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A comparison of locally analytic group cohomology and Lie algebra cohomology for p-adic Lie groups

An alternative approach to Lazard's isomorphism

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Introduction

The main result of this work is a new proof and generalization of Lazard's comparison theorem of locally analytic group cohomology with Lie algebra cohomology for K-Lie groups, where K is a finite extension of $\mathbb{Q}_p$.

Main Theorem. *Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{G}$ be a K-Lie group. Then there exists an open subgroup $\mathcal{U}$ of $\mathcal{G}$ such that the Lazard morphism*

$$\Phi_L : H^n_{la}(\mathcal{U}, K) \to H^n(\mathrm{Lie}(\mathcal{U}), K)$$

induced by differentiating cochains is an isomorphism (c.f. Theorem 4.1.7).

The proof of this theorem is independent of the proof of Lazard's comparison result in [Laz65].

Lazard's comparison theorem was one of the main results in his work on p-adic groups [Laz65]. It relates locally analytic group cohomology with Lie algebra cohomology for $\mathbb{Q}_p$-Lie groups in two steps. First Lazard worked out an isomorphism between locally analytic group cohomology and continuous group cohomology and secondly between continuous group cohomology and Lie algebra cohomology. The latter is obtained from a difficult isomorphism between the saturated group ring and the saturated universal enveloping algebra.

Huber and Kings showed in [HK11] that one can directly define a map from locally analytic group cohomology to Lie algebra cohomology by differenting cochains and that in the case of smooth algebraic group schemes H over $\mathbb{Z}_p$ with formal group $\mathcal{H} \subset H(\mathbb{Z}_p)$ the resulting map

$$\Phi : H^n_{la}(\mathcal{H}, \mathbb{Q}_p) \to H^n(\mathfrak{h}, \mathbb{Q}_p)$$

coincides, after identifying continuous group cohomology with locally analytic group cohomology, with Lazard's comparison isomorphism ([HK11, Theorem 4.7.1]). Serre mentioned to the aforementioned authors that this was clear to him at the time Lazard's paper was written, however it was not included in the published results. In their joint work with N. Naumann in [HKN11] they extended the comparison isomorphism for K-Lie groups attached to smooth group schemes with connected generic fibre over the integers of K ([HKN11, Theorem 4.3.1]). The aim of this thesis is to use this simpler map to obtain an independent proof of Lazard's result.

The context in which Huber and Kings worked out the new description of the Lazard isomorphism is the construction of a p-adic regulator map in complete analogy to Borel's regulator for the infinite prime. The van Est isomorphism between relative Lie algebra cohomology and continuous group cohomology is replaced by the Lazard isomorphism. Their aim is to use this construction of a p-adic regulator for attacking the Bloch-Kato conjecture for special values of Dedekind Zeta functions.

Our strategy to prove the comparison isomorphism between locally analytic group cohomology and Lie algebra cohomology is to trace it back to the case of a formal group law G. Hence the first step is to obtain an isomorphism of formal group cohomology with Lie algebra cohomology (Corollary 3.1.4)

$$\tilde{\Phi} : H^n(\tilde{G}, R) \to H^n(\mathfrak{g}, R),$$

where R is an integral domain of characteristic zero. The tilde over Φ and over the formal group law G indicate that one has to modify the formal group cohomology while working with coefficients in R. This means that if the ring of functions to G, called $\mathcal{O}(G)$, is given by a formal power series ring over R one has to allow certain denominators. However, we will prove in Lemma 4.2.13 that functions of this modified ring of functions $\tilde{\mathcal{O}}(G)$ still converge, with the same region of convergence as the exponential function.
Let $\mathfrak{g}$ be the Lie algebra associated to the formal group law G and $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Then an essential ingredient in to the first step is a morphism of complete Hopf algebras (Proposition 2.1.2)

$$\beta^{\star} : \mathcal{O}(G) \to \mathcal{U}(\mathfrak{g})$$

from the ring of functions to the universal enveloping algebra. We show in Proposition 2.4.1 that this morphism is an isomorphism if we consider the modified ring of functions $\tilde{\mathcal{O}}(G)$. Furthermore, we prove in Theorem 3.1.3 that this

isomorphism extends to a quasi-isomorphism of the corresponding complexes and hence to the above isomorphism $\tilde{\Phi}$.
The second step in the proof of the Main Theorem is a Comparison Theorem for standard K-Lie groups. These standard groups are K-Lie groups associated to a formal group law G, see Definition 4.1.2.

Comparison Theorem for standard groups [Thm.4.1.7]. *Let G be a formal group law over R and let $\mathcal{G}(h)$ be the m-standard group of level h to G with Lie algebra $\mathfrak{g} \otimes_R K$. Then the map*

$$\Phi_s : H^n_{la}(\mathcal{G}(h), K) \to H^n(\mathfrak{g}, K)$$

given by the continuous extension of

$$f_1 \otimes \cdots \otimes f_n \mapsto df_1 \wedge \cdots \wedge df_n,$$

for $n \geq 1$ and by the identity for $n = 0$ is an isomorphism for all $h > h_0 = \frac{1}{p-1}$.

The Main Theorem can then be deduced from the Comparison Theorem for standard K-Lie groups, since every K-Lie group contains an open subgroup, which is standard, see Lemma 4.1.10. Our approach to the proof of the Comparion Theorem for standard K-Lie groups is as follows. In a first step, we will show using the isomorphism $\tilde{\Phi}$ that the limit morphism

$$\Phi_\infty : H^n(\mathcal{O}^{la}(\mathcal{G}(0)^\bullet)_e, K) \to H^n(\mathfrak{g}, K),$$

associated to the ring of germs of locally analytic functions in e, denoted by $\mathcal{O}^{la}(\mathcal{G}(0))_e$, is an isomorphism. Then injectivity of Φ_s follows from a spectral sequence argument. The proof of this injectivity part will be analogous to the proof of Theorem 4.3.1 in [HKN11], however independent of the work of Lazard [Laz65]. For surjectivity we will again use the isomorphism statement for formal group cohomology of Corollary 3.1.4 in addition to the aforementioned fact that functions of $\tilde{\mathcal{O}}(G)$ still converge.

The thesis is organized as follows: In Chapter 1 we give a number of definitions and well-known facts concerning formal groups, Lie algebras, Hopf algebra structures and cohomology complexes. Chapter 2 deals, firstly, with the existence of a morphism of complete Hopf algebras between the ring of functions of a formal group law and the dual of the universal enveloping algebra. Secondly we consider the cases where this morphism is an isomorphism and thirdly, considering the modified ring of functions, we can prove that the morphism is in this modified case actually an isomorphism of complete Hopf algebras. In Chapter 3 we show that the isomorphism of Chapter 2 can be extended to a quasi-isomorphism of the corresponding complexes and in Section 3.1 we will give an explicit description, which will be identical to the description of the comparison map in [HKN11] and hence to Lazard's map. The last Chapter 4 gives the proof of the Main Theorem. We will begin by fixing some notation in Section 4.1, in order to formulate the Comparison Theorem for standard groups. The proof of this theorem will be given in the remaining two sections.

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Chapter 1

Preliminaries and Notation

The main objects we are dealing with are introduced in this chapter. First of all we define formal group laws and show how they can be constructed by completing group schemes at the identity. Afterwards we define their associated Lie algebras. The second section deals with Hopf algebra structures associated to these objects, and the last section contains the definitions of the different complexes which will be needed in the third chapter.

> Throughout the first three chapters R will be an integral domain of characteristic zero.

1.1 Formal group law, Lie algebra and Universal enveloping algebra

For details concerning the following definitions, the standard reference is M. Hazewinkels book about formal groups an their applications, [Haz78, Chap. I.1, Chap.II.14].

Definition 1.1.1. Let $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ be two sets of m variables. An *m-dimensional formal group law over R* is an m-tuple of power series

$$G(\mathbf{X}, \mathbf{Y}) = (G^{(1)}(\mathbf{X}, \mathbf{Y}), \ldots, G^{(m)}(\mathbf{X}, \mathbf{Y}))$$

with $G^{(j)}(\mathbf{X}, \mathbf{Y}) \in R[[\mathbf{X}, \mathbf{Y}]]$ such that for all $j = 1, \ldots, m$

(1) $G^{(j)}(\mathbf{X}, \mathbf{Y}) = X_j + Y_j + \sum_{l,k=1}^{m} \gamma_{lk}^{j} X_l Y_k + O(d \geq 3), \gamma_{lk}^{j} \in R$

(2) $G^{(j)}(G(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = G^{(j)}(\mathbf{X}, G(\mathbf{Y}, \mathbf{Z}))$,

where the notation $O(d \geq n)$ stands for a formal power series whose homogeneous parts vanish in degree strictly less than n. If in addition

$$G^{(j)}(\mathbf{X}, \mathbf{Y}) = G^{(j)}(\mathbf{Y}, \mathbf{X})$$

holds for all $j = \{1, \ldots, m\}$, then the formal group law is called commutative. The *ring of functions* $\mathcal{O}(G)$ to a formal group law G is the ring of formal power series in m variables $t^{(1)}, \ldots, t^{(m)}$ over R, i.e. $\mathcal{O}(G) = R[t^{(1)}, \ldots, t^{(m)}]$.

Proposition 1.1.2. *Let $G(\boldsymbol{X}, \boldsymbol{Y})$ be an m-dimensional formal group law over R. Then there exists a power series $s(\boldsymbol{X})$ such that $G(\boldsymbol{X}, s(\boldsymbol{X})) = 0$.*

Proof. See [Haz78, Appendix A.4.5]. □

Remark. 1.1.3. The proof of the existence of the power series $s(\mathbf{X})$ gives an explicit construction of this power series. The first step of the construction yields $s(\mathbf{X}) = -\mathbf{X}$ mod (degree 2), a fact we will need later in Section 2.1.

The following definition of a homomorphism of formal group laws will be needed in Chapter 4.

Definition 1.1.4. Let $G(\mathbf{X}, \mathbf{Y})$ and $G'(\mathbf{X}, \mathbf{Y})$ be m-dimensional formal group laws over R. A *homomorphism*

$$G(\mathbf{X}, \mathbf{Y}) \to G'(\mathbf{X}, \mathbf{Y})$$

over R is an m-tuple of power series $\alpha(\mathbf{X})$ in n indeterminantes such that $\alpha(\mathbf{X}) \equiv 0$ mod (degree 1) and

$$\alpha(G(\mathbf{X}, \mathbf{Y})) = G'(\alpha(\mathbf{X}), \alpha(\mathbf{Y})).$$

The homomorphism $\alpha(\mathbf{X})$ is an *isomorphism* if there exists a homomorphism $\beta(\mathbf{X}) : G'(\mathbf{X}, \mathbf{Y}) \to G(\mathbf{X}, \mathbf{Y})$ such that $\alpha(\beta(\mathbf{X})) = \beta(\alpha(\mathbf{X}))$.

Lemma 1.1.5. *The ring of functions $\mathcal{O}(G)$ to a formal group law G is complete with respect to the topology induced by the following descending filtration*

$$F^i\,\mathcal{O}(G) = \{f \in \mathcal{O}(G)\,|\,\textit{all monomials of } f \textit{ have total degree} \geq i\}\,. \tag{1.1}$$

Notation. 1.1.6. • We denote by the sign $\hat{\otimes}$ the completed tensor product with respect to the above topology. Thus we can identify $\mathcal{O}(G)^{\hat{\otimes} n}$ with the ring of formal power series in nm indeterminates

$$R[[t_1^{(1)}, \dots t_1^{(m)}, t_2^{(1)}, \dots, t_2^{(m)}, \dots, t_n^{(1)}, \dots, t_n^{(m)}]].$$

- For the elements $t_i^{(j)}$ of the ring of functions $\mathcal{O}(G)$ we will use equivalently the notation $1 \otimes \dots \otimes 1 \otimes t^{(j)} \otimes 1 \otimes \dots \otimes 1$ (with $t^{(j)}$ at the i-th entry) for all $j = 1, \dots, m$.
- For simplicity we write
  - $\underline{t}_1$ for $t_1^{(1)}, \dots, t_1^{(m)}$,
  - $\underline{t}_{1,\dots,n}$ for $t_1^{(1)}, \dots, t_1^{(m)}, t_2^{(1)}, \dots, t_2^{(m)}, \dots, t_n^{(1)}, \dots, t_n^{(m)}$.
- We use the general multi-index notation $\underline{j}$ for the tuple $(j_1, \dots, j_m)$.
- If $m = 1$ or $n = 1$ we skip the upper, respectively lower index and write t_i for $t_i^{(1)}$ respectively $t^{(j)}$ for $t_1^{(j)}$.

Example 1.1.7. Examples of formal group laws are:

(i) The additive formal group law: $G^{(j)}(\mathbf{X}, \mathbf{Y}) = X_j + Y_j$

(ii) The multiplicative formal group law ($m = 1$): $G(\mathbf{X}, \mathbf{Y}) = X + Y + XY$

(iii) The general linear formal group law ($m = 4$):

$$\begin{aligned}
G^{(1)}(\mathbf{X}, \mathbf{Y}) &= X_1 + Y_1 + X_1 Y_1 + X_2 Y_3 \\
G^{(2)}(\mathbf{X}, \mathbf{Y}) &= X_2 + Y_2 + X_1 Y_2 + X_2 Y_4 \\
G^{(3)}(\mathbf{X}, \mathbf{Y}) &= X_3 + Y_3 + X_3 Y_1 + X_4 Y_3 \\
G^{(4)}(\mathbf{X}, \mathbf{Y}) &= X_4 + Y_4 + X_3 Y_2 + X_4 Y_4,
\end{aligned}$$

see [Haz78, Chap. II.9.2]. The formulas can be obtained by calculating

$$\begin{pmatrix} 1 + X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} 1 + Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and writing X_1 for X_{11}, X_2 for X_{12}, X_3 for X_{21} and X_4 for X_{22} and similarly for Y.

Remark. 1.1.8. Formal group laws arising from smooth algebraic affine group schemes over R

Here we will see one situation in which a formal group law arises, see e.g. [Wat79, Chap.11]. Suppose $\mathbf{G}$ is a smooth algebraic affine group scheme over R. Then $\mathbf{G}$ is represented by a finitely generated Hopf algebra, say A with augmentation map ϵ and augmentation ideal I (i.e. $I = \ker(\epsilon : A \to R)$). By smoothness we get that I/I^2 is free on the generators $t^{(1)}, \ldots, t^{(m)}$ where m is the number of generators of A over R. Also I^n/I^{n+1} is free and generated by the monomials ${t^{(1)}}^{r_1} \cdots {t^{(m)}}^{r_m}$ with $\sum r_i = n$. Thus if we take the completion with respect to the I-adic topology, we get

$$\hat{A} = \varprojlim(A/I^n) = R[[t^{(1)}, \ldots, t^{(m)}]].$$

Let $\mu : A \to A \otimes A$ be the comultiplication. Then μ maps I into $I \otimes A + A \otimes I$. Hence there is an induced map on completions

$$\mu : R[[t^{(1)}, \ldots, t^{(m)}]] \to R[[t_1^{(1)}, \ldots, t_1^{(m)}]] \hat{\otimes} R[[t_2^{(1)}, \ldots, t_2^{(m)}]].$$

However, any such map is completely described by where each of the $t^{(j)}$ get sent to. Set $\mu(t^{(j)}) =: G^{(j)}(\underline{t}_1, \underline{t}_2)$, then the ϵ-axiom shows

$$G^{(j)}(\underline{t}_1, \underline{0}) = G^{(j)}(\underline{0}, \underline{t}_2) = t^{(j)}$$

and coassociativity yields the identity

$$G^{(j)}(G(\underline{t}_1, \underline{t}_2), \underline{t}_3) = G^{(j)}(\underline{t}_1, G(\underline{t}_2, \underline{t}_3)).$$

Hence $G(\underline{t}_1, \underline{t}_2) := (G^{(1)}(\underline{t}_1, \underline{t}_2), \ldots, G^{(m)}(\underline{t}_1, \underline{t}_2))$ is a formal group law of dimension m.

Example 1.1.9. We will see here that we named the formal group laws of Example 1.1.7 (i) and (ii) in a natural way.

(i) The additive formal group law G_a over R is given by completing the group scheme $\mathbf{G}_a$ over R in the following way: start with the representing Hopf algebra $R[t]$ with comultiplication given by

$$\begin{aligned} \mu : R[t] &\to R[t] \otimes R[t] \\ t &\mapsto t \otimes 1 + 1 \otimes t \end{aligned}$$

and augmentation map ϵ with $\epsilon(t) = 0$. Consider the element t which generates the augmentation ideal I. The completion with respect to the I-adic topology yields $\varprojlim R[t]/I^n = R[[t]]$. The comultiplication on $R[t]$ induces a comultiplication on the completion

$$\begin{aligned} \mu : R[[t]] &\rightarrow R[[t]] \otimes R[[t]] \\ t &\mapsto t \otimes 1 + 1 \otimes t. \end{aligned}$$

(ii) The multiplicative formal group law G_m over R is given by completing the group scheme $\mathbf{G}_m$ over R in the following way: start with the representing Hopf algebra $R[x, x^{-1}]$ with comultiplication given by

$$\begin{aligned} \mu : R[x, x^{-1}] &\rightarrow R[x, x^{-1}] \otimes R[x, x^{-1}] \\ x &\mapsto x \otimes x \end{aligned}$$

and consider the element $(x - 1)$ which generates the augmentation ideal I. Set $t := x - 1$, then the completion with respect to the I-adic topology yields $\varprojlim R[x, x^{-1}]/I^n = R[[t]]$. The comultiplication on $R[x, x^{-1}]$ induces a comultiplication on the completion

$$\begin{aligned} \mu : R[[t]] &\rightarrow R[[t]] \hat{\otimes} R[[t]] \\ t &\mapsto t \otimes 1 + 1 \otimes t + t \otimes t. \end{aligned}$$

$$\begin{aligned} (\mu(t) &= \mu(t + 1) - \mu(1) = \mu(x) - \mu(1) = x \otimes x - 1 \otimes 1 \\ &= (t + 1) \otimes (t + 1) - 1 \otimes 1 = t \otimes 1 + 1 \otimes t + t \otimes t.) \end{aligned}$$

Lie algebra

Let $R[\underline{t}_1]$ be the polynomial ring over R in m variables and let $\frac{\partial}{\partial t_1^{(j)}}$ for all $j \in \{1, \ldots, m\}$ be the j-th partial derivative. Let $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ denote the set of R-derivations of $\mathcal{O}(G)$. Then $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ is a free $\mathcal{O}(G)$-module on the basis $\frac{\partial}{\partial t_1^{(1)}}, \ldots, \frac{\partial}{\partial t_1^{(m)}}$. If $d, d_1, d_2 \in \mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ and $r \in R$, then the mappings rd, $d_1 + d_2$ and $[d_1, d_2] := d_1 d_2 - d_2 d_1$ are also derivations, see [Bou98a, Chap.III §10.4]. Thus, the set $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ is an R-module which is also a Lie algebra.

