

TENSOR STRUCTURE FOR NORI MOTIVES

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ABSTRACT. We construct a tensor product on Freyd's universal abelian category $\text{Ab}(C)$ attached to an additive tensor category or a \otimes -quiver and establish a universal property. This is used to give an alternative construction for the tensor product on Nori motives.

INTRODUCTION

In the late 1990's Nori made a spectacular proposal for an unconditional definition of an abelian category of motives and a motivic Galois group over a field of characteristic zero. It has two main inputs:

- (1) The existence of a universal abelian category attached to a fixed representation of a quiver.
- (2) His Basic Lemma (known earlier to Beilinson and Vilonen) which shows the existence of an algebraically defined "skeletal filtration" on an affine algebraic variety.

The first part is enough to give the definition of the category. The second is needed in order to establish the tensor structure. In a third step, we pass from effective motives to all motives and check rigidity.

The motivic Galois group is its Tannaka dual. However, all steps are intrinsically linked together. The proof of the existence of the abelian category is done by constructing a suitable coalgebra. The tensor product is defined by turning this coalgebra into a bialgebra. After localisation, it is shown to be even a Hopf algebra - the Hopf algebra of the motivic Galois group. Indeed, the proof given in full detail in [HMS] give as a byproduct a full proof of Tannaka duality.

Meanwhile there have been a couple of alternative approaches to the first step of the above program, see [BVCL], [BV], [BVP] and [I1]. They are more general and arguably simpler. However, these references did not address tensor products.

In this paper we explain how the approach of [BVP] can be used to handle tensor categories and tensor functors. We show that if (C, \otimes) is an additive tensor category then Freyd's universal abelian category $\text{Ab}(C)$ carries an induced right-exact tensor structure which is also universal in a certain sense (the exact statement is Proposition 1.8).

Given a module M on C (i.e., an additive functor into an abelian tensor category), this induces, under additional technical assumptions, a tensor structure on

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the universal abelian category $\mathcal{A}(M)$ for the module M . It is again universal, see Proposition 1.11. The results can also be reformulated in terms of representations of quivers, see Section 2, in particular Theorem 2.10, bringing it even closer to the shape of Nori's original results. Our results are a lot more general in allowing modules with values in quite general abelian categories. On the other hand, the technical restrictions make it more narrow. We get back Nori's case for representations in the category of representations in the category of modules over a Dedekind domain, but not for arbitrary Noetherian rings (see Remark 1.9 and Example 1.10).

We also show how to apply our results to Nori motives. This can be done by using his original quiver of good pairs. Alternatively, we start with the more canonical tensor category of geometric motives in the sense of Voevodsky. However, the functor H_B^0 used in the definition of Nori motives is *not* a tensor functor, in contrast with the graded functor H_B^* . It remains to check that the Künneth components are motivic. This problem is addressed abstractly in Section 3 below. The application to Nori motives in Section 4 relies on Nori's Basic Lemma. The motivic Galois group is then obtained as the Tannaka dual.

We feel that the nature of the argument and the role of the Basic Lemma become a lot clearer in this new description.

Notation. By a tensor category (C, \otimes) we mean a category C provided with a functor $\otimes : C \times C \rightarrow C$ satisfying an associativity constraint and with $\mathbf{1}$ a unit object; in addition, also a commutativity constraint can be required, e.g. see [DMT, §1]. By an additive (resp. abelian) tensor category we mean a tensor category (C, \otimes) such that C is additive (resp. abelian) and \otimes is a bi-additive functor, see [DMT, Def. 1.15]. Tensor functors between additive tensor categories are assumed to be additive. We denote by $\mathbb{Q}\text{-vsp.}$ the tensor category of \mathbb{Q} -vector spaces.

If \mathcal{A} is an abelian category, we denote by $\text{gr}\mathcal{A}$ the associated category of \mathbb{Z} -graded objects. If, in addition, (\mathcal{A}, \otimes) carries a tensor structure, we equip $(\text{gr}\mathcal{A}, \otimes)$ with the induced tensor structure. If the tensor product is commutative, we choose the commutativity constraint on $\text{gr}\mathcal{A}$ such that the product becomes graded anti-commutative.

For an additive category C we shall consider the additive functors from C to the category Ab of abelian groups as (left) C -modules. We shall denote by $C\text{-mod}$ the category of finitely presented C -modules, see e.g. [P1].

1. UNIVERSAL ABELIAN TENSOR CATEGORIES

Let C be an additive category. We denote by $\text{Ab}(C)$ the universal abelian category on C , see [F], also [P1, Chap. 4]. We may refer to it as Freyd's abelian category. It comes with a canonical fully faithful functor $C \hookrightarrow \text{Ab}(C)$, which we write $X \mapsto [X]$. Recall that this functor is universal with respect to additive functors into abelian categories, e.g. see [BVP, Thm 1.1].

Thus, for $M : C \rightarrow \mathcal{A}$ an additive functor into some abelian category \mathcal{A} , we obtain an induced exact functor $\widetilde{M} : \text{Ab}(C) \rightarrow \mathcal{A}$, unique to natural equivalence.

