if A is a Dedekind domain A. This is the number theoretic case.

Let $\mathcal{L}$ be an invertible sheaf. By assumption, there is an open cover $\{U_i\}_{i\in I}$ of X, such that $\mathcal{L}|_{U_i}$ is trivial. We choose isomorphisms

$$f_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$$

On $U_{ij} = U_i \cap U_j$ the composition

$$f_{ij} = \phi_j|_{U_{ij}} \circ \phi_i^{-i}|_{U_{ij}} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$$

is an isomorphism, hence an element of $\mathcal{O}(U_{ij})^*$. These f_{ij} satisfy the *cocycle condition* on $U_{ijk} = U_i \cap U_j \cap U_k$

$$f_{jk} \circ f_{ij} = f_{ik} \ .$$

Conversely every cocycle defines an invertible sheaf. When are the sheaves defined by different cocycles isomorphic? Equivalently: when does a cocycle define the trival sheaf?

Let $g : \mathcal{O}_X \to \mathcal{L}$ be an isomorphism. The commutative diagram on U_i

$$\begin{array}{ccc} \mathcal{O}_{U_i} & \xrightarrow{=} & \mathcal{O}_{U_i} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_i} \\ \mathcal{L}|_{U_i} & \xrightarrow{f_i} & \mathcal{O}_{U_i} \end{array}$$

means that

$$f_{ij} = f_j f_i^{-1} = g_j g_i^{-1}$$

Definition 11.2. *A cocycle is called* coboundary, *if there are $g_i \in \mathcal{O}(U_i)^*$ for $i \in I$ such that*

$$f_{ij} = g_j g_i^{-1}$$

The group

$$H^1(\{U_i\}, \mathcal{O}^*) = \textit{cocycles/coboundaries}$$

is called 1. Chech cohomologie group of the covering. *The group*

$$\check{H}^1(X, \mathcal{O}^*) = \varinjlim H^1(\{U_i\}, \mathcal{O}^*)$$

is called 1. Chech cohomology group of X *with coefficients* $\mathcal{O}^*$

The direct limit is taken with resept to refining morphism.

Definition 11.3. *Let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be open covers of X. We call $\mathfrak{V}$ a* refinement *of $\mathfrak{U}$, if for every $j \in J$ there is $i \in I$ such that $V_j \subset U_i$. The choice of such map $J \to I$ is called a* refining morphism.

One checks that the map $H^1(\{U_i\}, \mathcal{O}^*) \to H^1(\{V_j\}, \mathcal{O}^*)$ does not depend on the choice of the refining morphism.

Hence:

Lemma 11.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then*

$$\mathrm{Pic}(X) = \check{H}^1(X, \mathcal{O}_X^*)$$

Chech-cohomology

Let X be a topological space, $\mathcal{F}$ a preseheaf on X, $\mathfrak{U} = \{U_i\}_{i\in I}$ an open covering. For $J \subset I$ we put $U_J = \bigcap_{j\in J} U_j$.

Definition 11.5. *The* Chech complex *of $\mathfrak{U}$ with coefficients in $\mathcal{F}$ is defined as*

$$0 \to \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{d^0} \prod_{i_0,i_1\in I} \mathcal{F}(U_{i_0i_1}) \xrightarrow{d^1} \prod_{i_0,i_1,i_2} \mathcal{F}(U_{i_0i_1i_2}) \to \dots$$

with differential $d^n = \sum_{j=0}^n (-1)^j \partial_j$ *where* ∂_j *is induced from the inclusion*

$$U_{i_0\dots i_j \dots i_n} \to U_{i_0\dots \hat{i}_j \dots i_n}$$

The i-th Chech-cohomology $H^i(\mathfrak{U}, \mathcal{F})$ *is defined as the ith cohomology of the Chech complex. We put*

$$\check{H}^i(X, \mathcal{F}) = \varinjlim H^i(\mathfrak{U}, \mathcal{F})$$

the ith Chech cohomology von X*.*

Alternatively, one chooses an ordering of I and only considers tuples with $i_0 < i_1 < \dots < i_n$. The two complexes are quasi-isomorphis, i.e., the cohomology does not change. In the second version it becomes obvious that finite covers only have finitely many cohomology groups. On the other hand, compatibility with transition maps becomes hard to work out.