Let e_j denote the j-th partial derivative $\frac{\partial}{\partial t_1^{(j)}}$ evaluated at 0. We denote by $\mathfrak{g}$ the free R-module on the basis $e_1, \dots, e_m$. If $L = \sum_{j=1}^m r_i e_j \in \mathfrak{g}$ and $f \in \mathcal{O}(G)$, then we can apply L to f by

$$L(f) = \sum_{j=1}^{m} r_j \frac{\partial f}{\partial t_1^{(j)}}(0). \tag{1.2}$$

Hence we can identify $\mathfrak{g}$ with the set of R-derivations of $\mathcal{O}(G)$ into R, where R is considered as a $\mathcal{O}(G)$-module via evaluation at zero. We denote this set by $\mathrm{Der}_0(\mathcal{O}(G), R)$.

The elements γ_{lk}^j of Definition 1.1.1 of a formal group law G define a Lie algebra structure on $\mathfrak{g}$, as follows (see [Haz78, Chap.II, p.79]):

$$[e_l, e_k] = \sum_{j=1}^{m} (\gamma_{lk}^j - \gamma_{kl}^j) e_j. \tag{1.3}$$

However, $\mathfrak{g}$ inherits also a Lie algebra structure by the canonical bijection of $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ and $\mathrm{Der}_0(\mathcal{O}(G), R)$. We will see in Section 1.2 that both definitions of the Lie-bracket coincide.

Universal enveloping algebra

Let $\mathfrak{g}$ be an m-dimensional Lie algebra over R which is free as an R-module with basis $e_1, \dots, e_m$, let $T(\mathfrak{g})$ be the tensor algebra and $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. A reference for the following definition and properties is [Kna88, Chap.II.6].

Definition 1.1.10. The algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is given by the quotient

$$T(\mathfrak{g}) / \left(\begin{array}{l} \text{two sided ideal generated by all} \\ X \otimes Y - Y \otimes X - [X, Y], X, Y \in T^{(1)}(\mathfrak{g}) \end{array} \right)$$

and called the *universal enveloping algebra* of $\mathfrak{g}$. It is an associative algebra with identity.

Proposition 1.1.11. *Let A be a unitary associative algebra over R. Then the bracket $[a_1, a_2] = a_1 a_2 - a_2 a_1$ defines a Lie algebra structure on the underlying R-module of A. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and the canonical map $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$, given by the embedding of $\mathfrak{g}$ into $T^{(1)}(\mathfrak{g})$ and then passing to $\mathcal{U}(\mathfrak{g})$, have the following universal property: whenever A is a unitary associative* *algebra and* $\pi : \mathfrak{g} \to A$ *is a Lie algebra homomorphism, then there exists a unique algebra homomorphism* $\tilde{\pi} : \mathcal{U}(\mathfrak{g}) \to A$ *such that* $\tilde{\pi}(1) = 1$ *and*

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\pi"] \arrow[dr, "\iota"'] & & A \\ & \mathcal{U}(\mathfrak{g}) \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

commutes.

Proof. See [Bou98b, Chap.I.2.1, Prop. 1.]. □

Example 1.1.12. Let G_a be the additive m-dimensional formal group law with $G^{(j)}(\mathbf{X}, \mathbf{Y}) = X_j + Y_j$. Then a basis of $\mathfrak{g}$ is given by $e_1, \dots, e_m$, the partial derivatives evaluated at zero, and the universal enveloping algebra can be identified with the symmetric algebra on m generators, since the Lie bracket is zero. If $m = 1$, then $\mathcal{U}(\mathfrak{g}) \cong R[e_1]$.

Since the Lie algebra $\mathfrak{g}$ associated to a formal group law is a free R-module, the following theorem will give us more information about the structure of $\mathcal{U}(\mathfrak{g})$.

Theorem 1.1.13. *(Poincaré-Birkhoff-Witt)*
Let $\mathfrak{g}$ *be an* m*-dimensional Lie algebra over* R *which is free as an* R*-module with basis* $e_1, \dots, e_m$*. Then the monomials*

$$e_1^{j_1} \cdots e_m^{j_m}$$

with all $j_k \in \mathbb{N} \cup \{0\}$*, form a basis of* $\mathcal{U}(\mathfrak{g})$*. In particular, the canonical map* $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ *is injective.*

Proof. See [Bou98b, Chap.I.2.7, Thm. 1., Cor. 3.]. □

The theorem of Poincaré-Birkhoff-Witt says that the underlying set of $\mathcal{U}(\mathfrak{g})$ is the polynomial ring $R[e_1, \dots, e_m]$. Therefore we denote an arbitrary element of $\mathcal{U}(\mathfrak{g})$ by $\sum c_{\underline{j}} \underline{e}^{\underline{j}}$ with $c_{\underline{j}} \in R$.

1.2 Hopf algebra structures

The algebras $\mathcal{O}(G)$ and $\mathcal{U}(\mathfrak{g})$ carry a (complete) Hopf algebra structure. We will describe the maps defining these structures after recalling the basic set-up and fixing notation for the structure of (complete) Hopf algebras. As a reference one can take the book of M.E. Sweedler about Hopf algebras, [Swe69, Chap.I-VI.], or the book of Ch. Kassel about Quantum groups, [Kas95, Chap.III].

Definition 1.2.1. An *algebra* over R is an R-module A together with two R-module homomorphisms

$$\bigtriangledown : A \otimes A \to A \text{ (multiplication)}, \;\; \eta : R \to A \text{ (unit)}$$

such that $\bigtriangledown \circ (1 \otimes \bigtriangledown) = \bigtriangledown \circ (\bigtriangledown \otimes 1),\;\; \bigtriangledown \circ (1 \otimes \eta) = \mathrm{id} = \bigtriangledown \circ (\eta \otimes 1)$ (where we have identified $R \otimes A \simeq A \simeq A \otimes R$).

Definition 1.2.2. A *coalgebra* over R is an R-module C together with two R-module homomorphisms

$$\mu : C \to C \otimes C \text{ (comultiplication)}, \;\; \epsilon : C \to R \text{ (counit)}$$

such that $(1 \otimes \mu) \circ \mu = (\mu \otimes 1) \circ \mu,\;\; (\epsilon \otimes 1) \circ \mu = \mathrm{id} = (1 \otimes \epsilon) \circ \mu$ (where we have again identified $R \otimes C \simeq C \simeq C \otimes R$).

Definition 1.2.3. A *bialgebra* over R is an R-module B together with four R-module homomorphisms

$$\begin{aligned} &\bigtriangledown : B \otimes B \to B, \;\; \eta : R \to B \\ &\mu : B \to B \otimes B, \;\; \epsilon : B \to R \end{aligned}$$

such that the following conditions hold:

(i) $(B, \bigtriangledown, \eta)$ is an algebra over R

(ii) (B, μ, ϵ) is a coalgebra over R

(iii) μ and ϵ are R-algebra morphisms, i.e.

$$\begin{aligned} \mu \circ \bigtriangledown &= (\bigtriangledown \otimes \bigtriangledown) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mu \otimes \mu) \\ \epsilon \circ \bigtriangledown &= \epsilon \otimes \epsilon \\ \mu \circ \eta &= (\eta \otimes \eta) \\ \epsilon \circ \eta &= \mathrm{id}, \end{aligned}$$

where $\tau : B \otimes B \to B \otimes B$ is the switching morphism which interchanges the two factors.

Note that condition (iii) is equivalent to the condition that $\bigtriangledown$ and η are R-coalgebra morphisms.

Definition 1.2.4. Let B and B' be two bialgebras over R. Then an R-module homomorphism $\phi : B \to B'$ is a *morphism of bialgebras* if ϕ is a morphism of coalgebras and a morphism of algebras, i.e. if

$$(\phi \otimes \phi) \circ \mu = \mu' \circ \phi,\ \epsilon' \circ \phi = \epsilon,\ \phi \circ \eta = \eta',\ \phi \circ \bigtriangledown = \bigtriangledown' \circ (\phi \otimes \phi).$$

Definition 1.2.5. Let B be a bialgebra over R. An *antipode* for B is a morphism of R-modules $s : B \to B$ such that

$$\bigtriangledown \circ (s \otimes 1) \circ \mu = \eta \circ \epsilon = \bigtriangledown \circ (1 \otimes s) \circ \mu.$$

Definition 1.2.6. A *Hopf algebra* over R is a pair consisting of a bialgebra B with an antipode s.

Definition 1.2.7. Let H and H' be two Hopf algebras over R. Let s_H, $s_{H'}$, be the antipodes of H, H', respectively. A morphism of bialgebras $\phi : H \to H'$ which satisfies the condition

$$s_{H'} \circ \phi = \phi \circ s_H,$$

ϕ is called *morphism of Hopf algebras.*

Lemma 1.2.8. *Let H and H' be two Hopf algebras over R. Let s_H, $s_{H'}$, be the antipodes of H, H', respectively. If $\phi : H \to H'$ is a morphism of bialgebras, then it is a morphism of Hopf algebras.*

Proof. See [Swe69, Chap.IV, Lemma 4.0.4]. □

For the definition of a complete Hopf algebra we have to consider complete R-modules, i.e. topologized R-modules which are complete with respect to a given topology. In our cases this topology will come from a descending filtration $\{F^n M\}$ on the R-module M.

Definition 1.2.9. By replacing R-modules by complete R-modules, and also tensor products by complete tensor products, we define in the same way as above a *complete algebra*, a *complete coalgebra*, a *complete bialgebra*, and a *complete Hopf algebra* over R (and the corresponding morphisms).

Definition 1.2.10. Let M be a topologized R-module, where the topology is induced by a descending filtration $M = F^0M \supset F^1M \supset \cdots$ of submodules and let R carry the discrete topology. We denote by M° the set of continuous linear maps from M to R, i.e

$$M^\circ = \mathrm{Hom}_{\mathrm{cont}}(M, R) = \varinjlim \mathrm{Hom}_R(M/F^nM, R).$$

We call M° the *continuous dual* of M.

We will now describe the (complete) Hopf algebra structures on $\mathcal{O}(G)$, $\mathcal{U}(\mathfrak{g})$ and their duals. This is taken from [Haz78, Chap. VII.36] for the complete Hopf algebra structure on $\mathcal{O}(G)$ and from [Haz78, Chap.II.14.3] for the Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$.

Proposition 1.2.11. *Let G be a formal group law. Then the ring of functions on G carries a complete Hopf algebra structure $(\mathcal{O}(G), \bigtriangledown, \eta, \mu, \epsilon, s)$. The maps are given by*

$$\begin{array}{rccc} \bigtriangledown\colon & \mathcal{O}(G)\,\hat{\otimes}\,\mathcal{O}(G) & \to & \mathcal{O}(G) \\ & f \otimes g & \mapsto & f \cdot g \end{array} \qquad \begin{array}{rccc} \eta: & R & \to & \mathcal{O}(G) \\ & 1 & \mapsto & 1 \end{array}$$

$$\begin{array}{rccc} \mu\colon & \mathcal{O}(G) & \to & \mathcal{O}(G)\,\hat{\otimes}\,\mathcal{O}(G) \\ & t^{(i)} & \mapsto & G^{(i)}(t_1^{(1)}, \ldots, t_1^{(m)}, t_2^{(1)}, \ldots, t_2^{(m)}) \end{array} \qquad \begin{array}{rccc} \epsilon: & \mathcal{O}(G) & \to & R \\ & f & \mapsto & f(0) \end{array}$$

$$\begin{array}{rccc} s\colon & \mathcal{O}(G) & \to & \mathcal{O}(G) \\ & t^{(i)} & \mapsto & s(t^{(i)}), \end{array}$$

where the antipode map s is given by Proposition 1.1.3 by the condition that

$$G^{(i)}(t^{(1)}, \ldots, t^{(m)}, s(t^{(1)}), \ldots, s(t^{(m)})) = 0.$$

Proof. See [Haz78, Chap.VII.36]. Note that the map $\bigtriangledown$ is continuous so that it is enough to define this map on $f \otimes g \in \mathcal{O}(G) \otimes \mathcal{O}(G)$. $\square$

Example 1.2.12. (a) Let G_a be the one-dimensional additive formal group law with $G_a(X, Y) = X + Y$. Then comultiplication is given by

$$\mu(t_1) = G(t_1, t_2) = t_1 + t_2$$

and the j-th power of t_1 maps to

$$\mu(t_1^j) = (t_1 + t_2)^j = \sum_{k=0}^{j} \binom{j}{k} t_1^k t_2^{j-k}.$$

The antipode s maps in this case t_1 to $-t_1$.

(b) Let G_m be the one-dimensional multiplicative formal group law with $G_m(X, Y) = X + Y + XY$. Then comultiplication is given by

$$\mu(t_1) = G(t_1, t_2) = t_1 + t_2 + t_1 t_2$$

and the j-th power of t_1 maps to

$$\mu(t_1^j) = (t_1 + t_2 + t_1 t_2)^j = \sum_{k_1=0}^{j} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} \binom{j-k_1}{k_2} t_1^{j-k_1} t_2^{k_1+k_2}.$$

The antipode s maps in this case t_1 to $-t_1 + t_1^2 - t_1^3 + t_1^4 - t_1^5 + \ldots$.

The complete Hopf algebra structure on $\mathcal{O}(G)$ enables us to verify that the definition of the Lie bracket on $\mathfrak{g}$ given by Equation (1.3) is equal to the ordinary Lie bracket definition for derivations.

Lemma 1.2.13. *The Lie bracket definition on $\mathfrak{g}$ given by Equation (1.3) is equal to the ordinary Lie bracket definition for derivations if one identifies $\mathfrak{g}$ with $\mathrm{Der}_0(\mathcal{O}(G), R)$.*

Proof. Let $e_l, e_k \in \mathrm{Der}_0(\mathcal{O}(G), R)$. Then $D_l := (\mathrm{id} \otimes e_l) \circ \mu$ and $D_k := (\mathrm{id} \otimes e_k) \circ \mu$ with μ and ϵ given by the complete Hopf algebra structure on $\mathcal{O}(G)$ are elements of $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ and the canonical bijection of $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ and $\mathrm{Der}_0(\mathcal{O}(G), R)$ defines a Lie bracket on $\mathfrak{g}$ by

$$[e_l, e_k] = \epsilon \circ [D_l, D_k],$$

see [Wat79, Chap.12]. After the Lie bracket definition on $\mathrm{Der}_R(\mathcal{O}(G), \mathcal{O}(G))$ the right hand side is equal to $\epsilon \circ (D_l \circ D_k - D_k \circ D_l)$. Thus

$$[e_l, e_k](t^{(j)}) = \epsilon \circ (D_l \circ D_k - D_k \circ D_l)(t^{(j)}).$$

Look at the term $(D_l \circ D_k)(t^{(j)})$:

$$\begin{aligned}
(D_l \circ D_k)(t^{(j)}) &= D_l \circ (\mathrm{id} \otimes e_k)(\mu(t^{(j)})) \\
&= D_l \circ (\mathrm{id} \otimes e_k)(G^{(j)}(t^{(1)} \otimes 1, \dots, t^{(m)} \otimes 1, 1 \otimes t^{(1)}, \dots, 1 \otimes t^{(m)})) \\
&= D_l \circ (\mathrm{id} \otimes e_k)(t^{(j)} \otimes 1 + 1 \otimes t^{(j)} + \sum_{i,r=1}^{m} \gamma_{ir}^{j} t^{(i)} \otimes t^{(r)} + O(d \geq 3)).
\end{aligned}$$

According to the definition of e_k, given by

$$e_k(t^{(i)^s}) = \begin{cases} 0, & \text{if } i \neq k \text{ or } i = k, s \neq 1 \\ 1, & \text{if } i = k, s = 1, \end{cases} \tag{1.4}$$

the right hand side reduces to $D_l \circ (\sum_{i=1}^{m} \gamma_{ik}^{j} t^{(i)})$. Hence

$$\begin{aligned}
(D_l \circ D_k)(t^{(j)}) &= D_l \circ (\sum_{i=1}^{m} \gamma_{ik}^{j} t^{(i)}) = \sum_{i=1}^{m} \gamma_{ik}^{j} (\mathrm{id} \otimes e_l)(\mu(t^{(i)})) \\
&= \sum_{i=1}^{m} \gamma_{ik}^{j} (\mathrm{id} \otimes e_l)(G^{(i)}(t^{(1)} \otimes 1, \dots, t^{(m)} \otimes 1, 1 \otimes t^{(1)}, \dots, 1 \otimes t^{(m)})) \\
&= \sum_{i=1}^{m} \gamma_{ik}^{j} (\mathrm{id} \otimes e_l)(t^{(i)} \otimes 1 + 1 \otimes t^{(i)} + \sum_{s,t=1}^{m} \gamma_{st}^{i} t^{(s)} \otimes t^{(t)} + O(d \geq 3)) \\
&= \sum_{\substack{i=1 \\ i \neq l}}^{m} \gamma_{ik}^{j} (\mathrm{id} \otimes e_l)(t^{(i)} \otimes 1 + 1 \otimes t^{(i)} + \sum_{s,t=1}^{m} \gamma_{st}^{i} t^{(s)} \otimes t^{(t)} + O(d \geq 3)) \\
&\quad + \gamma_{lk}^{j} (\mathrm{id} \otimes e_l)(t^{(l)} \otimes 1 + 1 \otimes t^{(l)} + \sum_{s,t=1}^{m} \gamma_{st}^{l} t^{(s)} \otimes t^{(t)} + O(d \geq 3)) \\
&\overset{(1.4)}{=} \sum_{\substack{i=1 \\ i \neq l}}^{m} \gamma_{ik}^{j} (\sum_{s=1}^{m} \gamma_{sl}^{i} t^{(s)}) + \gamma_{lk}^{j} + \gamma_{lk}^{j} (\sum_{s=1}^{m} \gamma_{sl}^{l} t^{(s)})
\end{aligned}$$

and

$$\epsilon \circ (\sum_{\substack{i=1 \\ i \neq l}}^{m} \gamma_{ik}^{j} (\sum_{s=1}^{m} \gamma_{sl}^{i} t^{(s)}) + \gamma_{lk}^{j} + \gamma_{lk}^{j} (\sum_{s=1}^{m} \gamma_{sl}^{l} t^{(s)})) = \gamma_{lk}^{j}$$

such that

$$[e_l, e_k](t^{(j)}) = \epsilon \circ (D_l \circ D_k - D_k \circ D_l)(t^{(j)}) = \gamma^j_{lk} - \gamma^j_{kl},$$

which proves the lemma. □

Definition and Proposition 1.2.14. Let $\mathcal{D}$ be the continuous dual of $\mathcal{O}(G)$, i.e. $\mathcal{D} = \mathcal{O}(G)^\circ = \varinjlim \mathrm{Hom}_R(\mathcal{O}(G)\,/F^n\,\mathcal{O}(G), R)$, where the filtration was given in Lemma 1.1.5. Then the complete Hopf algebra structure on $\mathcal{O}(G)$ yields a Hopf algebra structure on its continuous dual $(\mathcal{D}, \bigtriangledown, \eta, \mu, \epsilon, s)$ given by dualizing the structure morphsims of $(\mathcal{O}(G), \bigtriangledown, \eta, \mu, \epsilon, s)$.