We denote by $\mathcal{A}(M)$ the quotient of $\text{Ab}(C)$ by the Serre subcategory which is the kernel of \widetilde{M} ; we also denote by $\widetilde{M} : \mathcal{A}(M) \rightarrow \mathcal{A}$ the induced faithful exact functor.

We shall refer to $\mathcal{A}(M)$ as the *universal abelian category defined by M* , according with [BVP, §1.1]. In fact, this abelian category $\mathcal{A}(M)$ is universal for all abelian categories together with a faithful exact functor into \mathcal{A} , which extends M . Note that,

in the case where \mathcal{A} is the category of finitely generated modules over a commutative noetherian ring R , this recovers Nori's abelian category (see [HMS, Chap. 7] and compare with [BVP, §1.2]).

For later use, we introduce:

1.1. Definition. Let C be additive, $C \rightarrow \text{Ab}(C)$ Freyd's abelian category. We denote by $\text{Ab}(C)^{\flat}$ the smallest full subcategory containing the objects in the image of C and closed under kernels.

1.2. Remark. The universal abelian category $\text{Ab}(C)$ can be constructed explicitly as the category $(C\text{-mod})\text{-mod}$. In this construction, $\text{Ab}(C)^{\flat}$ is precisely the image of $C\text{-mod}$ under the (contravariant) Yoneda embedding into $\text{Ab}(C)$. The above definition is independent of this description.

Let (C, \otimes) be an additive tensor category, see [DMT, §1]. Consider an (additive) tensor functor $M : (C, \otimes) \rightarrow (\mathcal{A}, \otimes)$ where (\mathcal{A}, \otimes) is an abelian tensor category. We want to equip the above universal abelian category $\mathcal{A}(M)$ with a natural tensor structure $(\mathcal{A}(M), \otimes)$ such that $\tilde{M} : (\mathcal{A}(M), \otimes) \rightarrow (\mathcal{A}, \otimes)$ is turned into a tensor functor. We proceed in several steps.

Multilinear functors. By definition, $\text{Ab}(C)$ has a universal property with respect to additive functors. In fact, this extends to bi-additive and even multi-additive functors, even though we lose some properties.

1.3. Proposition. *Let C_1, \dots, C_n be additive categories, \mathcal{A} an abelian category. Let $F : C_1 \times \dots \times C_n \rightarrow \mathcal{A}$ be a multilinear functor, i.e., additive in each argument. Then F extends uniquely to a multilinear functor*

$$\tilde{F} : \text{Ab}(C_1) \times \text{Ab}(C_2) \times \dots \times \text{Ab}(C_n) \rightarrow \mathcal{A}$$

which is right-exact in each argument. Fix j and for $i \neq j$ choose $X_i \in \text{Ab}(C_i)^{\flat}$ (see Definition 1.1). Then $\tilde{F}(X_1, \dots, -, \dots, X_n)$ is exact as a functor on $\text{Ab}(C_j)$.

Proof. We explain the case $n = 2$. The general case is shown in the same way. Recall that $\text{Ab}(C_i) = (C_i\text{-mod})\text{-mod}$ and that the universal functor factors

$$C_i \rightarrow (C_i\text{-mod})^{\text{op}} \rightarrow (C_i\text{-mod})\text{-mod}$$

where both steps are given by the Yoneda embedding. As pointed out in Remark 1.2 the subcategory $\text{Ab}(C_i)^{\flat}$ agrees with the image of $(C_i\text{-mod})^{\text{op}}$.

Let $X_i \in (C_i\text{-mod})^{\text{op}}$. By definition, these objects have an injective copresentation

$$0 \rightarrow X_i \rightarrow (A_i, -)^{\text{op}} \rightarrow (B_i, -)^{\text{op}}.$$

We apply F to the resolution and obtain a double complex in the abelian category \mathcal{A} . We define $F'(X_1, X_2)$ as its H^0 , i.e., the kernel of

$$F(A_1, A_2) \rightarrow F(A_1, B_2) \oplus F(A_2, B_1).$$

This is well-defined and functorial because any two injective resolutions, equivalently projective resolutions in $C_i\text{-mod}$, are homotopy equivalent. The functor F' is left-exact in both arguments.

We repeat the same construction for the second Yoneda extension. This time, every object is the cokernel of a morphism of representable objects. The extension F'' is right-exact in both arguments by construction.