Example. $\check{H}^0(X, \mathcal{F})$ was used in our construction of the sheafification. If $\mathcal{F}$ is sheaf, then

$$\mathcal{F}(X) = H^0(\mathfrak{U}, \mathcal{F})$$

For $i = 1$ the cocylces are the elements of the kernel, the coboundaries the elements in the image of the differential.
The nice thing about cohomology is that it is computable with long exact sequences.

Lemma 11.6. *Let* X *be a separated scheme,* $\mathfrak{U}$ *an open affine cover,*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

an exact sequence of coherent sheaves. Then there is a natural long exact sequence

$$\begin{aligned} 0 \to H^0(\mathfrak{U}, \mathcal{F}) \to H^0(\mathfrak{U}, \mathcal{G}) \to H^0(\mathfrak{U}, \mathcal{H}) \to \\ H^1(\mathfrak{U}, \mathcal{F}) \to H^1(\mathfrak{U}, \mathcal{G}) \to H^1(\mathfrak{U}, \mathcal{H}) \\ \to H^2(\mathfrak{U}, \mathcal{F}) \to \dots \end{aligned}$$

Proof. As X is separated, all U_I are affine. The global section functor is exact in the affine case, i.e.,

$$0 \to \mathcal{F}(U_I) \to \mathcal{G}(U_I) \to \mathcal{H}(U_I) \to 0$$

is exact. Direct products are exact. Hence we obtain a short exact sequence of Chech complexes. It induces a long exact sequence in cohomology. □

Variants for all topological spaces:

(i) A short exact sequence of presheaves induces long exact sequences for $H^*(\mathfrak{U}, \cdot)$ and $\check{H}^*(X, \cdot)$.

(ii) A short exact sequence of sheaves induces a long exact sequence for $\check{H}^*(X, \cdot)$.

Cartier-Divisors

Let A be a ring, $S \subset A$ the set of non-zero divisors. Then $K(A) = S^{-1}A$ is called *total ring of fractions* of A.

Let $(X, \mathcal{O}_X)$ be a scheme, $\mathcal{K}$ the sheaf of total rings of fractions, i.e., the unique sheaf associated to the presheaf $U = \mathrm{Spec} A \mapsto K(A)$.

Example. Let X be integral. Then X has function field K given by $K = Q(\mathcal{O}(U))$ for $U \subset X$ affine. Then $\mathcal{K}$ is the constant sheaf K.

Definition 11.7. *Let X be a scheme. A* Cartier-Divisor *on X is a global section of $\mathcal{K}^*/\mathcal{O}^*$. It is called a* principal divisor*, if it is in the image of $\mathcal{K}^*(X)$. Two Cartier-Divisors are called* linearly equivalent*, if their difference (=quotient) is a principal divisor. The group* $\mathrm{CaCl}(X)$ *of equivalence classes is called* Cartier-class group.

A Cartier-Divisor consist of local sections of $\mathcal{K}^*$, differeing by regular invertible functions. We write (U_i, f_i) with $f_i \in \mathcal{K}^*(U_i)$. The $f_{ij} = f_j f_i^{-i} \in \mathcal{O}^*(U_{ij})$ then define a cocycle, hence an invertible sheaf.

Proposition 11.8. *For every scheme there is an injective map*

$$\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$$

It is surjective, if X is integral.

Proof. Cohomological version: consider the exact sequence of sheaves

$$0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0$$

and the associated long exact sequence

$$\mathcal{O}^*(X) \to \mathcal{K}^*(X) \to \mathcal{K}^*/\mathcal{O}^*(X) \to \check{H}^1(X, \mathcal{O}^*) \to \check{H}^1(X, \mathcal{K}^*)$$

We get the inclusion by definition. If X is integral , then $\mathcal{K}^*$ is a constant sheaf. Hence its higher cohomology vanishes.

Explicitly: We have defined the map on the level of cocycles. If D is a Cartier-divisor, then $\mathcal{L}(D)$ is the sub-$\mathcal{O}_X$-module of $\mathcal{K}$, generated by f_i^{-1} on U_i. This is well-defined on U_{ij}. The line bundle is trivial, if there is a global generator, i.e., the Cartier divisor is a principal divisor.

In the image of this map we have all line bundles which are isomorphic to submodules of $\mathcal{K}$. Let X be integral, $\mathcal{L}$ a line bundle. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$ is a locally free $\mathcal{K}$-module of rank 1. As $\mathcal{K}$ is constant, it is even free. The choice of an isomorphism induces the choice of an embedding $\mathcal{L} \to \mathcal{K}$. □

Excercises

(i) Check that the morphism on $H^1(\mathcal{U}, \cdot)$ does not depend on the choice of refining morphism.