Proof. See [Haz78, Chap.VII.36] and note that $\mathcal{D}$ is in our case actually a Hopf algebra, since we didn't require that the antipode s is a $\mathcal{D}$-module homomorphism. □

Note that we have associated to a formal group law G the complete Hopf algebra $\mathcal{O}(G)$ and the Hopf algebra $\mathcal{D}$. These objects are dual to each other, where one gets from $\mathcal{O}(G)$ to $\mathcal{D}$ by taking continuous linear duals and from $\mathcal{D}$ to $\mathcal{O}(G)$ by taking linear duals. This duality extends to the categories formed by these objects and is known as Cartier duality, see for example [Die73, Chap.I.2].

Proposition 1.2.15. *Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Then $\mathcal{U}(\mathfrak{g})$ carries a Hopf algebra structure $(\mathcal{U}(\mathfrak{g}), \bigtriangledown, \eta, \mu, \epsilon, s)$. The maps are given by:*

$$\begin{array}{rccc} \bigtriangledown: & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) & \to & \mathcal{U}(\mathfrak{g}) \\ & x \otimes y & \mapsto & x \cdot y \end{array} \qquad \begin{array}{rccc} \eta: & R & \to & \mathcal{U}(\mathfrak{g}) \\ & 1 & \mapsto & 1 \end{array}$$

$$\begin{array}{rccc} \mu: & \mathfrak{g} & \to & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \\ & L & \mapsto & L \otimes 1 + 1 \otimes L \end{array} \qquad \begin{array}{rccc} \epsilon: & \mathfrak{g} & \to & R \\ & L & \mapsto & 0 \end{array}$$

$$\begin{array}{rccc} s: & \mathfrak{g} & \to & \mathcal{U}(\mathfrak{g}) \\ & L & \mapsto & -L. \end{array}$$

Proof. See [Haz78, Chap.II.14.3]. □

Remark. 1.2.16. Note that it is sufficient to define the maps μ, ϵ and s on $\mathfrak{g}$, because of the universal property of $\mathcal{U}(\mathfrak{g})$, mentioned in Proposition 1.1.11.

Let $\mathfrak{g}$ be an m-dimensional Lie algebra over R which is free as an R-module with basis $e_1, \ldots, e_m$ and let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then we denote by $\mathcal{U}^\star(\mathfrak{g}, R) := \mathrm{Hom}_R(\mathcal{U}(\mathfrak{g}), R)$ or by $\mathcal{U}^\star$ - if $\mathfrak{g}$ and R are clear from the context - the R-linear dual of $\mathcal{U}(\mathfrak{g})$.

Let $d^{j_1}t^{(1)} \cdots d^{j_m}t^{(m)}$ be the dual basis of $e_i^{j_1} \cdots e_m^{j_m}$ with $j_i \in \{1, \ldots n\}$ for all $i \in \{1, \ldots m\}$. Then $\mathcal{U}^\star$ has a ring structure with underlying set

$$R\{\{d\underline{t}\}\} := \prod_{\underline{j}} Rd^{j_1}t^{(1)} \cdots d^{j_m}t^{(m)} \tag{1.5}$$

and the two binary operations $+$ as usual addition and multiplication $\bullet$ given by the comultiplication of $\mathcal{U}(\mathfrak{g})$:

$$\begin{aligned} \bullet : \mathcal{U}^\star \otimes \mathcal{U}^\star &\to \mathcal{U}^\star \\ \psi \otimes \varphi &\mapsto [x \mapsto \rho(\psi \otimes \varphi)(\mu(x))], \end{aligned}$$

where $\rho : \mathcal{U}^\star \otimes \mathcal{U}^\star \to (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))^\star$ is the linear injection given by

$$\rho(\psi \otimes \varphi)(x \otimes y) = \psi(x) \cdot \varphi(y).$$

Analogous to $\mathcal{O}(G)$ we can define a filtration on $\mathcal{U}^\star$ by

$$\mathcal{F}^i\mathcal{U}^\star = \left\{\varphi \in \mathcal{U}^\star \,|\text{for all monomials } d^{\underline{j}}\underline{t} \text{ is } |\underline{j}| \geq i\right\}, \tag{1.6}$$

for all $i \in \mathbb{N}$, such that $\mathcal{U}^\star$ is a completed ring with respect to the topology induced by this filtration. And we can identify $\mathcal{U}^\star \hat{\otimes} \mathcal{U}^\star$ with $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))^\star$. The underlying set of $(\mathcal{U}^\star)^{\hat{\otimes} n}$ is given by $R\{\{d\underline{t}_1, \ldots, d\underline{t}_n\}\}$, where we use the multi-index notation $d^{\underline{r}}\underline{t}_i$ - or equivalently $1 \otimes \ldots \otimes 1 \otimes d^{\underline{r}}\underline{t} \otimes 1 \otimes \ldots \otimes 1$ with $d^{\underline{r}}\underline{t}$ at the i-th entry - for $d^{r_1}t_i^{(1)} \cdots d^{r_m}t_i^{(m)}$.

Proposition 1.2.17. *Let $\mathfrak{g}$ be a free Lie algebra, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{U}^\star$ the linear dual of $\mathcal{U}(\mathfrak{g})$. Then $\mathcal{U}^\star$ carries a complete Hopf algebra structure $(\mathcal{U}^\star, \bullet, \eta, \mu, \epsilon, s)$. The maps are given by:*

$$\begin{aligned} \bullet : \mathcal{U}^\star \hat{\otimes} \mathcal{U}^\star &\to \mathcal{U}^\star & \qquad \eta : R &\to \mathcal{U}^\star \\ \psi \otimes \varphi &\mapsto [x \mapsto \rho(\psi \otimes \varphi)(\mu(x))] & 1 &\mapsto [x \mapsto \epsilon(x)] \end{aligned}$$

$$\begin{array}{rccc} \mu\colon & \mathcal{U}^\star & \to & \mathcal{U}^\star \hat{\otimes} \mathcal{U}^\star \\ & \varphi & \mapsto & [x \otimes y \mapsto \varphi(\nabla(x \otimes y))] \end{array} \qquad \begin{array}{rccc} \epsilon: & \mathcal{U}^\star & \to & R \\ & \varphi & \mapsto & \varphi(1) \end{array}$$

$$\begin{array}{rccc} s\colon & \mathcal{U}^\star & \to & \mathcal{U}^\star \\ & \varphi & \mapsto & [x \mapsto \varphi(s(x))]. \end{array}$$

Proof. Note first that again since $\bullet$ is a continuous map it is enough to define this map on $\varphi \otimes \psi \in \mathcal{U}^\star \otimes \mathcal{U}^\star$. After Definition 1.2.6 we have to check that $(\mathcal{U}^\star, \bullet, \eta, \mu, \epsilon)$ is a complete bialgebra. For this see [Swe69, Chap.I-IV] and note that the finiteness condition in Sweedlers book can in our case be replaced by the identification

$$(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))^\star \cong \mathcal{U}^\star \hat{\otimes} \mathcal{U}^\star .$$

Secondly we have to show that s is actually an antipode, i.e. that

$$\bullet \circ (s \otimes 1) \circ \mu = \eta \circ \epsilon = \bullet \circ (1 \otimes s) \circ \mu$$

but this can be easily verified from the antipode condition of $\mathcal{U}(\mathfrak{g})$. $\square$

In the following lemma, we will provide explicit formulas for the multiplication and comultiplication in $\mathcal{U}^\star$. Especially the explicit formula for the multiplication will play an essential role in the next chapter.

Lemma 1.2.18. *Let $\mathfrak{g}$ be an m-dimensional Lie algebra over R which is free as an R-module with basis $e_1, \ldots, e_m$ and let $\mathcal{U}^\star$ be the linear dual of the universal enveloping algebra of $\mathfrak{g}$. Let $d^{j_1} t^{(1)} \cdots d^{j_m} t^{(m)}$ be the dual basis of $e_i^{j_1} \cdots e_m^{j_m}$ with $j_i \in \{1, \ldots n\}$ for all $i \in \{1, \ldots m\}$, so that an element of $\mathcal{U}^\star$ is of the form $\sum_{\underline{j}} c_{\underline{j}} d^{j_1} t^{(1)} \cdots d^{j_m} t^{(m)}$ with $c_{\underline{j}} \in R$. Then multiplication and comultiplication in $\mathcal{U}^\star$ are given by the continuous R-linear extension of*

$$\begin{array}{rccc} \bullet\colon & \mathcal{U}^\star \otimes \mathcal{U}^\star & \to & \mathcal{U}^\star \\ & d^{\underline{r}}\underline{t} \otimes d^{\underline{s}}\underline{t} & \mapsto & \binom{\underline{r}+\underline{s}}{\underline{r}} d^{\underline{r}+\underline{s}}\underline{t} \end{array} \quad \text{and} \quad \begin{array}{rccc} \mu\colon & \mathcal{U}^\star & \to & \mathcal{U}^\star \hat{\otimes} \mathcal{U}^\star \\ & d^{\underline{r}}\underline{t} & \mapsto & \sum_{\underline{l}+\underline{k}=\underline{r}} d^{\underline{l}}\underline{t} \otimes d^{\underline{k}}\underline{t}. \end{array}$$

Proof. We prove the formula for the multiplication by evaluating $d^{\underline{r}}\underline{t} \otimes d^{\underline{s}}\underline{t}$ at an arbitrary element of $\mathcal{U}(\mathfrak{g})$, see also [Ser06, Chap.V.6].

$$\bullet\Big(d^{\underline{r}}\underline{t} \otimes d^{\underline{s}}\underline{t}\Big)\Big(\sum c_{\underline{l}} \underline{e}^{\underline{l}}\Big) = \Big(d^{\underline{r}}\underline{t} \otimes d^{\underline{s}}\underline{t}\Big)\Big(\mu(\sum c_{\underline{l}} \underline{e}^{\underline{l}})\Big)$$

Looking at the right hand side, $\mu(\sum c_{\underline{l}}\underline{e}^{\underline{l}})$ can be rewritten as follows:

$$
\begin{aligned}
\mu\left(\sum c_{\underline{l}}\underline{e}^{\underline{l}}\right) &= \sum c_{\underline{l}}\mu(\underline{e}^{\underline{l}}) \\
&= \sum c_{\underline{l}}\mu(e_1)^{l}\cdots\mu(e_m)^{l_m} \\
&= \sum c_{\underline{l}}\left(\left(\sum_{k_1=0}^{l_1}\binom{l_1}{k_1}e_1^{k_1}\otimes e_1^{l_1-k_1}\right)\cdots\left(\sum_{k_m=0}^{l_m}\binom{l_m}{k_m}e_m^{k_m}\otimes e_m^{l_m-k_m}\right)\right) \\
&= \sum c_{\underline{l}}\left(\sum_{k_1=0}^{l_1}\cdots\sum_{k_m=0}^{l_m}\binom{l_1}{k_1}\cdots\binom{l_m}{k_m}\left(e_1^{k_1}\cdots e_m^{k_m}\otimes e_1^{l_1-k_1}\cdots e_m^{l_m-k_m}\right)\right).
\end{aligned}
$$

Thus we get that

$$
\begin{aligned}
&\bullet\left(d^{\underline{r}}\underline{t}\otimes d^{\underline{s}}\underline{t}\right)\left(\sum c_{\underline{l}}\underline{e}^{\underline{l}}\right) \\
&= \sum c_{\underline{l}}\left(\sum_{k_1=0}^{l_1}\cdots\sum_{k_m=0}^{l_m}\binom{l_1}{k_1}\cdots\binom{l_m}{k_m}d^{\underline{r}}\underline{t}(e_1^{k_1}\cdots e_m^{k_m})\cdot d^{\underline{s}}\underline{t}(e_1^{l_1-k_1}\cdots e_m^{l_m-k_m})\right).
\end{aligned}
$$

We will now analyse the product $d^{\underline{r}}\underline{t}(e_1^{k_1}\cdots e_m^{k_m})\cdot d^{\underline{s}}\underline{t}(e_1^{l_1-k_1}\cdots e_m^{l_m-k_m})$. For the first factor we get

$$
\begin{aligned}
d^{\underline{r}}\underline{t}(e_1^{k_1}\cdots e_m^{k_m}) &= (d^{r_1}t^{(1)}\cdots d^{r_m}t^{(m)})(e_1^{k_1}\cdots e_m^{k_m}) \\
&= \begin{cases} 1, & \text{if } r_1 = k_1, \ldots, r_m = k_m \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

and the second factor reduces to

$$
\begin{aligned}
d^{\underline{s}}\underline{t}(e_1^{l_1-k_1}\cdots e_m^{l_m-k_m}) &= (d^{s_1}t^{(1)}\cdots d^{s_m}t^{(m)})(e_1^{l_1-k_1}\cdots e_m^{l_m-k_m}) \\
&= \begin{cases} 1, & \text{if } s_1 = l_1-k_1, \ldots, s_m = l_m-k_m \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Thus the product in total can be simplified to

$$
d^{\underline{r}}\underline{t}(e_1^{k_1}\cdots e_m^{k_m})\cdot d^{\underline{s}}\underline{t}(e_1^{l_1-k_1}\cdots e_m^{l_m-k_m}) = \begin{cases} 1, & \text{if } l_1 = r_1+s_1, \ldots, l_m = r_m+s_m \\ 0, & \text{otherwise,} \end{cases}
$$

which implies that

$$
\bullet\left(d^{\underline{r}}\underline{t}\otimes d^{\underline{s}}\underline{t}\right)\left(\sum c_{\underline{l}}\underline{e}^{\underline{l}}\right) = \sum c_{\underline{r}+\underline{s}}\binom{r_1+s_1}{r_1}\cdots\binom{r_m+s_m}{r_m}.
$$

We will check the second claim in a similar fashion:

$$\begin{aligned}
\mu\Big(d^{\underline{r}}\underline{t}\Big)\Big(\sum c_{\underline{i}}\underline{e}^{\underline{i}} \otimes \sum d_{\underline{j}}\underline{e}^{\underline{j}}\Big) &= \Big(d^{\underline{r}}\underline{t}\Big)\Big(\triangledown\Big(\sum c_{\underline{i}}\underline{e}^{\underline{i}} \otimes \sum d_{\underline{j}}\underline{e}^{\underline{j}}\Big)\Big) \\
&= \Big(d^{\underline{r}}\underline{t}\Big)\Big(\sum\sum c_{\underline{i}}d_{\underline{j}}\underline{e}^{\underline{i}+\underline{j}}\Big) \\
&= \sum\sum c_{\underline{i}}d_{\underline{j}}(d^{r_1}t^{(1)}\cdots d^{r_m}t^{(m)})(e_1^{i_1+j_1}\cdots e_m^{i_m+j_m}).
\end{aligned}$$

Since

$$d^{r_1}t^{(1)}\cdots d^{r_m}t^{(m)}(e_1^{i_1+j_1}\cdots e_m^{i_m+j_m}) = \begin{cases} 1, & \text{if } r_1 = i_1+j_1, \ldots r_m = i_m+j_m \\ 0, & \text{otherwise,} \end{cases}$$

we get the explicit formula for the comultiplication. □

Example 1.2.19. Let $m = 1$. Then the explicit formulas for multiplication and comultiplication amount to

$$\underbrace{dt \bullet \cdots \bullet dt}_{n-\text{times}} = \underbrace{dt \bullet \cdots \bullet dt}_{(n-2)-\text{times}} \bullet 2d^2t = n!d^nt$$

$$\mu(dt) = 1 \otimes dt + dt \otimes 1.$$

1.3 Cohomology complexes

This section introduces all complexes we are dealing with and certain relations between them. Chapter 2 can be read without knowledge of all these complexes. We will need them in Chapter 3 where we will describe a morphism between the complex of inhomogeneous n-cochains of G and the complex of n-cochains of $\mathfrak{g}$.

Definition and Proposition 1.3.1. Let $(H, \triangledown, \eta, \mu, \epsilon, s)$ be a complete Hopf algebra over R (Definition 1.2.9). Set $T^n(H) = H^{\hat{\otimes} n}$ if $n > 0$ and $T^0(H) = R$. We define linear maps $\partial_n^0, \ldots, \partial_n^{n+1}$ from $T^n(H)$ to $T^{n+1}(H)$ by the continuous extension of

$$\begin{aligned}
\partial_n^0(x_1 \otimes \cdots \otimes x_n) &= 1 \otimes x_1 \otimes \cdots \otimes x_n, \\
\partial_n^{n+1}(x_1 \otimes \cdots \otimes x_n) &= x_1 \otimes \cdots \otimes x_n \otimes 1, \\
\partial_n^i(x_1 \otimes \cdots \otimes x_n) &= x_1 \otimes \cdots \otimes x_{i-1} \otimes \mu(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n,
\end{aligned}$$

if $1 \leq i \leq n$. If $n = 0$, we set $\partial_0^0(1) = \partial_0^1(1) = 1$. We have $\partial_{n+1}^j \partial_n^i = \partial_{n+1}^i \partial_n^{j-1}$ for all integers i, j such that $0 \leq i < j \leq n+2$. We define the differential $\partial : T^n(H) \to T^{n+1}(H)$ by

$$\partial = \sum_{i=0}^{n+1} (-1)^i \partial_n^i. \tag{1.7}$$

Then $\partial \circ \partial = 0$ and we obtain a cochain complex $(T^\bullet(H), \partial)$ called *cobar complex* of the complete Hopf algebra H.