We return to $X_1 \in (C_1\text{-mod})^{\text{op}}$ with resolution as above and claim that $F''(X_1, -)$ is exact. It suffices to show that it agrees with the universal extension of the functor $G : C_2 \rightarrow \mathcal{A}$ with $G(-) = F''(X_1, -)$ to $\text{Ab}(C_2)$ because the latter is known to be exact. For $X_2 \in (C_2\text{-mod})^{\text{op}}$ with resolution as above, we have by construction

$$\begin{aligned} G'(X_2) &= \text{Ker}((G'((A_2, -)^{\text{op}}) \rightarrow G'((B_2, -)^{\text{op}})) \\ &= \text{Ker}((F'(X_1, (A_2, -)^{\text{op}}) \rightarrow F'(X_2, (B_2, -)^{\text{op}})) = F'(X_1, X_2) \end{aligned}$$

by left-exactness of F' in the second argument. Now let $Y_2 \in \text{Ab}(C_2)$ with resolution

$$(-, X_2) \rightarrow (-, X'_2) \rightarrow Y_2 \rightarrow 0.$$

By construction

$$\begin{aligned} G''(Y_2) &= \text{Coker}(G'(X_2) \rightarrow G'(X'_2)) \\ &= \text{Coker}(F'(X_1, X_2) \rightarrow F'(X_1, X'_2)) \\ &= F''(X_1, Y_2). \end{aligned}$$

□

1.4. *Remark.* Unexpectedly the extension F'' fails to be exact in each argument. For a counterexample, see Example 1.10 below.

This applies in particular to additive tensor categories.

1.5. **Definition.** Let (C, \otimes) be an additive tensor category. We extend the functor $\otimes : C \times C \rightarrow C$ defining

$$\otimes : \text{Ab}(C) \times \text{Ab}(C) \rightarrow \text{Ab}(C)$$

as the universal bi-additive extension of $C \times C \rightarrow C \hookrightarrow \text{Ab}(C)$, using Proposition 1.3.

1.6. **Proposition.** Let $(\text{Ab}(C), \otimes)$ be Freyd's category together with the functor in Definition 1.5. Then

- (1) $(\text{Ab}(C), \otimes)$ is an abelian tensor category.
- (2) The tensor product is right-exact by construction. The objects in $\text{Ab}(C)^{\flat}$ are flat, i.e., acyclic with respect to \otimes .
- (3) If the tensor structure on C is commutative then so is the tensor structure on $\text{Ab}(C)$.

Proof. Right-exactness and acyclicity are special cases of Proposition 1.3. Let $\mathbf{1}$ be the unit object of C . By definition it comes with a transformation of functors $u : \mathbf{1} \otimes - \rightarrow \text{id}$ on C . Let $[\mathbf{1}]$ be its image in $\text{Ab}(C)$. In explicit formulas this means $[\mathbf{1}] = ((\mathbf{1}, -), -)$. Then $[\mathbf{1}]$ with the induced transformation is the unit of $\text{Ab}(C)$. The equivalences used to express the associativity constraint on C^3 (see [DMT, §1]) induce equivalences on $\text{Ab}(C)^3$. Similarly for the commutativity constraint if there is one on C . □

1.7. **Definition.** For an abelian tensor category, with a right exact tensor product, a \flat -subcategory is a full additive subcategory of flat objects (i.e., acyclic with respect to the tensor product) which is closed under kernels. If (\mathcal{A}, \otimes) is such an abelian tensor category we shall denote by $\mathcal{A}^{\flat} \subseteq \mathcal{A}$ a \flat -subcategory.

As a consequence of Proposition 1.6 we have that $\text{Ab}(C)^{\flat} \subset \text{Ab}(C)$ as in Definition 1.1 is a \flat -subcategory.

1.8. Proposition (Universal property). *Let C be an additive tensor category. Let \mathcal{A} be an abelian tensor category with a right exact tensor product. Let $M : (C, \otimes) \rightarrow (\mathcal{A}, \otimes)$ be a tensor functor. In addition, assume that M factors via $\mathcal{A}^b \subseteq \mathcal{A}$ a b-subcategory (see Definition 1.7). Then $\widetilde{M} : (\text{Ab}(C), \otimes) \rightarrow (\mathcal{A}, \otimes)$ is a tensor functor. The triple $(\text{Ab}(C), \text{Ab}(C)^b, \otimes)$ is universal with this property, in particular unique.*

Proof. Let $M : (C, \otimes) \rightarrow (\mathcal{A}, \otimes)$ be a tensor functor. We have to compare

$$\text{Ab}(C) \times \text{Ab}(C) \rightarrow \text{Ab}(C) \rightarrow \mathcal{A}$$

and

$$\text{Ab}(C) \times \text{Ab}(C) \rightarrow \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}.$$

Both are right-exact in each argument (this is where right-exactness of the tensor product on \mathcal{A} is used) and agree on $C \times C$.

As in the proof of Proposition 1.3, we extend M in two steps: first to $(C\text{-mod})^{\text{op}}$, then to $\text{Ab}(C) = (C\text{-mod})\text{-mod}$. The second step is unproblematic as it only uses the right-exactness. In the first step, we need to check the action on (certain) kernels. Let $X_1, X_2 \in (C\text{-mod})^{\text{op}}$ with resolutions

$$0 \rightarrow X_i \rightarrow (A_i, -)^{\text{op}} \xrightarrow{f} (B_i, -)^{\text{op}}.$$