(ii) Deduce the beginning of the long exact sequence (H^0 und H^1) for a short exact sequence of sheaves.

(iii) Make Cartier-divisors and line bundles explicit in the case of the spectrum of a Dedekind domain.

(iv) Ha II 6.10

Chapter 12

Weil divisors

Ha II 6

All schemes noetherian, separated.

Definition 12.1. *Let X be a scheme. A* Weil divisor *(or simply divisor) is a formal linear combination*

$$\sum_{i=1}^{n} a_i D_i$$

with $n \geq 0$, $a_i \in \mathbb{Z}$ and $D_i \subset X$ irreducible closed subset of codimension 1 ("prime divisor"). The set of divisors is a free abelian group over the prime divisors.

Divisors are also called *(algebraic)* 1*-cycles*. A different point of view: prime divisors are points $x \in X$ with $\mathcal{O}_{X,x}$ of dimension 1. We write X^1 for the set of these points. For the generalisation to higher codimension, see Ha Appendix A.

The next step is to introduce *principal divisors*, i.e., the zero and pole divisor of a meromorphic function f. This does not work for all X.

From now on:

Assumption: X noetherian, separated, integral and all local ring of codimension 1 are regular (i.e, discrete valuation rings).

The assumption is satisfied if X is normal (affne: integrally closed), e.g., smooth over a field.

If X is a variety of dimension 1, this means that X is smooth and connected. Then a divisor is a formal linear combination of closed points.

Definition 12.2. *Let X be as above with function field K, $f \in K^*$. The* divisor *of f is defined as*

$$(f) = \sum_{x \in X^1} v_x(f)x$$

where $v_x : K^* \to \mathbb{Z}$ is the discrete valuation of the point x. Divisors of the form

(f) *are called* principal divisors. *The group* $\mathrm{Cl}(X)$ *of divisors modulo principal divisors is called* divisor class group.

Example. Let $X = \mathbb{A}^1_k$, x the origin, $f = P/Q$ with $P, Q \in k[X] \smallsetminus 0$. Then $v_x(f) = v_x(P) - v_x(Q)$ where $v_x(P)$ is the exact power of X in P. If $v = v_x(f) > 0$, then f has a zero of order v. If $v < 0$, then f has a pole of order $-v$. Alternatively: The function f has a Laurent series in the point 0

$$\sum_{i=v}^{\infty} a_i X^i$$

with $a_v \neq 0$.

Example. Let A a Dedekind ring, e.g., the ring of integers of a number field. Then a divisor defines a fractional ideal. The class group is the class group of number theory. $\mathrm{Cl}(A) = 0$ is equivalent to A being a principal ideal domain.

Example. $X = \mathbb{P}^n_k$ for k a field. We compute $\mathrm{Cl}(\mathbb{P}^n_k)$. The irreducible closed subvarieties of codimension 1 are defined by $g \in S = k[X_0, \ldots, X_n]$ irreducible and homogenous of d. (Krull's principal ideal theorem). We write v_g for the corresponding valuation. For $f \in S$ homogenous the number $v_g(f)$ is the exact power of g dividing f. Let f in the function field. Wir decompose it into factors $g_1^{n_1} \ldots g_r^{n_r}$. Hence

$$(f) = \sum n_i V(g_i)$$

The degree of the divisor $\sum a_i D_i$ is defined as $\sum a_i \deg D_i$. For principal divisor we obtain

$$\deg(f) = \sum n_i \deg V(g_i) = \sum n_i \deg(g_i) = 0.$$

Two divisors can only be equivalent, if they have the same degree. Conversely, let $D = \sum n_i V(g_i)$ a divisor of degree d. Then

$$D \sim dV(X_0)$$

because the difference is the principal divisor for $g_1^{n_1} \ldots g_r^{n_r} X_0^{-r}$. The degree defines an isomorphism

$$\mathrm{Cl}(\mathbb{P}^n_k) \to \mathbb{Z}$$

.

We need some computation rules.