Proof. See [Kas95, Chap.XVIII.5]. $\square$

In the case of the ring of functions $\mathcal{O}(G)$ to a formal group law we will use the following notation.

Definition 1.3.2. Let G be a formal group law. An *inhomogeneous n-cochain* of G with coefficients in R is an element of $\mathcal{O}(G)^{\hat{\otimes} n}$. We will denote the set of inhomogeneous n-cochains also by $K^n(G, R)$. The coboundary homomorphisms $\partial^n : K^n(G, R) \to K^{n+1}(G, R)$ of definition 1.3.1 transform into:

$$\begin{aligned}
\partial^n(f)(\underline{t}_{1,\ldots,n+1}) = f(\underline{t}_{2,\ldots,n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i f(\underline{t}_1, \ldots, G^{(1)}1(\underline{t}_i, \underline{t}_{i+1}), \ldots, G^{(m)}(\underline{t}_i, \underline{t}_{i+1}), \ldots, \underline{t}_{n+1}) \\
&+ (-1)^{n+1} f(\underline{t}_{1,\ldots,n}).
\end{aligned}$$

We obtain a cochain complex $(K^\bullet(G, R), \partial)$ whose cohomology group $H^n(G, R)$ is called *n-th group cohomology of G with coefficients in R.*

In the case of the dual of the universal enveloping algebra of $\mathfrak{g}$ we will use the following notation.

Definition and Proposition 1.3.3. Let $\mathfrak{g}$ be a Lie algebra over R and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra. An *inhomogeneous n-cochain* of $\mathcal{U}(\mathfrak{g})$ with coefficients in R is an element of $\mathrm{Hom}_R((\mathcal{U}(\mathfrak{g}))^{\otimes n}, R)$. The coboundary homomorphisms $\partial_u^n : \mathrm{Hom}_R((\mathcal{U}(\mathfrak{g}))^{\otimes n}, R) \to \mathrm{Hom}_R((\mathcal{U}(\mathfrak{g}))^{\otimes n+1}, R)$ given by

$$\begin{aligned}
\partial_u^n(u_1, \ldots u_{n+1}) = f(u_2, \ldots, u_{n+1}) \quad &+ \quad \sum_{i=1}^{n} (-1)^i f(u_1, \ldots, u_i u_{i+1}, \ldots, u_{n+1}) \\
&+ \quad (-1)^{n+1} f(u_1, \ldots u_n)
\end{aligned}$$

define a cochain complex $(\mathrm{Hom}_R(U^\bullet, R), \partial_u)$.

Proof. See [NSW00, Chap. I.2]. □

Remark. 1.3.4. Since the set of inhomogeneous n-cochains can be identified with $\mathcal{U}^{\star\hat{\otimes}n}$, the definition of the coboundary homomorphisms of Definition 1.3.3 is equivalent to the definition of the differential defined by (1.7) of Definition 1.3.1 for the complete Hopf algebra $\mathcal{U}^{\star}$.

Definition and Proposition 1.3.5. A *homogeneous n-cochain* of $\mathcal{U}(\mathfrak{g})$ with coefficients in R is an element of $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}((\mathcal{U}(\mathfrak{g}))^{\otimes n+1}, R)$, where $\mathcal{U}(\mathfrak{g})^{\otimes n}$ is considered as an $\mathcal{U}(\mathfrak{g})$-module via the following operation:

$$u.(u_0, \dots u_{n-1}) = (uu_0, \dots u_{n-1}).$$

The map

$$\begin{array}{rcl} \iota^n : \mathrm{Hom}_R(\mathcal{U}(\mathfrak{g})^{\otimes n}, R) & \to & \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g})^{\otimes n+1}, R) \\ \varphi & \mapsto & [(u_0, \dots, u_n) \mapsto \varphi(u_1, \dots u_n)] \\ [(u_1, \dots, u_n) \mapsto \varphi(1, u_1, \dots u_n)] & \leftarrow\!\shortmid & \varphi \end{array}$$

is an isomorphism from the set of inhomogeneous to the set of homogeneous n-cochains. If we consider the following coboundary homomorphisms

$$\partial^n_{u_h} : \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}((\mathcal{U}(\mathfrak{g}))^{\otimes n+1}, R) \to \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}((\mathcal{U}(\mathfrak{g}))^{\otimes n+2}, R)$$

defined by

$$\partial^n_{u_h} = \iota^{n+1} \circ \partial^n_u \circ (\iota^n)^{-1}$$

we obtain a complex $(\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^{\bullet}, R), \partial_{u_h})$ of *homogeneous n-cochains* and ι^n yields an isomorphism $\iota : (\mathrm{Hom}_R(U^{\bullet}, R), \partial_u) \to (\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^{\bullet}, R), \partial_{u_h})$ of complexes.

Proof. It is enough to prove that ι is an isomorphism, since the remaining statement can be easily deduced from this. We check first that $\iota^n(\varphi)$ is $\mathcal{U}(\mathfrak{g})$-invariant:

$$(u.\iota^n(\varphi))(u_0, \dots u_n) = \iota^n(\varphi)(uu_0, \dots u_n) = \varphi(u_1, \dots u_n) = \iota^n(\varphi)(u_0, \dots u_n).$$

Secondly we show that ι^n and $(\iota^n)^{-1}$ are inverse to each other:

$$\begin{aligned} (\iota^n)^{-1} \circ \iota^n(\varphi)(u_1, \dots, u_n) &= \iota^n(\varphi)(1, u_1, \dots, u_n) = \varphi(u_1, \dots, u_n) \\ \iota^n \circ (\iota^n)^{-1}(\varphi)(u_0, \dots, u_n) &= (\iota^n)^{-1}(\varphi)(u_1, \dots, u_n) = \varphi(1, u_1, \dots, u_n) \\ (\varphi \text{ is homogeneous}) \qquad &= (u_0.\varphi)(1, u_1, \dots, u_n) = \varphi(u_0, u_1, \dots, u_n). \end{aligned}$$

□

For the following definitions let $\mathfrak{g}$ be an m-dimensional Lie algebra over R which is free as an R-module with basis $e_1, \ldots, e_m$. Let $\bigwedge^n \mathfrak{g}$ be the n-fold exterior product of $\mathfrak{g}$ with basis $\{e_{i_1} \wedge \ldots \wedge e_{i_n} \mid i_1 < \ldots < i_n\}, i_j \in \{1, \ldots, m\}$. We endow R with the trivial $\mathfrak{g}$-action.

Definition and Proposition 1.3.6. The set $\mathrm{Hom}_R(\bigwedge^n \mathfrak{g}, R)$ is called the set of *n-cochains* of $\mathfrak{g}$ with coefficients in R and denoted by $C^n(\mathfrak{g}, R)$. Note that the rank of $C^n(\mathfrak{g}, R)$ over R is $\binom{m}{n}$. The boundary operators $\partial'^n : C^n \to C^{n+1}$ are given by the formula

$$\partial'^n(\omega)(e_{i_1} \wedge \ldots \wedge e_{i_{n+1}}) = \sum_{1 \le r < s \le n+1} (-1)^{r+s} \omega([e_{i_r}, e_{i_s}] \wedge e_{i_1} \wedge \ldots \wedge e_{i_{n+1}})_{r,s},$$

where the notation $([e_{i_r}, e_{i_s}] \wedge e_{i_1} \wedge \ldots \wedge e_{i_{n+1}})_{r,s}$ indicates that the elements e_{i_r} and e_{i_s} are omitted. We thus obtain, after assuring ourself that $\partial'^2 = 0$, a complex $(C^\bullet(\mathfrak{g}, R), \partial')$ whose cohomology group $H^n(\mathfrak{g}, R)$ is called *n-the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in R.*

Proof. See [Kna88, Chap. IV.3]. □

In particular we have $K^0(G, R) \cong R \cong C^0(\mathfrak{g}, R)$ and the boundary operators ∂^0, ∂_u^0 and ∂'^0 are zero maps.

For details about the following complex see [CE56, Chap. XIII.7] or [Kna88, Chap. IV.3].

Definition 1.3.7. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Set

$$V_i(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes \bigwedge^i \mathfrak{g}$$

for all $i \in \{0, 1, 2, \ldots\}$ with the $\mathfrak{g}$-module structure induced by the action on the first factor. The differential $d^{n-1} : V_n(\mathfrak{g}) \to V_{n-1}(\mathfrak{g})$ is given by the formula

$$\begin{aligned} d^{n-1}(u \otimes e_{i_1} \wedge \ldots \wedge e_{i_n}) = &\sum_{1 \le k < l \le n} (-1)^{k+l} (u \otimes [e_{i_k}, e_{i_l}] \wedge e_{i_1} \wedge \ldots \wedge e_{i_n})_{k,l} \\ &+ \sum_{j=1}^{n} (-1)^{j+1} (u e_{i_j} \otimes e_{i_1} \wedge \ldots \wedge e_{i_n})_j, \end{aligned}$$

where the notation $(e_{i_1} \wedge \ldots \wedge e_{i_n})_{k,l}$, respectively $(e_{i_1} \wedge \ldots \wedge e_{i_n})_j$ again indicates that the elements e_{i_k} and e_{i_l}, respectively e_{i_j} are omitted. This leads, after assuring ourselves that $d^2 = 0$, to a complex, called *Koszul complex.*

The following two propositions relate the Koszul complex first to the standard homogeneous complex of $\mathcal{U}(\mathfrak{g})$ and secondly to the Lie algebra complex.

Proposition 1.3.8. *Let* $(\mathrm{V}(\mathfrak{g})^\bullet, d)$ *be the Koszul complex defined above. Then the map* $\nu : \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^\bullet, R) \to \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V(\mathfrak{g})^\bullet, R)$ *induced by the anti-symmetrisation map*

$$as_n : \bigwedge^n \mathfrak{g} \to \mathcal{U}^{\otimes n}$$

given by

$$as_n(e_{i_1} \wedge \ldots \wedge e_{i_n}) = \sum_{\alpha \in S_n} \mathrm{sgn}(\alpha) e_{i_{\alpha(1)}} \otimes \cdots \otimes e_{i_{\alpha(n)}},$$

with $i_j \in \{1, \ldots, m\}$*, is a quasi-isomorphism of complexes.*

Proof. See [CE56, Chap. XIII.7, Theorem 7.1]. □

Proposition 1.3.9. *Let* $(\mathrm{V}(\mathfrak{g})^\bullet, d)$ *be the Koszul complex defined above. Then the map* $\kappa : \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V(\mathfrak{g})^\bullet, R) \to C^\bullet(\mathfrak{g}, R)$ *given by*

$$\begin{array}{rcl} \kappa^n : \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V_n(\mathfrak{g}), R) & \to & \mathrm{Hom}_R(\bigwedge^n \mathfrak{g}, R) \\ f & \mapsto & [(e_{i_1} \wedge \cdots \wedge e_{i_n}) \mapsto f(1 \otimes e_{i_1} \wedge \ldots \wedge e_{i_n})] \end{array}$$

is an isomorphism of complexes.

Proof. See [CE56, Chap. XIII.8] or for a more detailed version [Kna88, Chap. IV.3-6]. □

Chapter 2

Isomorphism of complete Hopf algebras

> Throughout this chapter R will be an integral domain of characteristic zero.

After the description of the (complete) Hopf algebra structures in Chapter 1, we will show in Section 2.1 that for an abitrary formal group law G there exists a homomorphism of complete Hopf algebras between the ring of functions $\mathcal{O}(G)$ and the dual of the universal enveloping algebra $\mathcal{U}^\star$. In the special cases of the m-dimensional additive or a one-dimensional formal group law we will prove in the second section, by giving an explicit description of the morphism, that this morphism is actually an isomorphism of complete Hopf algebras if $\mathbb{Q} \subset R$. This observation leads us to the introduction of a modified ring $\tilde{\mathcal{O}}(G)$ in Section 2.3. In Section 2.4 we will show that we actually obtain an isomorphism from $\tilde{\mathcal{O}}(G)$ to $\mathcal{U}^\star$ for all formal group laws G over R.

2.1 Homomorphism between $\mathcal{O}(G)$ and $\mathcal{U}^\star$

Let G be an m-dimensional formal group law and $\mathcal{O}(G)$ the ring of functions to G (see Definition 1.1.1). Let $\mathfrak{g}$ be the associated free m-dimensional Lie algebra over R to G which is free as an R-module with basis $e_1, \dots, e_m$ (see description around (1.3)) and let $\mathcal{U}^\star$ be the linear dual of the universal enveloping algebra of $\mathfrak{g}$. This section describes the desired map from $\mathcal{O}(G)$ to $\mathcal{U}^\star$ as a composition of maps $\mathcal{O}(G) \to \mathcal{D}^\star \to \mathcal{U}^\star$, where $\mathcal{D}^\star$ is the linear dual of $\mathcal{D}$ and the latter map is defined by dualizing the map $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}$. The existence of the map $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}$ can also be found in [Haz78, Chap. VII.37.4] or [Ser06, Chap.V.6]. Since however we are more interested in the dual map $\mathcal{D}^\star \to \mathcal{U}^\star$, and since in particular we wish to show it is a complete Hopf algebra morphism, we give a complete proof of Proposition 2.1.2.

Definition 2.1.1. Let H be a (complete) Hopf algebra. An element $x \in H$ is called *primitive* if its comultiplication is given by

$$\mu(x) = x \otimes 1 + 1 \otimes x.$$

Proposition 2.1.2. *Let G be an m-dimensional formal group law, $\mathcal{O}(G)$ the ring of functions to G, $\mathfrak{g}$ the associated free m-dimensional Lie algebra over R to G which is free as an R-module with basis $e_1, \ldots, e_m$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{U}^\star$ the linear dual of $\mathcal{U}(\mathfrak{g})$. There are natural homomorphisms of (complete) Hopf algebras*

$$\beta : \mathcal{U}(\mathfrak{g}) \to \mathcal{D} \text{ and } \beta^\star : \mathcal{O}(G) \to \mathcal{U}^\star$$

induced by the pairing

$$\begin{array}{rcl} \mathfrak{g} \otimes \mathcal{O}(G) & \to & R \\ L \otimes f & \mapsto & L(f), \end{array}$$

where $L(f)$ was defined by $L(f) = \sum_{j=1}^m r_j \frac{\partial f}{\partial t_1^{(j)}}(0)$ if $L = \sum_{j=1}^m r_i e_j \in \mathfrak{g}$, see Equation (1.2) in Section 1.1.

Proof. Let $\mathcal{D}$ be the linear dual of $\mathcal{O}(G)$ and let $\gamma : \mathfrak{g} \to \mathcal{D}$ be defined by the pairing of the proposition. Let $\beta : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}$ be the unique algebra homomorphism given by the universal property of $\mathcal{U}(\mathfrak{g})$ in Proposition 1.1.11. To prove that the map β is a Hopf algebra homomorphism, it remains to check that the map β is a coalgebra morphism, i.e. that

1) $\epsilon \circ \beta = \epsilon$

2) $(\beta \otimes \beta) \circ \mu = \mu \circ \beta$.

Then according to Lemma 1.2.8 the map β is already a Hopf algebra homomorphism. To check the first condition let $x = \sum_{\underline{j}} c_{\underline{j}} \underline{e}^{\underline{j}} \in \mathcal{U}(\mathfrak{g})$. Then

$$\epsilon \circ \beta(x) = \beta(x)(1) = x(1) = c_{\underline{0}} = \epsilon(x).$$

Since $\mathcal{U}(\mathfrak{g})$ is as an algebra generated by elements of the Lie algebra $\mathfrak{g}$, the second condition is equivalent to the condition that the map β maps elements of the Lie algebra to primitive elements, which we will now check. Let $L = \sum a_i e_i \in \mathfrak{g}$. Then

$$\begin{aligned}
\mu(\beta(L))(f \otimes g) &= \beta(L)(fg) \\
&= \sum a_i e_i(fg) \\
&= \sum a_i e_i(f) \cdot g(0) + \sum a_i e_i(g) \cdot f(0) \\
&= \beta(L)(f) \cdot \beta(1)(g) + \beta(L)(g) \cdot \beta(1)(f) \\
&= (\beta(L) \otimes 1 + 1 \otimes \beta(L))(f \otimes g).
\end{aligned}$$

The map

$$\begin{array}{rcl}
\beta^\star : \mathcal{O}(G) & \to & \mathcal{U}^\star \\
f & \mapsto & [x \mapsto \beta(x)(f)]
\end{array}$$

is given by the following composition of maps

$$\begin{array}{ccccc}
\mathcal{O}(G) & \to & \mathcal{D}^\star & \to & \mathcal{U}^\star \\
f & \mapsto & [\varphi \mapsto \varphi(f)] & & \\
 & & \psi & \mapsto & [x \mapsto \psi(\beta(x))],
\end{array}$$

where $\mathcal{D}^\star$ is the linear dual of $\mathcal{D}$ and inherits therefore by dualizing the structure morphisms a complete Hopf algebra structure. The latter map is defined as the dual of the map β. One sees as follows that the map $\beta^\star$ is a homomorphism of completed Hopf algebras:

1) the map $\mathcal{O}(G) \to \mathcal{D}^\star$ is a homomorphism of complete Hopf algebras, which can be easily verified from the complete Hopf algebra structures of $\mathcal{O}(G)$ and $\mathcal{D}^\star$,

3) the dual of a Hopf algebra homomorphism is a homomorphism of complete Hopf algebras and

3) the composition of homomorphisms of complete Hopf algebras is a homomorphisms of complete Hopf algebras.

□

Notation. 2.1.3. We will denote the image of e_i under the map

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\gamma} & \mathcal{D} \\ L & \mapsto & [f \mapsto L(f)] \end{array}$$

by $\phi^{(i)}$, to that we have $\phi^{(i)}(f) = e_i(f) = \frac{\partial f}{\partial t_1^{(i)}}(0)$. The image of e_i^k of $\mathcal{U}(\mathfrak{g})$ under the map β is therefore denoted by $(\phi^{(i)})^k$, where the multiplication is given by the Hopf algebra structure on $\mathcal{D}$. For $(\phi^{(i)})^0$ one has that $(\phi^{(i)})^0(f) = f(0)$.