By definition

$$0 \rightarrow M'(X_i) \rightarrow M(A_i) \rightarrow M(B_i)$$

is exact. By assumption $M(A_i), M(B_i)$ and hence also $M'(X_i)$ are in \mathcal{A}^b . In particular, consider the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M'(X_1) \otimes M'(X_2) & \longrightarrow & M(X_1) \otimes M(A_2) & \longrightarrow & M'(X_1) \otimes M(B_2) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M(A_1) \otimes M'(X_2) & \longrightarrow & M(A_1) \otimes M(A_2) & \longrightarrow & M(A_1) \otimes M(B_2) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M(B_1) \otimes M'(X_2) & \longrightarrow & M(B_1) \otimes M(A_2) & \longrightarrow & M(B_1) \otimes M(B_2) \end{array}$$

All rows and columns are exact because they arise by tensoring an exact sequence with a flat object. This implies

$$\begin{aligned} M'(X_1) \otimes M'(X_2) &= \text{Ker}(M(A_1) \otimes M(A_2) \rightarrow (M(A_1) \otimes M(B_2)) \oplus (M(B_1) \otimes M(A_2))) \\ &= M'(X_1 \otimes X_2). \end{aligned}$$

The triple $(\text{Ab}(C), \text{Ab}(C)^b, \otimes)$ satisfies itself the assumptions of the universal property, hence it is universal and as such unique. \square

1.9. Remark. There are a number of interesting cases where the assumptions of Proposition 1.8 and Definition 1.7 are satisfied. However, they are not as general as one could hope for.

- (1) If \otimes is exact on \mathcal{A} , then $\mathcal{A}^b = \mathcal{A}$ clearly satisfies the assumptions.

- (2) If $C_1 \rightarrow C_2$ is a \otimes -functor between additive tensor categories, by composition, we may consider $M : C_1 \rightarrow \mathcal{A}^b = \text{Ab}(C_2)^b \subset \mathcal{A} = \text{Ab}(C_2)$ which satisfies the assumptions; then, by the universal property, we get an exact tensor functor $\widetilde{M} : \text{Ab}(C_1) \rightarrow \text{Ab}(C_2)$.
- (3) The assumptions are satisfied if $\mathcal{A} = R\text{-mod}$ for a Dedekind domain R where \mathcal{A}^b is the b -subcategory of projective finitely-generated R -modules, i.e., torsion free finitely-generated modules, and $M : C \rightarrow \mathcal{A}^b$ any tensor functor. In particular this is true for $R = \mathbb{Z}$.
- (4) They are not satisfied for $\mathcal{A} = R\text{-mod}$ for a general Noetherian commutative ring R and the subcategory of projective finitely-generated R -modules, which is not a b -subcategory if the global dimension of R is > 2 . See the example below.

1.10. *Example.* Let C be the category with objects $(\mathbb{Z}/4\mathbb{Z})^n$ for $n \geq 0$ and morphisms given by homomorphisms of abelian groups.

Let $\mathcal{A} = \mathbb{Z}/4\mathbb{Z}\text{-mod}$. In this case is possible to compute all objects explicitly. The functor $M \mapsto M^\vee = \text{Hom}(M, \mathbb{Z}/4\mathbb{Z})$ is an antiequivalence of C with itself. We have $C\text{-mod} \cong \mathbb{Z}/4\mathbb{Z}\text{-mod}$ with $C \rightarrow \mathbb{Z}/4\text{-mod}$ given by $M \mapsto M^\vee$. Hence $\text{Ab}(C)$ is the category of finitely presented presheaves on $\mathbb{Z}/4\mathbb{Z}\text{-mod}$. Objects are uniquely determined by the values of these presheaves on the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Direct computation will show:

- (1) \otimes is not biexact on $\text{Ab}(C)$.
- (2) The tensor functor $\text{Ab}(C) \rightarrow \mathcal{A}$ induced by the inclusion functor $C \rightarrow \mathcal{A}$ is not a tensor functor.

By Auslander-Reiten theory, see e.g. [ASS, §IV.6, p. 149], the simple objects of the category $\text{Ab}(C)$ have the form $(X, -)/\text{rad}(X, -)$ for X an indecomposable $\mathbb{Z}/4$ -module. So there are two simple objects, say S and T say, and these are such that $S(\mathbb{Z}/4) = \mathbb{Z}/2$, $S(\mathbb{Z}/2) = 0$ and $T(\mathbb{Z}/4) = 0$, $T(\mathbb{Z}/2) = \mathbb{Z}/2$. Noting the exact sequence $0 \rightarrow \mathbb{Z}/2 \xrightarrow{j} \mathbb{Z}/4 \xrightarrow{p} \mathbb{Z}/2 \rightarrow 0$ and considering the maps $(p, -)$ and $(j, -)$ in $\text{Ab}(C)$, it can be easily checked that $\text{rad}(\mathbb{Z}/4, -) = (\mathbb{Z}/2, -)$ and that $(\mathbb{Z}/2, -)$ has length 2, with socle S . The remaining indecomposable objects of $\text{Ab}(C)$ may then be computed (see, for example, [P2, 4.3]): there are 5 of them, all of them subquotients of the two representable functors. They are $(\mathbb{Z}/4, -)$, $(\mathbb{Z}/2, -)$, the two simples S , T and $(\mathbb{Z}/4, -)/S$.