Lemma 12.3. *Let* $Z \subset X$ *closed,* $U = X \smallsetminus Z$. *Then*

$$\mathrm{Cl}(X) \to \mathrm{Cl}(U)$$

is surjective. If $\operatorname{codim} Z \geq 2$, *then the map is an somorphism. If* Z *is irreducible of codimension* 1, *we have an exact sequence*

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

Proof. The map is defined by $D \mapsto D \cap U$. A divisor is mapped to zero, if $D \cap U = \emptyset$, i.e., $D \subset Z$. The function fields are the same, hence the assignment respects principal divisors. $\square$

Example. Let $Y \subset \mathbb{P}^2$ be an irreducible curve of degree d, U the open complement. Then $\mathrm{Cl}(U) = \mathbb{Z}/d\mathbb{Z}$, because $\mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z}$ and the class of Y ist d. In particular $\mathrm{Cl}(\mathbb{A}^1) = 0$.

Proposition 12.4. $\mathrm{Cl}(X) = \mathrm{Cl}(X \times \mathbb{A}^1)$.

Proof. Ha II 6.6 $\square$

Connection with Cartier-divisors

We continue with the restricted situation fixed above. $\mathcal{K}$ is the constant sheaf K and $\mathrm{CaCl}(X) = \mathrm{Pic}(X)$.

Let D be a Cartier-divisor. Then there is covering $\{U_i\}$ of X on which D is given by $f_i \in K^*$. On U_i the function f_i defines a Weil divisor. On $U_i \cap U_j$ they differ by the divisor of $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$, hence not at all. By gluing we get a Weil divisor on X. The assignment respects principal divisors, hence we get a map

$$\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$$

The image are the Weil divisors that can be written locally as principal divisors. Hence "Cartier" can be viewed as property of a Weil-divisor.

We now consider the converse. When is an irreducible subvariety of codimension 1 locally given by a principal divisor?

Proposition 12.5. *Let X be a scheme as above such that in addition every local ring has unique prime factorisation. Then the group of Weil-divisors is equal to the group Cartier-divisors.*

Proof. We consider the prime divisor D in a neighbourhood of x, more precisely, we restrict to $\mathrm{Spec}\mathcal{O}_{X,x}$. Then $D = V(f)$ for a prime element f. Then the same is true in some open neighbourhood U of x, where we view f in $\mathcal{O}(U) \subset K$. On the intersection of two such neighbourhoods, we have two functions defining the same subscheme. Hence their quotient is a unit. We have defined a Cartier-divisor $\square$

Corollary 12.6. $\mathrm{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$ *with generator* $\mathcal{O}(1)$.

Proof. $\mathbb{Z} = \mathrm{Cl}(\mathbb{P}^n_k) = \mathrm{CaCl}(\mathbb{P}^n_k) = \mathrm{Pic}(\mathbb{P}^n_k)$. We identify the generators. It is $V(X_0)$ as a Weil-divisor. Locally on $U_i = \mathbb{P}^n \smallsetminus V(X_i)$ it is defined by the rational function $f_i = X_0/X_i$. The corresponding cocylce is $f_{ij} = f_i^{-1} f_j = X_j/X_i$.

On the other hand, the line bundle $\mathcal{O}(1)$ is generated by X_i on U_i. The cocycle is the same. Alternatively: $\mathcal{L}(V(X_0))$ is defined as the subsheaf of $\mathcal{K}$ generated by X_i/X_0 onU_i. $\square$

Corollary 12.7. *Let A be Dedekind ring. Then*

$$\mathrm{Cl}(A) = \mathrm{Pic}(A)$$

Excercises

Ha II 6.1. 6.6, this needs Ex.. 6.10.2, Ex. 6.5.2 and 6.11.3

Chapter 13

The Weil Conjectures

Ha App. C

Let $\mathbb{F}_p$ be a finite field of characteristic p. We work with varieties over $\mathbb{F}_p$, i.e., separated reduced schemes of finite type over $\mathbb{F}_p$.

Definition 13.1. *Let $X/\mathbb{F}_p$ be a variety. Put*

$$N_r(X) = |X(\mathbb{F}_{p^r})| = |\operatorname{Mor}_{\mathbb{F}_p}(\operatorname{Spec}\mathbb{F}_{p^r}, X)|$$