2.2 The additive and one-dimensional cases

In this section we will consider the special cases, i.e. the additive formal group law G_a and arbitrary one-dimensional formal group laws. By giving an explicit description of the map $\beta^\star$ we will prove that $\beta^\star$ is in these cases an isomorphism of complete Hopf algebras, under the assumption that $\mathbb{Q} \subset R$.

Theorem 2.2.1. *Let $\beta^\star : \mathcal{O}(G) \to \mathcal{U}^\star$ be the map defined in Proposition 2.1.2 and let G be either the m-dimensional additve or a one-dimensional formal group law with $G(X,Y) = X + Y + \sum_{i,j \geq 1} c_{ij} X^i Y^j$. Assume $\mathbb{Q} \subset R$. Then the map $\beta^\star$ is an isomorphism of complete Hopf algebras.*

The proof of this theorem will be given in three steps. First we will look at the one-dimensional additive group G_a as an example, to see exactly what is going on in this easy case, i.e. what $\beta^\star(t)(x)$ is for $t \in \mathcal{O}(G)$ and $x \in \mathcal{U}(\mathfrak{g})$. In the second step we prove a lemma, which gives an explicit description of $\beta^\star(t)(x)$ for all one-dimensional formal group laws. Finally we generalize Example 2.2.2 for all m-dimensional additive formal group laws to gain the statement of Theorem 2.2.1.

Example 2.2.2. In the case of the one-dimensional additive formal group law G_a, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is according to Example 1.1.12 of the form $\mathcal{U}(\mathfrak{g}) \cong R[e_1]$. Let $x = \sum a_k e_1^k \in \mathcal{U}(\mathfrak{g}), t \in \mathcal{O}(G)$. Then

$$
\begin{aligned}
&\beta^\star(t)(x) = \beta(\textstyle\sum a_k e_1^k)(t)\\
&\quad = \textstyle\sum a_k (\phi^{(1)})^k(t)\\
&\quad = \textstyle\sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\ldots(\mu \otimes \mathrm{id}(\mu(t)))\ldots))\\
&\quad = \textstyle\sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\ldots(\mu \otimes \mathrm{id}(t \otimes 1 + 1 \otimes t))\ldots))\\
&\quad = \textstyle\sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(t \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes t)\\
&\quad = a_1,
\end{aligned}
$$

where the last equation holds because $\phi^{(1)}(t) = 1$ and $\phi^{(1)}(1) = 0$. Thus the map $\beta^\star$ is explicitly given by:

$$
\begin{aligned}
\beta^\star : \mathcal{O}(G) &\to \mathcal{U}^\star\\
t &\mapsto dt.
\end{aligned}
$$

If we consider the n-th power of t we get as shown in Example 1.2.19 that

$$\beta^\star(t^n) = \underbrace{dt \bullet \ldots \bullet dt}_{n \text{ times}} = n! d^n t.$$

Since the map $\beta^\star$ is continuous and R-linear we can conclude in the case of the one-dimensional additive formal group law that $\beta^\star$ is an isomorphism of complete Hopf algebras under the assumption that $\mathbb{Q} \subset R$.

Lemma 2.2.3. *Let $\beta^\star$ be the map defined in Proposition 2.1.2 and let G be an arbitrary one-dimensional formal group law with*

$$G(X, Y) = X + Y + \sum_{i,j \geq 1} c_{ij} X^i Y^j.$$

Let $c := c_{11}$. Then

$$\beta^\star(t) = \begin{cases} dt, & \text{if } G = G_a \\ \frac{1}{c}\exp(cdt) - 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

Proof. Let $x = \sum a_k e_1^k \in \mathcal{U}(\mathfrak{g})$ and $t \in \mathcal{O}(G)$. Then, as in Example 2.2.2, $\beta^\star(t)(x)$ is given by the following sum

$$\sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\ldots(\mu \otimes \mathrm{id}(\mu(t)))\ldots)), \tag{2.2}$$

where $\mu(t)$ is in this case given by

$$\mu(t) = t \otimes 1 + 1 \otimes t + \sum_{i,j \geq 1} c_{ij} t^i \otimes t^j.$$

Consider the expression

$$\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-3 \text{ times}}(\ldots(\mu \otimes \mathrm{id}(\mu(t)))\ldots)))$$

on the right hand side of (2.2). After applying μ, three types of factors arise, namely:

$$\star \otimes \ldots \otimes \star \otimes 1, \; 1 \otimes \ldots \otimes 1 \otimes t \text{ and } \star \otimes \ldots \otimes \star \otimes t^j,$$

where the symbol $\star$ stands for an arbitrary element in the formal power series ring $R[[t]]$. The first type comes from the factor $t \otimes 1$ in $\mu(t)$, the second from $1 \otimes t$ in $\mu(t)$. Since we apply only the identity on t^j of the last factor in $\mu(t)$, we get the third type. From the definition of $\phi^{(1)}$ in 2.1.3 we know that

$$\phi^{(1)}(t^j) = \begin{cases} 1, & \text{if } j = 1 \\ 0, & \text{otherwise.} \end{cases}$$

This shows that we can assume that $c_{ij} = 0$ for $j \geq 2$, since all factors $c_{ij} t^i \otimes t^j$ with $c_{ij} \neq 0$ for $j \geq 2$ provide no contribution to the calculation of $\beta^\star(t)(x)$. Since comultiplication is associative (2.2) is equal to

$$\sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}} \otimes \mu(\ldots(\mathrm{id} \otimes \mu(\mu(t)))\ldots))$$

and by the same argument as above, it follows that we can assume that $c_{ij} = 0$ for $i \geq 2$. Hence we can restrict ourselves to the case where the formal group law is given by $G(X,Y) = X + Y + cXY$ and we get

$$\begin{aligned}
\beta^\star(t)(x) &= \sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\ldots(\mu \otimes \mathrm{id}(\mu(t)))\ldots))) \\
&= \sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\ldots \\
&\qquad \ldots(\mu \otimes \mathrm{id}(t \otimes 1 + 1 \otimes t + ct \otimes t))\ldots)) \\
&= \sum a_k \underbrace{(\phi^{(1)} \otimes \ldots \otimes \phi^{(1)})}_{k \text{ times}}(\mu \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{k-2 \text{ times}}(\ldots(t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 \\
&\quad + ct \otimes t \otimes 1 + 1 \otimes 1 \otimes t + ct \otimes 1 \otimes 1 + c \otimes t \otimes 1 + c^2 t \otimes t \otimes t))\ldots)) \\
&= \sum a_k c^{k-1}.
\end{aligned}$$

Thus

$$\beta^\star(t) = \sum_{i=1}^{\infty} c^{i-1} d^i t \overset{1.2.18}{=} \sum_{i=1}^{\infty} c^{i-1} \frac{1}{i!}(dt)^i = \begin{cases} dt, & \text{if } c = 0 \\ \frac{1}{c}\exp(cdt) - 1, & \text{if } c \neq 0. \end{cases}$$

□

Proof of Theorem 2.2.1. At first we will consider the case where G is one-dimensional. Since the map $\beta^\star$ is continuous and R-linear, it is given by the continuous R-linear extension of (2.1). Hence injectivity of the map $\beta^\star$ is obvious. For surjectivity we have to show that there exists an element f of $\mathcal{O}(G)$ such that $\beta^\star(f) = d^j t$ for all j. This element f is given by

$$f = \begin{cases} \frac{1}{j!} t^j, & \text{if } G = G_a \\ \frac{1}{cj!}\log(ct + c)^j, & \text{otherwise.} \end{cases}$$

Since we assumed that $\mathbb{Q} \subset R$ the coefficients of f lie in R.
Consider now the case where G is the m-dimensional additive formal group law G_a. Then the map $\beta^\star$ is via a computation analogous to that of Example 2.2.2 given by the continuous R-linear extension of:

$$\begin{aligned}
\beta^\star : \mathcal{O}(G) &\to \mathcal{U}^\star \\
t^{(j)} &\mapsto dt^{(j)} = d^{\mathbf{e}(\mathbf{j})} \underline{t},
\end{aligned}$$

where the multi-index $\mathbf{e(j)}$ is $(0,\dots,0,1,0,\dots,0)$ with 1 in the j-th entry and therefore

$$\begin{aligned}\beta^\star(\underline{t}^{\underline{j}}) &= \underbrace{d^{\mathbf{e(1)}}\underline{t}\bullet\ldots\bullet d^{\mathbf{e(1)}}\underline{t}}_{j_1\text{ times}}\bullet\ldots\bullet\underbrace{d^{\mathbf{e(m)}}\underline{t}\bullet\ldots\bullet d^{\mathbf{e(m)}}\underline{t}}_{j_m\text{ times}}\\ &\overset{1.2.18}{=} j_1!d^{(j_1,0\ldots,0)}\underline{t}\bullet\ldots\bullet j_m!d^{(0,\ldots,0,j_m)}\underline{t}\\ &= \underline{j}!d^{\underline{j}}\underline{t}.\end{aligned}$$

Hence injectivity is again obvious and since

$$\beta^\star(\frac{1}{\underline{j}!}\underline{t}^{\underline{j}}) = d^{\underline{j}}\underline{t}$$

we can conclude that the map $\beta^\star$ is surjective if $\mathbb{Q}\subset R$. □

2.3 The modified ring $\tilde{\mathcal{O}}(G)$

In the proof of Theorem 2.2.1 we have seen that the strong assumption $\mathbb{Q}\subset R$ is really neccessary. However we can weaken this restriction, since we only need certain divisibility conditions for the coefficients of the formal power series of $\mathcal{O}(G)$. This leads to the definition of the following modified ring $\tilde{\mathcal{O}}(G)$.

Definition 2.3.1. Let $Q(R)$ be the quotient field of R. To each m-dimensional formal group law G we associate an extension of the ring of functions $\mathcal{O}(G)$ called the *modified ring of functions* $\tilde{\mathcal{O}}(G)$ which is defined by

$$\tilde{\mathcal{O}}(G) := \left\{\sum_{\underline{j}} b_{\underline{j}}\underline{t}^{\underline{j}}\in Q(R)[[\underline{t}]] \mid \underline{j}!b_{\underline{j}}\in R\right\}.$$

Remark. 2.3.2. Note that the modified ring $\tilde{\mathcal{O}}(G)$ is actually a ring and that $\mathcal{O}(G)\subset\tilde{\mathcal{O}}(G)$. If one adds two elements $f=\sum_{\underline{j}} b_{\underline{j}}\underline{t}^{\underline{j}}$ and $g=\sum_{\underline{j}} c_{\underline{j}}\underline{t}^{\underline{j}}$ of $\tilde{\mathcal{O}}(G)$ the sum is given by

$$f+g=\sum_{\underline{j}}(b_{\underline{j}}+c_{\underline{j}})\underline{t}^{\underline{j}}$$

and obviously

$$\underline{j}!(b_{\underline{j}}+c_{\underline{j}})\in R.$$

If one multiplies f and g the product is given by

$$f \cdot g = \sum_{\underline{j}} \sum_{\underline{r}+\underline{s}=\underline{j}} b_{\underline{r}} c_{\underline{s}} \underline{t}^{\underline{j}}$$

and the divisibility condition is fulfilled since

$$\underline{j}! b_{\underline{r}} c_{\underline{s}} = (\underline{r}+\underline{s})! b_{\underline{r}} c_{\underline{s}} = \binom{\underline{r}+\underline{s}}{\underline{r}} \underline{r}! b_{\underline{r}} \underline{s}! c_{\underline{s}} \in R.$$

Lemma 2.3.3. *The modified ring of functions $\tilde{\mathcal{O}}(G)$ is a complete ring with respect to the topology induced by the following descending filtration*

$$F^i\,\tilde{\mathcal{O}}(G) = \left\{ f \in \tilde{\mathcal{O}}(G) \,|\, \textit{all monomials of } f \textit{ have a total degree} \geq i \right\}, \qquad (2.3)$$

for all $i \in \mathbb{N}$.

Proposition 2.3.4. *Let G be an m-dimensional formal group law and $\tilde{\mathcal{O}}(G)$ the associated modified ring of Definition 2.3.1. Then $\tilde{\mathcal{O}}(G)$ carries a complete Hopf algebra structure $(\tilde{\mathcal{O}}(G), \bigtriangledown, \eta, \mu, \epsilon, s)$.*

Proof. We claim that the morphisms $\bigtriangledown, \eta, \mu, \epsilon$ and s of the complete Hopf algebra structure of $\mathcal{O}(G)$ of Proposition 1.2.11 can be taken to get a complete Hopf algebra structure on $\tilde{\mathcal{O}}(G)$. To see this we have to verify the divisibility conditions, i.e. we have to check that if $f = \sum_{\underline{j}} b_{\underline{j}} \underline{t}^{\underline{j}}$ is an element of $\tilde{\mathcal{O}}(G)$, then

(i) $\epsilon(f) \in R$, but this is obvious since $\epsilon(f) = f(0) = b_{\underline{0}}$ and $\underline{0}! b_{\underline{0}} = b_{\underline{0}} \in R$

(ii) $\mu(f) \in \tilde{\mathcal{O}}(G) \,\hat{\otimes}\, \tilde{\mathcal{O}}(G)$, but this is true since

$$\begin{aligned} \mu(f) = \mu(\sum_{\underline{j}} b_{\underline{j}} \underline{t}^{\underline{j}}) &= \sum_{\underline{j}} b_{\underline{j}} G(\underline{t}_1, \underline{t}_2)^{\underline{j}} \\ &= \sum_{\underline{j}} b_{\underline{j}} G^{(1)}(\underline{t}_1, \underline{t}_2)^{j_1} \cdots G^{(m)}(\underline{t}_1, \underline{t}_2)^{j_m} \end{aligned}$$

where the coefficients of the formal group law lie in R. Consider the factors $G^{(i)}(\underline{t}_1, \underline{t}_2)$. These are given by

$$G^{(i)}(\underline{t}_1, \underline{t}_2) = t_1^{(i)} + t_2^{(i)} + O(d \geq 2),$$

see Definition 1.1.1. Via the binomial formula $G^{(i)}(\underline{t}_1, \underline{t}_2)^{j_i}$ can be written as

$$G^{(i)}(\underline{t}_1, \underline{t}_2)^{j_i} = \sum_{l_i=0}^{j_i} \sum_{k_i=0}^{j_i - l_i} \binom{j_i}{l_i} \binom{j_i - l_1}{k_i} \left(t_1^{(i)}\right)^{j_i - l_i - k_i} \cdot \left(t_2^{(i)}\right)^{l_i} \cdot O(d \geq 2)^{k_i}.$$

Hence to check that $\mu(f) \in \tilde{\mathcal{O}}(G) \hat{\otimes} \tilde{\mathcal{O}}(G)$ we must prove that

$$b_{\underline{j}} \cdot \prod_{i=1}^{m} \binom{j_i}{l_i} \binom{j_i - l_i}{k_i} (j_i - l_i - k_i)! l_i! k_i! \tag{2.4}$$

lies in R. However this can be seen be rewriting the binomials:

$$\begin{aligned} (2.4) &= b_{\underline{j}} \cdot \prod_{i=1}^{m} \frac{j_i!}{l_i! \cdot (j_i - l_i)!} \cdot \frac{(j_i - l_i)!}{k_i! \cdot (j_i - l_i - k_i)!} (j_i - l_i - k_i)! l_i! k_i! \\ &= b_{\underline{j}} \cdot \prod_{i=1}^{m} j_i! = b_{\underline{j}} \underline{j}! \in R \end{aligned}$$

(iii) $s(f) \in \tilde{\mathcal{O}}(G)$, this can be shown by an analogous computation to that in (ii) since

$$s(\underline{t})^{\underline{j}} = \prod_{i=1}^{m} s(t^{(i)})^{j_i} = \prod_{i=1}^{m} (-t^{(i)} + O(d \geq 2))^{j_i},$$

see Remark 1.1.3.

□

Proposition 2.3.5. *The map $\beta^\star$ of Proposition 2.1.2 extends to a homomorphism $\tilde{\beta}^\star : \tilde{\mathcal{O}}(G) \to \mathcal{U}^\star$ of complete Hopf algebras.*

Proof. Consider the pairing

$$\begin{array}{rcl} \mathfrak{g} \otimes \mathcal{O}(G) & \to & R \\ L \otimes f & \mapsto & L(f) \end{array}$$

which induced the homomorphism $\beta^\star : \mathcal{O}(G) \to \mathcal{U}^\star$ of complete Hopf algebras of Proposition 2.1.2. This pairing can be extended to a pairing

$$\begin{array}{rcl} \mathfrak{g} \otimes \tilde{\mathcal{O}}(G) & \to & R \\ L \otimes f & \mapsto & L(f) \end{array}$$

since $L(f) \in R$ even if $f \in \tilde{\mathcal{O}}(G)$, because the linear terms of f still lie in R. With these modifications, the proof of Proposition 2.1.2 still goes through. □

2.4 General case

We have proven in Section 2.2 that under the assumptions that $\mathbb{Q} \subset R$ and that G is either the m-dimensional additive or a one-dimensional formal group law, the map $\beta^\star : \mathcal{O}(G) \to \mathcal{U}^\star$ of Proposition 2.1.2 is an isomorphism of complete Hopf algebras, see Theorem 2.2.1. Our goal is now to generalize this result in two directions. Firstly we will drop the assumption $\mathbb{Q} \subset R$ and secondly we will consider all m-dimensional formal group laws. Therefore we consider the modified ring of functions $\tilde{\mathcal{O}}(G)$ instead of $\mathcal{O}(G)$. No analogue of the explicit description of the map $\beta^\star$ as in Lemma 2.2.3 will be provided in the general case. Note that although the following Theorem is a generalization of Theorem 2.2.1, its proof is independent of but inspired by the one of Theorem 2.2.1.