Now consider the exact functor $\widetilde{\mathbb{Z}/4} : \text{Ab}(C) \rightarrow \mathbb{Z}/4\text{-mod}$. This is evaluation of a functor at \mathbb{Z}_4 , hence is 0 only on T among those five indecomposables. Therefore its kernel is the Serre subcategory which consists of direct sums of copies of T . In order to compute $T \otimes T$, we apply the definition of the tensor product on $\text{Ab}(C)$ using the projective presentation $(\mathbb{Z}/4, -) \xrightarrow{(j, -)} (\mathbb{Z}/2, -) \xrightarrow{\pi_T} T \rightarrow 0$ of T and, checking that $(\text{id}_{\mathbb{Z}/2}, -) \otimes (j, -) = 0$, we obtain $T \otimes T = (\mathbb{Z}/2, -)$, which is not in the kernel of $\widetilde{\mathbb{Z}/4}$, so this is not a tensor functor. As part of the computation of $T \otimes T$ one sees that $T \otimes (\mathbb{Z}/2, -) = (\mathbb{Z}/2, -)$. So applying $T \otimes -$ to the monomorphism $(\mathbb{Z}/2, -) \xrightarrow{(p, -)} (\mathbb{Z}/4, -)$ gives $(\mathbb{Z}/2, -) \rightarrow T$ which is not monic, showing that \otimes is not exact on $\text{Ab}(C)$.

This implies that we cannot expect a different, exact, tensor product on $\text{Ab}(C)$ extending the tensor product on C – by the universal property the identity would have to be a tensor functor.

Tensor structures on $\mathcal{A}(M)$. Consider (\mathcal{A}, \otimes) an abelian tensor category with a right-exact tensor product.

For the sake of exposition we now drop explicit reference to \otimes if unnecessary.

1.11. Proposition. *Let C be an additive tensor category, \mathcal{A} an abelian tensor category with a right exact tensor product, and $M : C \rightarrow \mathcal{A}$ an additive tensor functor. Further assume that M factors through a \mathfrak{b} -subcategory $\mathcal{A}^{\mathfrak{b}} \subset \mathcal{A}$ (see Definition 1.7).*

- (1) *Then $\mathcal{A}(M)$ carries a canonical tensor structure such that the faithful exact functor $\widetilde{M} : \mathcal{A}(M) \rightarrow \mathcal{A}$ is a tensor functor.*
- (2) *If in addition, the tensor structures on C and \mathcal{A} are commutative and the tensor functor is symmetric, then the tensor product on $\mathcal{A}(M)$ is symmetric.*
- (3) *If in addition, the tensor structure on C is rigid and the tensor product and the Hom-functor on \mathcal{A} are exact in both arguments, then the same is true for $\mathcal{A}(M)$.*

Proof. We need to check that the tensor functor on $\text{Ab}(C)$ (see Proposition 1.6) factors via an induced tensor structure on $\mathcal{A}(M)$. We have a commutative diagram

$$\begin{array}{ccc} \text{Ab}(C) \times \text{Ab}(C) & \xrightarrow{\otimes} & \text{Ab}(C) \\ \widetilde{M} \times \widetilde{M} \downarrow & & \downarrow \widetilde{M} \\ \mathcal{A} \times \mathcal{A} & \xrightarrow{\otimes} & \mathcal{A} \end{array}$$

by Proposition 1.8. This implies that the kernel of $\text{Ab}(C) \rightarrow \mathcal{A}$ is a \otimes -ideal. Hence the tensor product induces one on $\mathcal{A}(M)$. Associativity, unit and symmetry are immediate from the properties of the tensor structure on $\text{Ab}(C)$.

We turn to rigidity. By assumption, every object X of C has a strong dual. By the criterion formulated in [Lev, Part I, IV, Proposition 1.1.9] the existence of a dual for X can be characterized by the existence of unit and counit maps satisfying some compatibilities. In particular, this property is functorial, hence $[X] \in \mathcal{A}(M)$ also has a strong dual. Consider the full subcategory of $\mathcal{A}(M)$ consisting of objects with a strong dual. It contains all objects in the image of C . Under our assumptions on \mathcal{A} , the tensor product on $\mathcal{A}(M)$ is exact in both arguments and hence the subcategory is closed under kernels and cokernels. Hence it is an abelian subcategory of $\mathcal{A}(M)$ containing the image of C , hence it agrees with $\mathcal{A}(M)$ \square

1.12. Remark. This is a version of Nori’s result on the tensor structure on his abelian category, see [HMS, Proposition 8.1.5]. It is much stronger in allowing general abelian categories \mathcal{A} as target. On the other hand, his result covers functors $C \rightarrow R\text{-proj}$ (where the latter is the category of finitely generated projective modules over a noetherian ring R). We get it back only for Dedekind domains.

2. UNIVERSAL \otimes -REPRESENTATION

We want to extend our results to representations of quivers. Given the results of the previous section, this means to extend tensor structures from a quiver to the additive category generated by it.