Example. $X = \mathbb{A}^1$. Then

$$\mathbb{A}^1(\mathbb{F}_{p^r}) = \operatorname{Mor}_{\mathbb{F}_p}(\operatorname{Spec}\mathbb{F}_{p^r}, \operatorname{Spec}\mathbb{F}_p[t]) = \operatorname{Mor}(\mathbb{F}_p[t], \mathbb{F}_{p^r}) = \mathbb{F}_{p^r}$$

and hence

$$N_r(\mathbb{A}^1) = p^r$$

Example. Consider $p \neq 2, 3$, $X = V(X^2 - 3)$. Then

$$X(\mathbb{F}_{q^r}) = \{x \in \mathbb{F}_{q^r} | x^2 = 3\}$$

We have $N_r(X) = 2$, if the equation is a solution in $\mathbb{F}_{p^r}$ and $N_r(X) = 0$, if not. There are two cases: 3 is a square in $\mathbb{F}_p$ (by quadratic reciprocity this holds if $(-1)^{\frac{(p-1)}{2}}\left(\frac{p}{3}\right) = 1$, e.g. $p = 13$)), then then $N_r = 2$ for all r. If 3 is not a square in $\mathbb{F}_p$ (e.g. $p = 5$), we get $N_r = 0$ for odd r, $N_r = 2$ for even r.

Example. For $X = \operatorname{Spec}(\mathbb{F}_{p^n})$, we have $N_r(X) = 0$ if r is not a multiple of n and $N_r(X) = n$ if it is.

We encode these numbers in a series.

Definition 13.2.

$$Z(X, t) = \exp\left(\sum_{r \geq 1} N_r(X) \frac{t^r}{r}\right) \in \mathbb{Q}[[t]]$$

is called Zeta-function of X.

Example. (i) $X = \mathbb{A}^1$.

$$Z(\mathbb{A}^1, t) = \exp\left(\sum_{r\geq 1} p^r \frac{t^r}{r}\right) = \exp(-\log(1-pt)) = \frac{1}{1-pt}$$

(ii) $X = \mathbb{P}^1$, $N_r = p^r + 1$

$$Z(\mathbb{P}^1, t) = \exp(\left(\sum_{r\geq 1} \frac{(pt)^r}{r} + \sum_{r\geq 1} \frac{t^r}{r}\right)$$
$$= \exp(-\log(1-pt) - \log(1-t)) = \frac{1}{(1-pt)(1-t)}$$

(iii) $X = V(X^2 - 3)$ for $p = 13$.

$$Z(X, t) = \exp\left(\sum_{2r\geq 1} 2\frac{t^{2r}}{2r}\right) = \exp(-\log(1-t^2)) = \frac{1}{1-t^2}$$

(iv) $X = \mathrm{Spec}(\mathbb{F}_{p^n})$:

$$Z(X, t) = \frac{1}{1-t^n}$$

The formula looks arbitrary, but this is not the case. Let X be of finite type over $\mathbb{Z}$, $|X|$ the set of closed points. For $x \in |X|$ the residue field is $\kappa(x)$ is of finite type over $\mathbb{Z}$, hence finite. We call $N(x) = |\kappa(x)|$ the norm.

Definition 13.3. *Let X of finite type over $\mathbb{Z}$. Put*

$$\zeta(X, s) = \prod_{x\in|X|} \frac{1}{1 - N(x)^{-s}}$$

Example. (i) For $X = \mathrm{Spec}\mathbb{Z}$ we get the Riemann Zeta-function $\zeta(s)$. More generally, for the ring of integers of a number field, we get its Dedekind Zeta-function.

(ii) For $X = \mathrm{Spec}(\mathbb{F}_{p^n})$, we have

$$\zeta(X, s)) = \frac{1}{1 - p^{-ns}} = Z(X, p^{-s}).$$

(iii) For $X/\mathbb{F}_p$ we have $N(x) = p^r$. By taking logarithms, a bit of computation yields

$$\zeta(X, s) = Z(X, p^{-s})$$

This computation also shows that $Z(X, t) \in \mathbb{Z}[[t]]$.

From examples as abvoe, Weil came to his conjectures. They are now theorems.

Theorem 13.4. *Let $X/\mathbb{F}_q$ be a smooth projective variety of dimension d, then $Z(X,t)$ is rational*

$$Z(X,t) = \frac{P_1(X,t)P_3(X,t)\cdots P_{2d-1}(X,t)}{P_0(X,t)\cdots P_{2d}(X,t)}$$

with $P_i(X,t) \in \mathbb{Q}[t]$. In addition:

(i) (functional equation) Let E be the self intersection number of the diagonal in $X \times X$. Then we have the functional equation

$$Z(X, 1/p^d t) = \pm p^{dE/2} t^E Z(X,t)$$

(ii) (Riemmann hypothesis) $P_0 = 1 - t$, $P_{2d} = 1 - p^n t$ and for every i

$$P_i(X,t) = \prod (1 - \alpha_{ij} t) \in \mathbb{Z}[t]$$

where the algebraic numbers α_{ij} have absolute value $p^{i/2}$.