Proposition 2.4.1. *Let G be an m-dimensional formal group law over R. Let $\tilde{\mathcal{O}}(G)$ be the modified ring of functions (Definition 2.3.1). Then the map $\tilde{\beta}^\star : \tilde{\mathcal{O}}(G) \to \mathcal{U}^\star$, defined by Proposition 2.3.5, is an isomorphism of complete Hopf algebras.*

Remark. 2.4.2. M. Hazewinkel shows in an analogous way in [Haz78, Chap. VII.37.4] that the map $\beta : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}$ is an algebra isomorphism if $\mathcal{O}(G)$ and $\mathcal{U}^\star$ are $\mathbb{Q}$-algebras and that β respects the comultiplication and counits.

Proof of Proposition 2.4.1. Consider the filtrations on $\tilde{\mathcal{O}}(G)$ and $\mathcal{U}^\star$ given in (2.3) and (1.6) by

$$F^i\,\tilde{\mathcal{O}}(G) = \left\{ f \in \tilde{\mathcal{O}}(G)\,|\text{all monomials of } f \text{ have a total degree} \geq i \right\}$$

and

$$\mathcal{F}^i\,\mathcal{U}^\star = \left\{ \varphi \in \mathcal{U}^\star\,|\text{for all monomials } d^{\underline{j}}\underline{t} \text{ is } |\underline{j}| \geq i \right\}$$

for all $i \in \mathbb{N}$. Let $\underline{t}^{\underline{j}}$ be a basis element of $F^i\,\tilde{\mathcal{O}}(G)$, i.e. $|\underline{j}| \geq i$. We claim that

$$\tilde{\beta}^\star(\underline{t}^{\underline{j}}) \equiv \underline{j}!\,d^{\underline{j}}\underline{t} \mod \mathcal{F}^{i+1}\,\mathcal{U}^\star\,.$$

To prove this we will first consider $\tilde{\beta}^\star(t^{(l)})(x)$ with $x = \sum a_{\underline{k}} \underline{e}^{\underline{k}} \in \mathcal{U}(\mathfrak{g})$.

$$\begin{aligned}
\tilde{\beta}^\star(t^{(l)})(x) &= \beta(\textstyle\sum a_{\underline{k}} \underline{e}^{\underline{k}})(t^{(l)}) \\
&= \textstyle\sum a_{\underline{k}}\, \beta(\underline{e}^{\underline{k}})(t^{(l)}) \\
&= \sum_{|k|\geq 1} a_{\underline{k}}\, \beta(\underline{e}^{\underline{k}})(t^{(l)}) \qquad \text{(since } (\phi^{(k)})^0(t^{(l)}) = 0) \\
&= a_{\mathbf{e(1)}} \phi^{(1)}(t^{(l)}) + \ldots + a_{\mathbf{e(m)}} \phi^{(m)}(t^{(l)}) + \sum_{|k|\geq 2} a_{\underline{k}}\, \beta(\underline{e}^{\underline{k}})(t^{(l)}) \\
&= a_{\mathbf{e(l)}} + \sum_{|k|\geq 2} a_{\underline{k}}\, \beta(\underline{e}^{\underline{k}})(t^{(l)}) \qquad \text{(since } \phi^{(k)}(t^{(l)}) = 0, \text{ if } k \neq l).
\end{aligned}$$

This shows that $\tilde{\beta}^\star(t^{(l)}) \equiv dt^{(l)} \mod (\mathcal{F}^2\,\mathcal{U}^\star)$. If we look at the monomial $\underline{t}^{\underline{j}} \in F^i\,\tilde{\mathcal{O}}(G)$ we get that $\tilde{\beta}^\star(\underline{t}^{\underline{j}})$ is of the form

$$(dt^{(1)} + \mathcal{F}^2\,\mathcal{U}^\star)^{j_1} \bullet \cdots \bullet (dt^{(m)} + \mathcal{F}^2\,\mathcal{U}^\star)^{j_m},$$

and due to the explicit formula for $\bullet$ in Lemma 1.2.18 this means that

$$\begin{aligned}
\tilde{\beta}^\star(\underline{t}^{\underline{j}}) &\equiv j_1! d^{j_1} t^{(1)} \cdots j_m! d^{j_m} t^{(m)} \mod (\mathcal{F}^{i+1}\,\mathcal{U}^\star) \\
&\equiv \underline{j}! d^{\underline{j}} \underline{t} \mod (\mathcal{F}^{i+1}\,\mathcal{U}^\star).
\end{aligned}$$

Since $\tilde{\beta}^\star$ is R-linear and continuous it follows that $\tilde{\beta}^\star$ induces isomorphisms

$$F^i\,\tilde{\mathcal{O}}(G)\,/F^{i+1}\,\tilde{\mathcal{O}}(G) \to \mathcal{F}^i\,\mathcal{U}^\star\,/\mathcal{F}^{i+1}\,\mathcal{U}^\star$$

for all $i \in \mathbb{N}$ and hence, since $\tilde{\mathcal{O}}(G)$ and $\mathcal{U}^\star$ are both complete with respect to these filtrations, that $\tilde{\beta}^\star : \tilde{\mathcal{O}}(G) \to \mathcal{U}^\star$ is an isomorphism of complete Hopf algebras. $\square$

Chapter 3

Quasi-isomorphism of complexes

> Throughout this chapter R will be an integral domain of characteristic zero.

The main purpose of this chapter is to show that the isomorphism

$$\tilde{\beta}^\star : \tilde{\mathcal{O}}(G) \to \mathcal{U}^\star$$

defined in Proposition 2.3.5 extends to a quasi-isomorphism $\tilde{\phi}$ of the corresponding complexes. The underlying morphism $\beta^\star : \mathcal{O}(G) \to \mathcal{U}^\star$ was already established in a paper of Huber and Kings concerning a p-adic analogue of the Borel regulator and the Bloch-Kato exponential map, see [HK11]. They showed that one can directly define a map from locally analytic group cohomology to Lie algebra cohomology by differenting cochains, and that the resulting map is Lazard's comparison isomorphism ([HK11, Proposition 4.2.4]). In Section 3.2 we will give an explicit description of this quasi-isomorphism and we will see that this description coincides with the one of Huber and Kings, [HK11, Definition 1.4.1], and hence with Lazard's map.

3.1 General construction

We have proven in Proposition 2.3.4 that the modified ring of functions $\tilde{\mathcal{O}}(G)$ to an m-dimensional formal group law G, see Definition 2.3.1, has a complete Hopf algebra structure. This structure leads, according to Proposition 1.3.1, to a cobar complex. We will use in the case of the modified ring of functions $\tilde{\mathcal{O}}(G)$ the following notation.

Definition 3.1.1. Let G be a formal group law. Let $\tilde{\mathcal{O}}(G)$ be the associated modified ring of functions which inherits by Proposition 2.3.4 a complete Hopf algebra structure. We denote by $\tilde{\mathrm{K}}^n(G,R)$ the n-fold complete tensor product of $\tilde{\mathcal{O}}(G)$, i.e.

$$\tilde{\mathrm{K}}^n(G,R) = \tilde{\mathcal{O}}(G)^{\hat{\otimes} n}$$

and hence by $(\tilde{\mathrm{K}}^\bullet(G,R), \partial)$ the cobar complex given by Proposition 1.3.1. The corresponding cohomology group, denoted by $H^n(\tilde{G},R)$, is called *n-th modified group cohomology of G with coefficients in R.*

Remark. 3.1.2. Note that we can identify $\mathcal{O}(G)^{\hat{\otimes} n}$ with $\mathcal{O}(G^n)$ as well as $\tilde{\mathcal{O}}(G)^{\hat{\otimes} n}$ with $\tilde{\mathcal{O}}(G^n)$.

Theorem 3.1.3. *Let R be an integral domain of characteristic zero and let G be a formal group law over R. Let $(K^\bullet(G,R), \partial)$, $(\tilde{\mathrm{K}}^\bullet(G,R), \partial)$ and $(C^\bullet(G,R), \partial')$ be the complexes defined in 1.3.2, 3.1.1 and 1.3.6. Then $\tilde{\beta}^\star : \tilde{\mathcal{O}}(G) \to \mathcal{U}^\star$, defined in Proposition 2.3.5, extends to a quasi-isomorphism*

$$\tilde{\phi} : (\tilde{\mathrm{K}}^\bullet(G,R), \partial) \to (C^\bullet(\mathfrak{g},R), \partial')$$

given by the following composition of maps

$$\begin{aligned}(\tilde{\mathrm{K}}^\bullet(G,R), \partial) \xrightarrow{\tilde{\beta}^\star} (\mathrm{Hom}_R(U^\bullet,R), \partial_u) &\xrightarrow[1.3.5]{\iota} (\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^\bullet,R), \partial_{u_h}) \\ &\xrightarrow[1.3.8]{\nu} (\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V(\mathfrak{g})^\bullet,R), d) \xrightarrow[1.3.9]{\kappa} (C^\bullet(\mathfrak{g},R), \partial').\end{aligned}$$

In particular, the morphism $\beta^\star : \mathcal{O}(G) \to \mathcal{U}^\star$ extends to a morphism

$$\phi : (K^\bullet(G,R), \partial) \to (C^\bullet(\mathfrak{g},R), \partial').$$

Proof. The proof is essentially the conjunction of all our previous results. In particular, we will use the statements of Proposition 2.4.1 and 1.3.5 and of Proposition 1.3.8 and 1.3.9 concerning the Koszul complex.

Consider the following composition of maps of the theorem

$$\begin{aligned}(\tilde{\mathrm{K}}^\bullet(G,R), \partial) \xrightarrow{\tilde{\beta}^\star} (\mathrm{Hom}_R(U^\bullet,R), \partial_u) &\xrightarrow[1.3.5]{\iota} (\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^\bullet,R), \partial_{u_h}) \\ &\xrightarrow[1.3.8]{\nu} (\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V(\mathfrak{g})^\bullet,R), d) \xrightarrow[1.3.9]{\kappa} (C^\bullet(\mathfrak{g},R), \partial').\end{aligned} \tag{3.1}$$

We will recall from Propositions 1.3.5, 1.3.8 and 1.3.9 that the maps ι, ν and κ of (3.1) are (quasi-)isomorphisms, and we will show that the first map is induced by the map $\tilde{\beta}^\star$ of Proposition 2.3.5 and therefore is even an isomorphism of complexes.

As a first step, recall from Proposition 2.4.1 that there exists an isomorphism

$$\tilde{\beta}^\star : \tilde{\mathcal{O}}(G) \to \mathcal{U}^\star$$

of complete Hopf algebras. This isomorphism $\tilde{\beta}^\star$ extends naturally to a morphism of the complexes

$$(\tilde{\mathrm{K}}^\bullet(G, R), \partial) \xrightarrow{\tilde{\beta}^\star} (\mathrm{Hom}_R(U^\bullet, R), \partial_u).$$

To see this, note first that we can identify $(\mathcal{U}^\star)^{\hat{\otimes} n}$ with $\mathrm{Hom}_R(\mathcal{U}(\mathfrak{g})^{\otimes n}, R)$, compare Remark 1.3.4, and secondly that the differentials of the complexes $(\tilde{\mathrm{K}}^\bullet(G, R), \partial)$ and $(\mathrm{Hom}_R(U^\bullet, R), \partial_u)$ are given by the comultiplication of $\tilde{\mathcal{O}}(G)$ and $\mathcal{U}^\star$. The latter means that $\tilde{\beta}^\star$ commutes with these differentials, since $\beta^\star$ is in particular a coalgebra morphism.

Secondly recall Proposition 1.3.5, which stated that there exists an isomorphism between the inhomogeneous and homogeneous complex of $\mathcal{U}(\mathfrak{g})$, hence we get an isomorphism of complexes:

$$(\mathrm{Hom}_R(U^\bullet, R), \partial_u) \xrightarrow{\iota} (\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^\bullet, R), \partial_{u_h}).$$

Finally we can conclude the proof of the theorem by recalling both Propositions 1.3.8 and 1.3.9 concerning the Koszul complex, whose combined statement is that the map

$$(\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(U_h^\bullet, R), \partial_{u_h}) \to (C^\bullet(\mathfrak{g}, R), \partial')$$

is a quasi-isomorphism. □

Corollary 3.1.4. *Let R be an integral domain of characteristic zero and let G be a formal group law over R. Let $\mathfrak{g}$ be the associated Lie algebra to G. Then there exists an isomorphism*

$$\tilde{\Phi} : H^n(\tilde{G}, R) \to H^n(\mathfrak{g}, R)$$

between the modified group cohomolgy of G with coefficients in R and Lie algebra cohomology of $\mathfrak{g}$ with coefficients in R given by $f \mapsto \tilde{\phi}(f)$, with $\tilde{\phi}(f)$ defined in Theorem 3.1.3.

Proof. This is a direct consequence of Theorem 3.1.3. □

3.2 Explicit description

This section will provide an explicit description of the map

$$\phi^n : K^n(G,R) \to C^n(\mathfrak{g},R)$$

of Theorem 3.1.3. We will see that the explicit description coincides with the one of Huber and Kings in [HK11, Definition 1.4.1] and hence with that of Lazard.

Proposition 3.2.1. *Let R be an integral domain of characteristic zero and let G be a formal group law over R. Let $f \in K^n(G,R)$ be given by $f = f_1 \otimes \cdots \otimes f_n$ with $f_i = \sum_{\underline{j}} b^i_{\underline{j}} \underline{t}_i^{\underline{j}}$. Then the map $\phi^n : K^n(G,R) \to C^n(\mathfrak{g},R)$ of Theorem 3.1.3 can be described by the continuous extension of*

$$f_1 \otimes \cdots \otimes f_n \mapsto df_1 \wedge \cdots \wedge df_n$$

for $n \geq 1$ and by the identity for $n = 0$.

Proof. The map ϕ^n is by Theorem 3.1.3 given by the following composition of maps

$$\begin{aligned} K^n(G,R) &\xrightarrow{\beta^{\star n}} \mathrm{Hom}_R(\mathcal{U}(\mathfrak{g})^{\otimes n}, R) \xrightarrow{\iota^n} \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g})^{\otimes n+1}, R) \\ &\xrightarrow{\nu^n} \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V_n(\mathfrak{g}), R) \xrightarrow{\kappa^n} \mathrm{Hom}_R(\bigwedge^n \mathfrak{g}, R). \end{aligned} \tag{3.2}$$

We will provide explicit descriptions of these maps. Consider first the last map of the above sequence

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V_n(\mathfrak{g}), R) \xrightarrow{\kappa^n} \mathrm{Hom}_R(\bigwedge^n \mathfrak{g}, R),$$

which is (see Proposition 1.3.9) given by

$$f \mapsto [(e_{i_1} \wedge \cdots \wedge e_{i_n}) \mapsto f(1 \otimes e_{i_1} \wedge \ldots \wedge e_{i_n})].$$

The penultimate map

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g})^{\otimes n+1}, R) \xrightarrow{\nu^n} \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V_n(\mathfrak{g}), R)$$

is according to Proposition 1.3.8 induced by the anti-symmetrisation map, and therefore given by

$$f \mapsto [(u \otimes e_{i_1} \wedge \ldots \wedge e_{i_n}) \mapsto \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha) f(u \otimes e'_{i_{\alpha(1)}} \otimes \cdots \otimes e'_{i_{\alpha(n)}})].$$

Note that the substitution of the elements e_{i_j} by the elements e'_{i_j} defined by the partial derivative $\frac{\partial}{\partial t_j^{(i_{\alpha(j)})}}$ evaluated at zero is just a formal consequence of the fact that we introduced the coordinates $\underline{t}_1, \ldots, \underline{t}_n$ on $\mathcal{O}(G)^{\hat{\otimes} n}$. The explicit description of the map $\operatorname{Hom}(\mathcal{U}(\mathfrak{g})^{\otimes n}, R) \xrightarrow{\iota^n} \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g})^{\otimes n+1}, R)$ was given in Section 1.3 of Chapter 1 by

$$\begin{aligned} \iota^n : \operatorname{Hom}(\mathcal{U}(\mathfrak{g})^{\otimes n}, R) &\rightarrow \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g})^{\otimes n+1}, R) \\ f &\mapsto [(u_0, \ldots, u_n) \mapsto f(u_1, \ldots u_n)] \\ [(u_1, \ldots, u_n) \mapsto f(1, u_1, \ldots u_n)] &\leftarrow f. \end{aligned}$$

We will now combine these maps to get:

$$\begin{aligned} \phi^n(f)(e_{i_1} \wedge \ldots \wedge e_{i_n}) &= \kappa^n(\nu^n(\iota^n(\beta^{\star n}(f))))(e_{i_1} \wedge \ldots \wedge e_{i_n}) \\ &= \nu^n(\iota^n(\beta^{\star n}(f)))(1 \otimes e_{i_1} \wedge \ldots \wedge e_{i_n}) \\ &= \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha) \iota^n(\beta^{\star n}(f))(1 \otimes e'_{i_{\alpha(1)}} \otimes \cdots \otimes e'_{i_{\alpha(n)}}) \\ &= \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha) \beta^{\star n}(f)(e'_{i_{\alpha(1)}} \otimes \cdots \otimes e'_{i_{\alpha(n)}}) \\ &= \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha)\, \beta^\star(f_1)(e'_{i_{\alpha(1)}}) \cdots \beta^\star(f_n)(e'_{i_{\alpha(n)}}) \\ &\overset{2.1.2}{=} \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha)(e'_{i_{\alpha(1)}})(f_1) \cdots (e'_{i_{\alpha(n)}})(f_n) \\ &= df_1 \wedge \cdots \wedge df_n(e_{i_1} \wedge \ldots \wedge e_{i_n}). \end{aligned} \tag{3.3}$$

Since the map $\phi^n : K^n(G, R) \to \operatorname{Hom}_R(\bigwedge^n \mathfrak{g}, R)$ is continuous and R-linear the explicit description of ϕ^n can be obtained by the continuous R-linear extension of the above description (3.3).

In the case of $n = 0$ the sequence (3.2) reduces to

$$R \xrightarrow{\beta^{\star 0}} R \xrightarrow{\iota^0} \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), R) \xrightarrow{\nu^0} \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), R) \xrightarrow{\kappa^0} R$$

and the maps are defined as

$$1 \mapsto 1 \mapsto (u \mapsto 1) \mapsto (u \mapsto 1) \mapsto 1$$

with the usual identification of $\mathrm{Hom}_R(R, R)$ and R. $\square$

Chapter 4

Locally analytic group cohomology

This chapter will provide a comparison isomorphism of locally analytic group cohomology with Lie algebra cohomology for K-Lie groups, where K is a finite extension of $\mathbb{Q}_p$. The main idea of the proof is its reduction to standard groups associated to formal group laws. Before stating the Comparison Theorem for standard groups 4.1.7 we will give some definitions and fix notation in Section 4.1. Section 4.2 covers the so called limit morphism, a preliminary step in the proof of the Comparison Theorem. The final steps to the proof of the Comparison Theorem 4.1.7 will be given in Section 4.3.