Recall from [Bo, Def. 5.1.5] or [GR] the concept of a quiver “with relations”, i.e., a quiver with a set of commutativity conditions or linear relations. In this sense:

2.1. Definition. A \otimes -quiver is a quiver D with relations, including the following data $(\text{id}, \otimes, \alpha, \beta, \beta', \mathbf{1}, u)$

- (1) for every vertex v a distinguished self-edge $\text{id} : v \rightarrow v$;
- (2) for every pair of vertices (v, w) a vertex denoted $v \otimes w$ in D ;
- (3) for every edge $e : v \rightarrow v'$ and vertex w a path $e \otimes \text{id} : v \otimes w \rightarrow v' \otimes w$ and a path $\text{id} \otimes e : w \otimes v \rightarrow w \otimes v'$;
- (4) for every pair of vertices u, v a distinguished edge $\alpha_{u,v} : u \otimes v \rightarrow v \otimes u$;
- (5) for every triple of vertices u, v, w a distinguished edge $\beta_{u,vw} : u \otimes (v \otimes w) \rightarrow (u \otimes v) \otimes w$ and also $\beta'_{u,vw} : (u \otimes v) \otimes w \rightarrow u \otimes (v \otimes w)$;
- (6) a distinguished vertex $\mathbf{1}$;
- (7) for every vertex distinguished edges $u_v : v \rightarrow \mathbf{1} \otimes v$ and $u'_v : \mathbf{1} \otimes v \rightarrow v$;

and the relations

- (1) $\text{id}_v \otimes \text{id}_v = \text{id}_{v \otimes v}$;
- (2) $\text{id}_v = e_v$ where e_v is the empty path for every vertex v ;
- (3) $(e \otimes \text{id}) \circ (\text{id} \otimes e') = (\text{id} \otimes e') \circ (e \otimes \text{id})$ for all pairs of edges e, e' ;
- (4) $\alpha_{v,w} \circ \alpha_{w,v} = \text{id}$ for all vertices v, w ;
- (5) $(\text{id} \otimes \gamma) \circ \alpha = \alpha \circ (\gamma \otimes \text{id})$ and $(\gamma \otimes \text{id}) \circ \alpha = \alpha \circ (\text{id} \otimes \gamma)$ for all edges γ ;
- (6) $\beta_{u,vw} \circ \beta'_{uv,w} = \text{id}$, $\beta'_{uv,w} \circ \beta_{u,vw} = \text{id}$;
- (7) $\beta \circ (\gamma \otimes (\text{id} \otimes \text{id})) = ((\gamma \otimes \text{id}) \otimes \text{id}) \circ \beta$ for all edges γ and analogously in the second and third argument;
- (8) (pentagon axiom) for all vertices x, y, z, t the relation

$$\begin{array}{ccc}
 x \otimes (y \otimes (z \otimes t)) & \xrightarrow{\beta} & (x \otimes y) \otimes (z \otimes t) \xrightarrow{\beta} & ((x \otimes y) \otimes z) \otimes t \\
 \text{id} \otimes \beta \downarrow & & & \uparrow \beta \otimes \text{id} \\
 x \otimes ((y \otimes z) \otimes t) & \xrightarrow{\beta} & & (x \otimes (y \otimes z)) \otimes t
 \end{array}$$

- (9) for all vertices x, y, z the relation

$$\begin{array}{ccc}
 x \otimes (y \otimes z) & \xrightarrow{\beta} & (x \otimes y) \otimes z \xrightarrow{\alpha} & z \otimes (x \otimes y) \\
 \text{id} \otimes \alpha \downarrow & & & \downarrow \beta \\
 x \otimes (z \otimes y) & \xrightarrow{\beta} & (x \otimes z) \otimes y \xrightarrow{\alpha \otimes \text{id}} & (z \otimes x) \otimes y
 \end{array}$$

- (10) $u_v \circ u'_v = \text{id}$ and $u'_v \circ u_v = \text{id}$ for all vertices v ;
- (11) for all edges $e : v \rightarrow v'$ the relation

$$\begin{array}{ccc}
 v' & \xrightarrow{u} & \mathbf{1} \otimes v' \\
 e \uparrow & & \uparrow \text{id} \otimes e \\
 v & \xrightarrow{u} & \mathbf{1} \otimes v
 \end{array}$$

2.2. Remark. This data is modeled after the notion of a commutative product structure on a diagram with identities, see [HMS, Def. 8.1.3] and the variant in loc.cit. Remark 8.1.6. The axioms for the associativity and commutativity constraint and unitality are the usual ones for a commutative tensor category, see in [DMT, §1].

Recall [BVP] where a universal representation $\Delta : D \rightarrow \text{Ab}(D)$ is constructed for any quiver D . It is given by the composition

$$\Delta : D \rightarrow \mathcal{P}(D) \rightarrow \mathbb{Z}D \rightarrow \mathbb{Z}D^+ \rightarrow \text{Ab}(\mathbb{Z}D^+) = \text{Ab}(D)$$

where (in the notation of [BVP, §1]) $\mathcal{P}(D)$ is the path category, $\mathbb{Z}D$ the preadditive enrichment of $\mathcal{P}(D)$ and $\mathbb{Z}D^+$ its additive completion.