(iii) (Betti numbers) If X is of the form $\mathcal{X} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ $\mathcal{X}$ smooth projective, then

$$\deg P_i = \dim H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})$$

The curve case was established by Weil himself. Rationality was shown independently by Dwork and Grothendieck. The Riemann hypothesis is due to Deligne. Already Weil suggested a strategy using a suitable cohomology. This cohomology was developled by Grothendieck.

We consider a special case first.

Example. Let E be an elliptic curve. The Betti numbers are $0, 2, 1$, hence we expect

$$Z(X,t) = \frac{P_1}{(1-t)(1-pt)}$$

with quadratic P_1

$$P_1(t) = (1 - \alpha t)(1 - \beta t) = 1 - (\alpha + \beta)t + \alpha\beta t^2$$

where $|\alpha| = |\beta| = \sqrt{p}$. Then $\alpha = \overline{\beta}$ and hence $\alpha\beta = |\alpha|^2 = p$ and $|\alpha + \beta| \leq 2\sqrt{p}$. Inserting into the formula this gives

$$(1 - \alpha t)(1 - \beta t) = (1-t)(1-pt) \exp\left(\sum_{r \geq 1} N_r \frac{t^r}{r} \right)$$

For the linear term this yields

$$-\alpha - \beta = -1 - p + N_1$$

and hence

$$|N_r - p - 1| \leq 2\sqrt{p}$$

This estimate is called Hasse bound!

The proof of the Weil conjectures in the special case is done in Silverman's book. We have

$$P_1(t) = \det(1 - \mathrm{Fr}_p t | V_l(E))$$

where $V_l(E) = T_l(E) \otimes \mathbb{Q}_l$, $T_l(E) = \varprojlim E[l^i]$ is the l-adic Tate-module, l a prime number different from p. The same proof works for curves via the Tate module of the Jacobian $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$.

The general proof uses etale cohomology:

$$P_i(t) = \det(1 - \mathrm{Fr}_p t | H^i(\overline{X}, \mathbb{Q}_l))$$

Rationality follows from the Lefschetz trace formula, the functional equation from Poincaré duality and the Betti-numbers appear because of a comparison theorem with singular cohomology. We discuss the connection to fixed points now.

Let X be a variety over $\mathbb{F}_p$. Then Frobenius Fr_p operates on $X(\overline{\mathbb{F}_p})$ by operating on the coordinates. A point is in $X(\mathbb{F}_p^r)$ if and only if it is a fixed point of $\mathbb{F}_p^r$.

$$\log Z(X,t) = \sum_{r=1}^{\infty} |X(\overline{\mathbb{F}_p})^{\mathrm{Fr}_p^r}| \frac{t^r}{r}$$

Proposition 13.5 (Fixed point formula)**.** *Let X be smooth projective over $k = \overline{\mathbb{F}}_p$, $f : X \to X$ a morphism with isolated simple fixed points. Then*

$$|X^f| = \sum (-1)^i \mathrm{Tr}(f^* | H^i(X, \mathbb{Q}_l)$$

We insert this into our formula.

$$\log Z(X,t) = \sum_r \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}_p^r | H^i(X, \mathbb{Q}_l) \frac{t^r}{r}$$

Lemma 13.6. *Let V be a finite dimensional vector space, $f : V \to V$ an endomorphism. Then*

$$\sum_r \mathrm{Tr}(f^r | V) \frac{t^r}{r} = -\log \det(1 - ft|V)$$

Proof. Without loss of generality, the ground field is algebraically closed, f given by an upper triangular matrix. This reduces the question to the 1-dimensional case, i.e., $f = a \in k$. Then the formula is the series for log. □

Hence:

$$\log Z(X,t) = \sum_i (-1)^{i+1} \log((1 - \mathrm{Fr}_p t | H^i(X, \mathbb{Q}_l))$$

and

$$Z(X,t) = \prod_i P_i(t)^{(-1)^{i+1}}.$$

Remark. The integrality of the $P_i(t)$ follows from the Riemann hypothesis - there is not cancellation and the product is integral. This also implies that the $P_i(t)$ are independent of the choice of l used in their construction.

This part of the conjectures is still open for singular or non-complete varieties!

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