Throughout this chapter let K be a finite extension of $\mathbb{Q}_p$.

4.1 K-Lie groups and standard groups

Before stating the Comparison Theorem for standard groups, we give some definitions, especially the one of a standard group, see Definition 4.1.2, and we state some elementary facts about these groups. We refer to [Ser06], [Bou98b] or [DdSMS99] for the background on K-Lie groups.

Let $|\ | : K \to \mathbb{R}_+$ be the non-archimedean absolute value on K which extends the p-adic absolute value on $\mathbb{Q}_p$ and let $v_p(x)$ denote the corresponding valuation on K, normalized by $v_p(p) = 1$, which satisfies $|x| = p^{-v_p(x)}$.

We denote by R the *valuation ring*

$$R = \{x \in K : \ |x| \leq 1\} = \{x \in K : \ v_p(x) \geq 0\}$$

and by $\mathfrak{m}$ the *maximal ideal*

$$\mathfrak{m} = \{x \in K : \ |x| < 1\} = \{x \in K : \ v_p(x) > 0\}$$

of R.

Definition 4.1.1. (i) Let $U \subset K^n$ be open and let $f : U \to K$ be a function. Then f is called *locally analytic in U* if for each $x \in U$ there is a ball $B_r(x) := \{y \in U \mid |y - x| < r\} \subset U$ and a formal power series F such that F converges in $B_r(x)$ and for $h \in B_r(x)$,

$$f(h) = F(h - x),$$

compare [Laz65, Chap.III, 1.3.2].

(ii) Let $U \subset K^n$ be open and let $f = (f_1, \dots, f_n) : U \to K^n$. Then f is called *locally analytic in U* if f_i is locally analytic for $1 \leq i \leq n$.

(iii) Let M be a topological space. A *chart for M* is a triple (U, φ, K^n) consisting of an open subset $U \subset M$ and a map $\varphi : U \to K^n$ such that $\varphi(U)$ is open in K^n and $\varphi : U \xrightarrow{\simeq} \varphi(U)$ is a homeomorphism.

(iv) A *locally analytic manifold over K* is a topological space M equipped with a maximal atlas, where the atlas is a set of charts for M any two of which are compatible, i.e. has locally analytic transition maps, and which cover M.

(v) A *K-Lie group* or a *K-analytic group* $\mathcal{G}$ is a locally analytic manifold over K which also carries the structure of a group such that

(1) the function $(x, y) \mapsto xy$ of $G \times G$ into G is locally analytic and

(2) the function $x \mapsto x^{-1}$ of G into G is locally analytic.

Definition and Proposition 4.1.2. Let G be an m-dimensional formal group law over the valuation ring R of K as in Definition 1.1.1. For $h \in \mathbb{R}$ we set

$$\mathcal{G}(h) := \{\underline{z} \in R^m \mid |\underline{z}| < p^{-h}\}.$$

We define a multiplication on $\mathcal{G}(h)$ by:

$$\underline{z_1} \cdot \underline{z_2} = G(\underline{z_1}, \underline{z_2}). \tag{4.1}$$

Then $\mathcal{G}(h)$ is a K-Lie group.

Proof. See [Ser06, Chap. IV.8]. □

Definition 4.1.3. A K-Lie group constructed as in Proposition 4.1.2 will be called an *m-standard group of level h* if $h > 0$ and just an *m-standard group* if $h = 0$.

Definition 4.1.4. Let $\mathcal{G}$ be a K-Lie group. We denote by $L(\mathcal{G}) = T_e(\mathcal{G})$ the canonical Lie algebra of $\mathcal{G}$, see [Tu11, Chap. 4, §16.3].

Proposition 4.1.5. *Let G be a formal group law over R and let $\mathcal{G}(h)$ be the m-standard group of level h to G. Let $L(\mathcal{G}(h))$, respectively $\mathfrak{g}$ be the corresponding Lie algebras. Then*

$$L(\mathcal{G}(h)) \cong \mathfrak{g} \otimes_R K.$$

Proof. (Compare [Sch11, Prop. 17.3].) One can choose the local coordinates $(t^{(1)}, \dots, t^{(m)})$ in a neighborhood of the identity e (with coordinates $(0, \dots, 0)$), such that we get a natural basis $\frac{\partial}{\partial t^{(1)}}|_{(0)}, \dots, \frac{\partial}{\partial t^{(m)}}|_{(0)}$ of the tangent space $T_e(\mathcal{G}(h))$. Let the formal group law G be given by

$$G^{(j)}(\mathbf{X}, \mathbf{Y}) = X_j + Y_j + \sum_{l,k=1}^{m} \gamma_{lk}^{j} X_l Y_k + O(d \geq 3)$$

for all $j \in \{1, \dots m\}$. Then the structure coefficients, i.e. those elements $c_{ij}^k \in R$ such that

$$[\frac{\partial}{\partial t^{(i)}}|_{(0)}, \frac{\partial}{\partial t^{(j)}}|_{(0)}] = \sum c_{ij}^k \frac{\partial}{\partial t^{(k)}}|_{(0)},$$

of $L(\mathcal{G}(h))$ are given by $\sum_{j=1}^{m} (\gamma_{lk}^{j} - \gamma_{kl}^{j})$ since multiplication on $\mathcal{G}(h)$ is defined by the formal group law, see (4.1). Hence the definition of the Lie bracket in $L(\mathcal{G}(h))$ coincides with the definition of the Lie bracket in $\mathfrak{g}$, see (1.3). $\square$

Definition 4.1.6. Let $\mathcal{G}$ be a K-Lie group. We denote by $\mathcal{O}^{la}(\mathcal{G}, K)$ locally analytic functions on $\mathcal{G}$, i.e. those that can be locally written as a converging power series with coefficients in K, see Definition 4.1.1. The cobar complex $\mathcal{O}^{la}(\mathcal{G}^n, K)_{n \geq 0}$, where we identified $\mathcal{O}^{la}(\mathcal{G}^n, K)$ with $\mathcal{O}^{la}(\mathcal{G}, K)^{\hat{\otimes} n}$, with the usual differential as in Definition 1.3.2 leads to *locally analytic group cohomology* whose n-th cohomology group is denoted by $H_{la}^n(\mathcal{G}, K)$.

Let G be a formal group law over R. Then one can consider the same formal group law over K, denote by G_K, with Lie algebra $\mathfrak{g}_K$. Let $\mathcal{G}(h)$ be the m-standard group of level h associated to G. Then we obtain by assigning to each locally analytic function $f \in \mathcal{O}^{la}(\mathcal{G}(h), K)$ its local power series representation around e a morphism of complete Hopf algebras

$$\mathcal{O}^{la}(\mathcal{G}(h), K) \to \mathcal{O}(G_K)$$

and hence a map

$$\Phi_e : H^n_{la}(\mathcal{G}(h), K) \to H^n(G_K, K).$$

Recall that in the case of a formal group law over the field K the modified ring of functions $\tilde{\mathcal{O}}(G_K)$ coincides with the ring of functions $\mathcal{O}(G_K)$. According to Corollary 3.1.4 and since

$$\mathrm{Hom}_R(\bigwedge^n \mathfrak{g}, R) \otimes_R K \cong \mathrm{Hom}_K(\bigwedge^n \mathfrak{g}_K, K)$$

we get an isomorphism

$$\Phi_K : H^n(G_K, K) \to H^n(\mathfrak{g}, K).$$

This isomorphism is, as we have seen in Proposition 3.2.1, given by the continuous extension of differentiating cochains.

Theorem 4.1.7 (Comparison Theorem for standard groups)**.** *Let G be a formal group law over R and let $\mathcal{G}(h)$ be the m-standard group of level h to G with Lie algebra $\mathfrak{g} \otimes_R K$. Then the map*

$$\Phi_s : H^n_{la}(\mathcal{G}(h), K) \to H^n(\mathfrak{g}, K)$$

given by the continuous extension of

$$f_1 \otimes \cdots \otimes f_n \mapsto df_1 \wedge \cdots \wedge df_n,$$

for $n \geq 1$ and by the identity for $n = 0$ is an isomorphism for all $h > h_0 = \frac{1}{p-1}$.

Note that since the elements of the form $f_1 \otimes \cdots \otimes f_n$ form a basis of the dense subset $\mathcal{O}(G_K)^{\otimes n} \subset \mathcal{O}(G_K)^{\hat{\otimes} n}$ and since Φ_s is given by the composition $\Phi_K \circ \Phi_e$, we use the suggestive notation $f_1 \otimes \cdots \otimes f_n$ for an element of $\mathcal{O}^{la}(\mathcal{G}(h), K)$.

Remark. 4.1.8. Huber and Kings showed in [HK11] that one can directly define a map from locally analytic group cohomology to Lie algebra cohomology by differenting cochains, as in Theorem 4.1.7, and that in the case of smooth algebraic group schemes H over $\mathbb{Z}_p$ with formal group $\mathcal{H} \subset H(\mathbb{Z}_p)$ the resulting map

$$\Phi : H^n_{la}(\mathcal{H}, \mathbb{Q}_p) \to H^n(\mathfrak{h}, \mathbb{Q}_p)$$

coincides with Lazard's comparison isomorphism ([HK11, Theorem 4.7.1]). In their joint work with N. Naumann in [HKN11] they extended the comparison isomorphism for K-Lie groups attached to smooth group schemes with connected generic fibre over the integers of K ([HKN11, Theorem 4.3.1]).

Remark. 4.1.9. Let us sketch the argument of the proof. In the first step, we are going to show that if we restrict to $\mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}}$, the ring of germs of locally analytic functions on $\mathcal{G}(0)$ in e, we can show that the limit morphism

$$\Phi_\infty : H^n(\mathcal{O}^{la}(\mathcal{G}(0)^\bullet)_e, K) \to H^n(\mathfrak{g}, K)$$

is an isomorphism, see Lemma 4.2.9. Then injectivity of Φ_s follows from a spectral sequence argument, see Corollary 4.3.4 and 4.3.5. The proof of this part will be analogous to the proof of Theorem 4.3.1 in [HKN11], however independent of the work of Lazard [Laz65]. For surjectivity we will use the statement of Theorem 3.1.3.

The Main Theorem of the introduction - which stated that if K is a finite extension of $\mathbb{Q}_p$ and if $\mathcal{G}$ is a K-Lie group, then there exists an open subgroup $\mathcal{U}$ of $\mathcal{G}$ such that the Lazard morphism

$$\Phi_L : H^n_{la}(\mathcal{U}, K) \to H^n(\mathrm{Lie}(\mathcal{U}), K)$$

induced by differentiating cochains is an isomorphism - can now be deduced from the following Lemma 4.1.10.

Lemma 4.1.10. *Any K-Lie group contains an open subgroup which is an m-standard group.*

Proof. See [Ser06, Chap. IV.8]. □

4.2 Limit morphism

This section contains the first step in the proof of Theorem 4.1.7. We will prove that the limit morphism $\Phi_\infty : H^n(\mathcal{O}^{la}(\mathcal{G}(0)^\bullet)_e, K) \to H^n(\mathfrak{g}, K)$, mentioned in Remark 4.1.9, is an isomorphism. To do this we need some further definitions and a comparison of germs.

Proposition 4.2.1. *Let G be a formal group law over R and let $\mathcal{G}(h)$ be the m-standard group of level h to G. Then there exists an isomorphic m-dimensional formal group law G_h over R such that the associated m-standard group $\mathcal{G}_h(0)$ is equal to $\mathcal{G}(h)$.*

Proof. Let the formal group law G be given by

$$G^{(j)}(\mathbf{X}, \mathbf{Y}) = X_j + Y_j + \sum_{l,k=1}^{m} \gamma_{lk}^{j} X_l Y_k + O(d \geq 3)$$

for all $j \in \{1, \dots, m\}$ with $\mathbf{X} = (X_1, \dots, X_m)$, $\mathbf{Y} = (Y_1, \dots, Y_m)$ and with $\mathcal{O}(G) = R[[\underline{t}]]$. Consider the m-dimensional formal group law defined by

$$G_h^{(j)}(\mathbf{X}, \mathbf{Y}) = p^{-h} G^{(j)}(p^h \mathbf{X}, p^h \mathbf{Y}) \text{ for all } j \in \{1, \dots, m\}$$

with $p^h\mathbf{X} = (p^h X_1, \dots, p^h X_m)$ and $p^h\mathbf{Y} = (p^h Y_1, \dots, p^h Y_m)$. Then the ring of functions to G_h is given by $R[[p^{-h}\underline{t}]]$ where $p^{-h}\underline{t}$ is the short notation for $p^{-h}t^{(1)}, \dots, p^{-h}t^{(m)}$. Now the m-standard group

$$\mathcal{G}_{\mathrm{h}}(0) = \{\underline{z} \in R^m \mid |p^{-h}\underline{z}| < 1\}$$

can be rewritten as following

$$\mathcal{G}_{\mathrm{h}}(0) = \{\underline{z} \in R^m \mid p^h|\underline{z}| < 1\} = \{\underline{z} \in R^m \mid |\underline{z}| < p^{-h}\} = \mathcal{G}(h).$$

The homomorphism from G to G_h is given by the m-tuple

$$\alpha(\mathbf{X}) = (\alpha_1(\mathbf{X}), \dots, \alpha_m(\mathbf{X})) \text{ with } \alpha_j(\mathbf{X}) = p^{-h} X_j$$

and the homomorphism from G_h to G is given by the m-tuple

$$\beta(\mathbf{X}) = (\beta_1(\mathbf{X}), \dots, \beta_m(\mathbf{X})) \text{ with } \beta_j(\mathbf{X}) = p^h X_j,$$

compare Definition 1.1.4. Since they satisfy the condition $\alpha(\beta(\mathbf{X})) = \beta(\alpha(\mathbf{X}))$ the formal group laws G and G_h are isomorphic. $\square$

Example 4.2.2. Let $G_m(X,Y) = X + Y + XY$ be the multiplicative formal group law. Then the formal group law G_h such that $\mathcal{G}_{\mathrm{h}}(0) = \mathcal{G}_m(h)$ is given by

$$G_h(X,Y) = p^{-h}(p^h X + p^h Y + p^{2h} XY) = X + Y + p^h XY.$$

Lemma 4.2.3. *Let G be an m-dimensional formal group law over R. Then the associated m-standard groups $\mathcal{G}(h)$ of level h, $h \in \mathbb{N} \setminus \{0\}$, are open and normal subgroups of $\mathcal{G}(0)$ of finite index and they form a neighbourhood basis of e in $\mathcal{G}(0)$.*

Proof. The m-standard groups $\mathcal{G}(h)$ of level h are, by their definition in 4.1.2, obviously open and closed subgroups of $\mathcal{G}(0)$ and form a neighbourhood basis of e in $\mathcal{G}(0)$. Since R^m is compact, see [Gou93, Prop. 5.4.5vi], $\mathcal{G}(0)$ is compact and the open and closed subgroups $\mathcal{G}(h)$ of $\mathcal{G}(0)$ are of finite index. For the property that these subgroups are normal we have to show that

$$G(\mathbf{x}, G(\mathbf{y}, s(\mathbf{x}))) \equiv 0 \pmod{p^h},$$

for $\mathbf{x} \in \mathcal{G}(0)$ and $\mathbf{y} \in \mathcal{G}(h)$, i.e. $\mathbf{y} \equiv 0 \pmod{p^h}$, where $s(\mathbf{x})$ was defined by the condition that $G(\mathbf{x}, s(\mathbf{x})) = 0$, see Proposition 1.1.3. However since all terms containing $\mathbf{y}$ are reduced to 0 mod p^h we have that

$$G(\mathbf{x}, G(\mathbf{y}, s(\mathbf{x}))) \equiv G(\mathbf{x}, s(\mathbf{x})) \pmod{p^h} \equiv 0 \pmod{p^h}.$$

□

Definition 4.2.4. Let $\mathcal{G}$ be a K-Lie group. We denote by $\mathcal{O}^{la}(\mathcal{G})_e$, the *ring of germs of locally analytic functions on $\mathcal{G}$ in e.*

By Lemma 4.2.3, the ring of germs of locally analytic functions on $\mathcal{G}(0)$ in e is given by

$$\mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}} = \varinjlim_{h} \mathcal{O}^{la}(\mathcal{G}(h), K).$$

Definition 4.2.5. The noetherian R-algebra

$$R\{\underline{t}\} := \{f(\underline{t}) = \sum_{\underline{j}} b_{\underline{j}} \underline{t}^{\underline{j}} \mid b_{\underline{j}} \in R, |b_{\underline{j}}| \to 0 \text{ as } |\underline{j}| \to \infty\}$$

is called the *algebra of strictly convergent power series over R.*

Remark. 4.2.6. Recall from non-archimedean analysis that every f in $R[[\underline{t}]]$ converges on the open polydisc $\{z \in K^m \mid |\underline{z}| < 1\}$ and every f in $\mathfrak{m}[[\underline{t}]]$ converges on the closed polydisc $\{z \in K^m \mid |\underline{z}| \leq 1\}$. The algebra $R\{\underline{t}\}$ is the sub-algebra of $R[[\underline{t}]]$ consisting of those power series which converge on R^m, since an infinite sum converges in a non-archimedean field if and only if its terms tend to zero.

Definition 4.2.7. The K-algebra

$$K\langle\underline{t}\rangle := \{f(\underline{t}) = \sum_{\underline{j}} b_{\underline{j}}\underline{t}^{\underline{j}} \mid b_{\underline{j}} \in K, |b_{\underline{j}}| \to 0 \text{ as } |\underline{j}| \to \infty\}$$

is the algebra of power series over K which converge on R^m in K^m and is called *Tate algebra*. The elements of $K\langle\underline{t}\rangle$ are called *rigid analytic functions*.

Note that the convergence condition means that, for any $n \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that for $|\underline{j}| > j_0$, the coefficient $b_{\underline{j}}$ belongs to $\pi^n R$, where π is the uniformizing parameter, i.e. $(\pi) = \mathfrak{m}$. We have that $K\langle\underline{t}\rangle = R\{\underline{t}\} \otimes_R K$, see [Nic08].