We now repeat the same chain with tensor categories. Let (D, \otimes) be a \otimes -quiver. We define the \otimes -path category $\mathcal{P}(D)^\otimes$ as the quotient of the path category by the relations of (D, \otimes) . We define

$$\otimes : \mathcal{P}(D) \times \mathcal{P}(D) \rightarrow \mathcal{P}(D)$$

on objects as prescribed by the tensor structure. Let $\Gamma = \gamma_1 \circ \dots \circ \gamma_n$, $\Delta = \delta_1 \circ \dots \circ \delta_m$ be paths. We define

$$\Gamma \otimes \Delta = (\gamma_1 \otimes \text{id}) \circ \dots \circ (\gamma_n \otimes \text{id}) \circ (\text{id} \otimes \delta_1) \circ \dots \circ (\text{id} \otimes \delta_m).$$

E.g. for $n = m = 1$ and $\gamma : v \rightarrow v'$, $\delta : w \rightarrow w'$, we have

$$\begin{array}{ccc} v \otimes w & \xrightarrow{\text{id} \otimes \delta} & v \otimes w' \\ \downarrow \gamma \otimes \text{id} & \searrow \gamma \otimes \delta & \downarrow \gamma \otimes \text{id} \\ v' \otimes w & \xrightarrow{\delta \otimes \text{id}} & v' \otimes w' \end{array}$$

where we have by definition set the diagonal to be the path via the top right corner. In $\mathcal{P}(D)^\otimes$ this agrees with the path via the bottom left corner because of the relation (3).

2.3. Lemma. *Let (D, \otimes) be a \otimes -quiver. Then $\mathcal{P}(D)^\otimes$ is a tensor category.*

Proof. Property (3) of a tensor structure ensures that \otimes is a functor on $\mathcal{P}(D)^\otimes$. The other axioms make sure that the commutativity constraint α and the associativity constraint β are isomorphisms and satisfy the properties of a commutative tensor category. The relations on u_v ensure that $v \rightarrow \mathbf{1} \otimes v$ is an isomorphism and $\mathbf{1} \otimes _$ is an equivalence of categories. \square

2.4. Definition. Let (D, \otimes) be a \otimes -quiver. We put $\mathbb{Z}D^\otimes$ and $\mathbb{Z}D^{\otimes,+}$ the preadditive and additive hull of $\mathcal{P}(D)^\otimes$. Denote by $\text{Ab}(D)^\otimes$ Freyd's abelian category of $\mathbb{Z}D^{\otimes,+}$.

2.5. Proposition. *$\mathbb{Z}D^\otimes$, $\mathbb{Z}D^{\otimes,+}$ and $\text{Ab}(D)^\otimes$ are commutative tensor categories with the bilinear extension of \otimes . The canonical functor $\mathbb{Z}D^+ \rightarrow \mathbb{Z}D^{\otimes,+}$ induces a Serre quotient $\text{Ab}(D) \rightarrow \text{Ab}(D)^\otimes$.*

Proof. The statements on the additive and preadditive category are obvious. The statement on the abelian category is Proposition 1.6. The claim on the Serre quotient is granted by the following general fact. \square

2.6. Lemma. *Let D be a quiver with relations and \underline{D} the underlying quiver. Then $\pi : \text{Ab}(\underline{D}) \rightarrow \text{Ab}(D)$ is a Serre quotient.*

This is well-known but for the convenience of the reader we give the simple argument directly.

Proof. Consider $\text{Ab}(\underline{D})/\text{Ker}\pi$. By construction it is an exact subcategory of $\text{Ab}(D)$, hence it remains to check that the inclusion is full and essentially surjective.

The quiver \underline{D} has a canonical representation in $\text{Ab}(\underline{D})/\text{Ker}\pi$. All relations in D are satisfied, hence it is even a representation of D . By the universal property this yields an exact functor $\text{Ab}(D) \rightarrow \text{Ab}(\underline{D})/\text{Ker}\pi$. By the uniqueness part of the universal property, its composition with the inclusion into $\text{Ab}(D)$ is isomorphic to the identity. In particular, the inclusion is full and essentially surjective, hence an equivalence of categories. \square

We now turn to the universal property. The obvious approach is to consider representations $T : D \rightarrow \mathcal{A}$ where all relations in D are mapped to identities in \mathcal{A} . However, this is too rigid for most applications. We follow the approach of [HMS, Definition 8.1.3]

2.7. Definition. Let D be a \otimes -quiver, \mathcal{A} a commutative tensor category. A *tensor representation* or \otimes -*representation* for short, is a representation $T : D \rightarrow \mathcal{A}$ of the underlying quiver together with the choice of natural isomorphisms

$$\kappa : T(u) \otimes T(v) \xrightarrow{\simeq} T(u \otimes v)$$

for all vertices $u, v \in D$, functorial in each variable and compatible with the associativity and commutativity constraints and the unit in the obvious way.