Since germs of locally analytic functions are none other than germs of rigid analytic functions, we can identify $\mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}}$ with the limit of Tate algebras

$$\mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}} \cong \varinjlim_h K\langle p^{-h}\underline{t}\rangle.$$

Definition 4.2.8. The cobar complex $(\mathcal{O}^{la}(\mathcal{G}(0)^\bullet)_e, \partial)$ with

$$\mathcal{O}^{la}(\mathcal{G}(0)^n)_e := \varinjlim_h K\langle p^{-h}\underline{t}_{1\dots,n}\rangle$$

and with the usual differential as in Definition 1.3.2 leads to cohomology groups $H^n(\mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}}, K)$.

Lemma 4.2.9. *The limit morphism*

$$\Phi_\infty : H^n(\mathcal{O}^{la}(\mathcal{G}(0)^\bullet)_e, K) \to H^n(\mathfrak{g}, K)$$

is an isomorphism.

Remark. 4.2.10. A proof of this lemma can also be found in [HKN11, Lemma 4.3.3], however our proof will be independent of the work of Lazard [Laz65] which is the utmost concern of this thesis.

Definition 4.2.11. Let $\mathcal{G}(h)$ be an m-standard group of level h. We denote by $\mathcal{O}_c(\mathcal{G}(h))$ the *ring of convergent functions on* $\mathcal{G}(h)$, i.e. those formal power series in $R[[p^{-h}\underline{t}]]$ which are convergent on the closed polydisc $\{\underline{z} \in K^m \mid |\underline{z}| \leq p^{-h}\}$.

Remark. 4.2.12. Since $R\{\underline{t}\} = \{f \in R[[\underline{t}]] \mid f \text{ converges on } R^m\}$, due to Remark 4.2.6, we get for the ring of convergent functions on the m-standard group $\mathcal{G}(h)$ of level h that $\mathcal{O}_c(\mathcal{G}(h)) = R\{p^{-h}\underline{t}\}$.

The following Lemma 4.2.13 will not only be of interest for the proof of Lemma 4.2.9 but also for the proof of the Main Theorem. We will see that functions of this modified ring of functions $\tilde{\mathcal{O}}(G)$ still converge.

Lemma 4.2.13 (Lemma of Convergence)**.** *Let G be a formal group law over R, let $\mathcal{G}(h)$ be the m-standard group of level h to G and let $\mathcal{O}_c(\mathcal{G}(h))$ be the ring of convergent functions on $\mathcal{G}(h)$. Let G_h be the associated m-dimensional formal group law (see Proposition 4.2.1) and let*

$$\tilde{\mathcal{O}}(G_h) = \left\{ \sum_{\underline{j}} b_{\underline{j}} (p^{-h}\underline{t})^{\underline{j}} \in K[[p^{-h}\underline{t}]] \mid \underline{j}! b_{\underline{j}} \in R \right\},$$

see Definition 2.3.1, be its modified ring of functions. Then

$$\tilde{\mathcal{O}}(G_h) \subset \mathcal{O}_c(\mathcal{G}(k))$$

for $k > k_0 = h + \frac{1}{p-1}$.

Proof. Let $f \in \tilde{\mathcal{O}}(G_h)$. Then f can be written as

$$f = \sum_{\underline{j}} b_{\underline{j}} (p^{-h}\underline{t})^{\underline{j}},$$

with $\underline{j}! b_{\underline{j}} \in R$. We claim that $f \in \mathcal{O}_c(\mathcal{G}(k))$ for $k > k_0 = h + \frac{1}{p-1}$. To see this, let us rewrite f in the following way:

$$f = \sum_{\underline{j}} b_{\underline{j}} p^{-h|\underline{j}|} \underline{t}^{\underline{j}} = \sum_{\underline{j}} b_{\underline{j}} p^{(k-h)|\underline{j}|} (p^{-k}\underline{t})^{\underline{j}}.$$

We prove now that $b_{\underline{j}} p^{(k-h)|\underline{j}|} \in \mathfrak{m}$. Since $\underline{j}! b_{\underline{j}} \in R$ we know that

$$|b_{\underline{j}}| < p^{\frac{|\underline{j}|}{p-1}},$$

where we use that for a prime p we have $v_p(n!) < \frac{n}{p-1}$, see [Gou93, Lemma 4.3.3]. Thus we can conclude that

$$\begin{aligned} |b_{\underline{j}} p^{(k-h)|\underline{j}|}| &< p^{\frac{|\underline{j}|}{p-1}} p^{-(k-h)|\underline{j}|} \\ &= p^{\frac{|\underline{j}|}{p-1}} p^{-k|\underline{j}|} p^{h|\underline{j}|} \\ &< p^{\frac{|\underline{j}|}{p-1}} p^{-(h+\frac{1}{p-1})|\underline{j}|} p^{h|\underline{j}|} = 1. \end{aligned}$$

Using the observation of Remark 4.2.6 we know that f converges on the closed polydisc $\{z \in K^m \mid |\underline{z}| \leq p^{-k}\}$, i.e. $f \in \mathcal{O}_c(\mathcal{G}(k))$. $\square$

Corollary 4.2.14. *The map*

$$\mathfrak{o} : \varinjlim_h \mathcal{O}_c(\mathcal{G}(h)) \to \varinjlim_h \tilde{\mathcal{O}}(G_h)$$

is an isomorphism.

Proof. The map $\mathfrak{o}$ is injective since $\mathcal{O}_c(\mathcal{G}(h)) \subset \tilde{\mathcal{O}}(G_h)$ and since the direct limit is exact. For surjectivity let f be an element of $\varinjlim_h \tilde{\mathcal{O}}(G_h)$. Then there exists h such that $f \in \tilde{\mathcal{O}}(G_h)$. However, after Lemma 4.2.13 $f \in \mathcal{O}_c(\mathcal{G}(k))$ for $k > k_0 = h + \frac{1}{p-1}$. $\square$

Proof of Lemma 4.2.9. The proof is essentially the conjunction of Proposition 2.4.1 with all preceding results in this section. We already know that

$$\begin{aligned} \mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}} &= \varinjlim_h \mathcal{O}^{la}(\mathcal{G}(h), K) \\ &= \varinjlim_h K\langle p^{-h}\underline{t}\rangle \\ &\overset{4.2.12}{=} \varinjlim_h \mathcal{O}_c(\mathcal{G}(h)) \otimes_R K \\ &\overset{4.2.14}{\cong} \varinjlim_h \tilde{\mathcal{O}}(G_h) \otimes_R K. \end{aligned}$$

Let $\mathfrak{g}_h$ be the associated Lie algebra to G_h. Then we know from Proposition 2.4.1 that

$$\varinjlim_h \tilde{\mathcal{O}}(G_h) \otimes_R K \cong \varinjlim_h \mathcal{U}^\star(\mathfrak{g}_h, R) \otimes_R K.$$

However, $\varinjlim_h \mathcal{U}^\star(\mathfrak{g}_h, R) \cong \mathcal{U}^\star(\mathfrak{g}, R) \otimes K$ and by Propositions 1.3.5, 1.3.8 and 1.3.9 of Chapter 1 we obtain

$$H^n(\mathcal{O}^{la}(\mathcal{G}(0)^\bullet)_e, K) \cong H^n(\mathfrak{g}, K).$$

□

4.3 Proof of the Comparison Theorem for standard groups

We mentioned in Remark 4.1.9 that injectivity of the map

$$\Phi_s : H^n_{la}(\mathcal{G}(h), K) \to H^n(\mathfrak{g}, K)$$

follows from a spectral sequence argument as in the proof of Theorem 4.3.1 in [HKN11]. Hence we will first prove the extistence of the required spectral sequence.

Definition 4.3.1. Let $\mathcal{G}$ be a K-Lie group and $\mathcal{H}$ a closed subgroup of $\mathcal{G}$. We define $I_K := \mathrm{Ind}^{la}_{\mathcal{H}\to\mathcal{G}}(K)$ to be the space of locally analytic maps $f : \mathcal{G} \to K$ such that f is $\mathcal{H}$-equivariant.

Lemma 4.3.2 (Shapiro's Lemma)**.** *Let $\mathcal{G}$ be a K-Lie group and $\mathcal{H}$ a closed subgroup of $\mathcal{G}$. Then*

$$H^\bullet_{la}(\mathcal{G}, I_K) = H^\bullet_{la}(\mathcal{H}, K).$$

Proof. See [CW74, Prop. 3 (Shapiro's Lemma), Remark (2) and (3)]. □

The proof of the following theorem about the spectral sequence for locally analytic group cohomology can now, after we have seen that Shapiro's Lemma holds in the case of locally analytic group cohomology, be adopted from the proof for discrete groups, see e.g. [NSW00, Chap. II.1].

Theorem 4.3.3. *Let $\mathcal{G}$ be a K-Lie group and $\mathcal{H}$ be a closed normal subgroup of $\mathcal{G}$. Then there is a cohomological spectral sequence*

$$E_2^{pq} = H^p_{la}(\mathcal{G} / \mathcal{H}, H^q_{la}(\mathcal{H}, K)) \Rightarrow H^{p+q}_{la}(\mathcal{G}, K).$$

Corollary 4.3.4 (Injectivity). *Let G be a formal group law over R and let $\mathcal{G}(0)$ be the m-standard group to G with Lie algebra $\mathfrak{g} \otimes_R K$. Then the map*

$$\Phi_s : H^n_{la}(\mathcal{G}(0), K) \to H^n(\mathfrak{g}, K)$$

of Theorem 4.1.7 is injective.

Proof. (Compare [HKN11, Cor. 4.3.4]) Since all subgroups $\mathcal{G}(h)$ of $\mathcal{G}(0)$ are open, normal and of finite index (see Lemma 4.2.3), the spectral sequence of Theorem 4.3.3 degenerates to

$$H^n_{la}(\mathcal{G}(0), K) \cong H^n_{la}(\mathcal{G}(h), K)^{\mathcal{G}(0)/\mathcal{G}(h)}.$$

Hence the restriction maps

$$H^n_{la}(\mathcal{G}(0), K) \to H^n_{la}(\mathcal{G}(h), K)$$

are injective. As the system of open normal subgroups is filtered, this also implies that

$$H^n_{la}(\mathcal{G}(0), K) \to \varinjlim_h H^n_{la}(\mathcal{G}(h), K)$$

is injective. We can therefore conclude injectivity of the map Φ_s from the injectivity of Φ_∞, since the cohomology functor commutes with the direct limit, i.e. in our case $\varinjlim_h H^n_{la}(\mathcal{G}(h)), K) = H^n(\mathcal{O}^{\mathrm{la}}(\mathcal{G}(0))_{\mathrm{e}}, K)$. □

Note that the proof of injectivity uses that K is locally compact, meaning that the proof can not be carried over to the case of the completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

Corollary 4.3.5. *Let G be a formal group law over R and let $\mathcal{G}(h)$ be the m-standard group of level h associated to G with Lie algebra $\mathfrak{g}$. Then the map*

$$\Phi_s : H^n_{la}(\mathcal{G}(h), K) \to H^n(\mathfrak{g}, K)$$

of Theorem 4.1.7 is injective for all $h \geq 0$.

Proof. Proposition 4.2.1 tells us that there exists an isomorphic formal group law G_h to G such that $\mathcal{G}(h) = \mathcal{G}_h(0)$. By Corollary 4.3.4 we know that

$$H^n_{la}(\mathcal{G}(h), K) = H^n_{la}(\mathcal{G}_h(0), K) \to H^n(\mathfrak{g}_h, K)$$

is injective, where $\mathfrak{g}_h$ is the Lie algebra associated to G_h. However,

$$\mathrm{Hom}_R(\bigwedge^n \mathfrak{g}_h, R) \otimes_R K \cong \mathrm{Hom}_R(\bigwedge^n \mathfrak{g}, R) \otimes_R K$$

so that we get injectivity of the map Φ_s. $\square$

Proof of the Comparison Theorem for standard groups 4.1.7. Since we already know from Corollary 4.3.5 that

$$\Phi_s : H^n_{la}(\mathcal{G}(h), K) \to H^n(\mathfrak{g}, K)$$

is injective for $h \geq 0$ we now concentrate on surjectivity. Let $[c]$ be a cohomology class in $H^n(\mathfrak{g}, K)$. Then by Theorem 3.1.3 $[c]$ is represented by a cocyle $\tilde{c} \in \tilde{\mathcal{O}}(G)^{\hat{\otimes} n} \otimes_R K$. By Lemma 4.2.13 we know that

$$\tilde{\mathcal{O}}(G) \subset \mathcal{O}_c(\mathcal{G}(h)) \subset \mathcal{O}^{la}(\mathcal{G}(h)), \text{ for } h > h_0 = \frac{1}{p-1}.$$

Hence Φ_s is surjective for all $h > h_0 = \frac{1}{p-1}$. $\square$

Bibliography

[Bou98a] Nicolas Bourbaki. *Algebra I. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].

[Bou98b] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[CW74] W. Casselman and D. Wigner. Continuous cohomology and a conjecture of Serre's. *Invent. Math.*, 25:199–211, 1974.

[DdSMS99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 1999.

[Die73] J. Dieudonné. *Introduction to the theory of formal groups.* Marcel Dekker Inc., New York, 1973. Pure and Applied Mathematics, 20.

[Gou93] Fernando Q. Gouvêa. *p-adic numbers.* Universitext. Springer-Verlag, Berlin, 1993. An introduction.

[Haz78] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[HK11] Annette Huber and Guido Kings. A p-adic analogue of the Borel regulator and the Bloch-Kato exponential map. *J. Inst. Math. Jussieu*, 10(1):149–190, 2011.

[HKN11] Annette Huber, Guido Kings, and Niko Naumann. Some complements to the Lazard isomorphism. *Compos. Math.*, 147(1):235–262, 2011.

[Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kna88] Anthony W. Knapp. *Lie groups, Lie algebras, and cohomology*, volume 34 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1988.

[Laz65] Michel Lazard. Groupes analytiques p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (26):389–603, 1965.

[Nic08] Johannes Nicaise. Formal and rigid geometry: an intuitive introduction and some applications. *Enseign. Math. (2)*, 54(3-4):213–249, 2008.

[NSW00] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[Sch11] Peter Schneider. *p-adic Lie groups*, volume 344 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[Ser06] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.

[Swe69] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[Tu11] Loring W. Tu. *An introduction to manifolds.* Universitext. Springer, New York, second edition, 2011.

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1979.

List of Symbols

β	Bialgebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}$, p.30
$\beta^\star$	Hopf algebra homomorphism $\mathcal{O}(G) \to \mathcal{U}^\star$, p.30
$C^n(\mathfrak{g}, R)$	Set of n-cochains of $\mathfrak{g}$ with coeff. in R, p.26
$\mathcal{D}$	Continuous dual of $\mathcal{O}(G)$, p.19
$\mathcal{D}^\star$	Linear dual of $\mathcal{D}$, p.31
$e_1, \dots, e_n$	Basis of $\mathfrak{g}$ (n-dim), p.12
$\mathbf{e(j)}$	Multi-index $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the j-th entry, p.36
$\mathfrak{g}$	Lie algebra associated to the formal group law G, p.12
$\mathcal{G}$	K-Lie group, p.48
G_a	Additive formal group, p.10
G_m	Multiplicative formal group, p.11
$\mathcal{G}(h)$	Standard group of level h to a formal group law G, p.48
$G(\mathbf{X}, \mathbf{Y})$	Formal group law over R, p.7
$G_h(\mathbf{X}, \mathbf{Y})$	Formal group law such that $\mathcal{G}_{\mathrm{h}}(0) = \mathcal{G}(h)$, p.52
$G_K((\mathbf{X}, \mathbf{Y})$	Formal group law over K, p.49
$H(G, R)$	Group cohomology of G with coeff. in R, p.24
$H(\tilde{G}, R)$	Modified group cohomology of G with coeff. in R, p.42
$H(\mathfrak{g}, R)$	Lie algebra cohomology of $\mathfrak{g}$ with coeff. in R, p.26
$H_{la}(\mathcal{G}, K)$	Locally analytic group cohomology of $\mathcal{G}$ with coeff. in K, p.49
K	Finite extension of $\mathbb{Q}_p$, p.47
$K\langle \underline{t} \rangle$	Tate algebra, p.54
$(K^\bullet(G, R), \partial)$	Complex of inhomogeneous cochains of G with coeff. in R, p.24
$(\tilde{\mathrm{K}}^\bullet(G, R), \partial)$	Complex associated to $\tilde{\mathcal{O}}(G)$, p.42
$L(\mathcal{G})$	Lie algebra associated to $\mathcal{G}$, p.49
$\mathcal{O}(G)$	Function ring of a formal group G, p.8
$\tilde{\mathcal{O}}(G)$	Modified function ring of G, p.36
$\mathcal{O}_c(\mathcal{G}(h))$	Ring of convergent functions on $\mathcal{G}(h)$, p.55
$\mathcal{O}^{la}(\mathcal{G})$	Ring of locally analytic functions on $\mathcal{G}$, p.49

$\mathcal{O}^{la}(\mathcal{G})_e$	Ring of germs of locally analytic functions in e, p.53
$\phi^{(i)}$	Image of e_i under the map $\mathfrak{g} \to \mathcal{D}$, p.32
$(\phi^{(i)})^k$	Image of e_i^k under β, p.32
R	Integral domain (Chap.1-3), Valuation ring (Chap.4), p.7
$R\{\underline{t}\}$	Ring of strictly convergent power series over R, p.53
$R\{\{d\underline{t}_1, .., d\underline{t}_n\}\}$	Underlying set of $(\mathcal{U}^\star)^{\hat{\otimes} n}$, p.20
$\underline{t}_1$	Short notation for $t_1^{(1)}, \ldots, t_1^{(m)}$, p.9
$\underline{t}_{1,\ldots n}$	Short notation for $(t_1^{(1)}, \ldots, t_1^{(m)}, \ldots, t_n^{(1)}, \ldots, t_n^{(m)})$, p.9
$\mathcal{U}(\mathfrak{g})$	Universal algebra of a Lie algebra $\mathfrak{g}$, p.12
$\mathcal{U}^\star$	Dual of $\mathcal{U}(\mathfrak{g})$, p.20
$(\mathrm{V}(\mathfrak{g})^\bullet, d)$	Koszul complex, p.26