2.8. Proposition. *Let (D, \otimes) be a \otimes -quiver.*

- (1) $D \rightarrow \mathcal{P}(D)^\otimes$ is the universal \otimes -representation into a commutative tensor category.
- (2) $D \rightarrow \mathbb{Z}D^{\otimes,+}$ is the universal \otimes -representation into an additive commutative tensor category.

Proof. The universal properties for $\mathcal{P}(D)^\otimes$ and $\mathbb{Z}D^{\otimes,+}$ are obvious. \square

2.9. Theorem. *Let (D, \otimes) be a \otimes -quiver.*

- (1) The natural assignment $\Delta^\otimes : D \rightarrow \text{Ab}(D)^\otimes$ is a \otimes -representation into an abelian tensor category with right-exact tensor product.
- (2) It takes values in the subcategory $(\text{Ab}(D)^\otimes)^\flat$ of Definition 1.1. Moreover, this is a \flat -subcategory (see Definition 1.7).
- (3) The category $\text{Ab}(D)^\otimes$ is universal with this property.

In detail: Let $T : D \rightarrow \mathcal{A}$ be a \otimes -representation via κ in an abelian tensor category with a right exact tensor, which factors through a \flat -subcategory $\mathcal{A}^\flat \subseteq \mathcal{A}$. Then there is an induced exact tensor functor $\widetilde{M}^\otimes : \text{Ab}(D)^\otimes \rightarrow \mathcal{A}$.

Proof. $\mathcal{A} = \text{Ab}(D)^\otimes$ is an abelian tensor category by Proposition 2.5 and Δ^\otimes is a \otimes -representation by construction. It factors via the additive category $\mathbb{Z}D^{\otimes,+}$. Hence property (2) follows from Proposition 1.6. To see the second statement, note that if $(B, -) \xrightarrow{(f, -)} (A, -)$ is a morphism in $(\text{Ab}(D)^\otimes)^\flat$ then its kernel is $(C, -) \xrightarrow{(g, -)} (B, -)$ where $B \xrightarrow{g} C$ is the cokernel of $A \xrightarrow{f} B$.

The induced functor $M^\otimes : \mathbb{Z}D^{\otimes,+} \rightarrow \mathcal{A}$ satisfies the assumptions in Proposition 1.8. Thus it induces the tensor functor $\widetilde{M}^\otimes : \text{Ab}(D)^\otimes \rightarrow \mathcal{A}$ such that $T = \widetilde{M}^\otimes \Delta^\otimes$. \square

Recall the universal representation theorem stated in [BVP]. For $T : D \rightarrow \mathcal{A}$ any representation of a quiver in an abelian category \mathcal{A} there is an induced additive functor

$$M : \mathbb{Z}D^+ \rightarrow \mathcal{A}$$

and a corresponding $\widetilde{M} : \text{Ab}(D) \rightarrow \mathcal{A}$ in such a way that

$$\widetilde{T} : D \rightarrow \mathcal{A}(T) := \mathcal{A}(M) = \text{Ab}(D)/\text{Ker}\widetilde{M}$$

is the induced universal representation (see [BVP, §1.3]). For a \otimes -quiver D , together with a \otimes -representation T in an abelian tensor category \mathcal{A} , as in Theorem 2.9, we have now constructed a factorisation via an exact tensor functor \widetilde{M}^\otimes on $\text{Ab}(D)^\otimes$. Hence we get a tensorial refinement of the universal representation theorem. This also implies the existence of a tensor structure on the universal abelian category $\mathcal{A}(T)$ attached to the representation. Note that this is really $\mathcal{A}(T)$; in contrast to $\mathcal{P}(D)^\otimes$ etc. no \otimes -adornment is needed.

2.10. Theorem. *Let $T : D \rightarrow \mathcal{A}$ be a representation in an abelian tensor category with a right exact tensor, which factors through a \flat -subcategory $\mathcal{A}^\flat \subseteq \mathcal{A}$, with the following additional properties:*

- (i) (D, \otimes) is a \otimes -quiver and
- (ii) T is a \otimes -representation in $\mathcal{A}^\flat \subseteq \mathcal{A}$ via κ .

Then Nori's universal abelian category $\mathcal{A}(T)$ carries a right exact tensor product and $\widetilde{M} : \mathcal{A}(T) \rightarrow \mathcal{A}$ is a tensor functor (here M is the additive functor induced by T and \widetilde{M} is the faithful exact functor induced by M , see also Proposition 1.11). It is universal among such representations into abelian tensor categories \mathcal{B} compatible via a faithful exact tensor functor $\mathcal{A} \rightarrow \mathcal{B}$.

Proof. By the universal property in Theorem 2.9, there is a canonical exact tensor functor $\widetilde{M}^\otimes : \text{Ab}(D)^\otimes \rightarrow \mathcal{A}$. Hence $\text{Ker}\widetilde{M}^\otimes$ is a Serre subcategory and a tensor ideal. Denoting by $\mathcal{A}(T)^\otimes$ the Serre quotient $\text{Ab}(D)^\otimes/\text{Ker}\widetilde{M}^\otimes$ we have obtained a tensor category with the universal property as claimed. Furthermore, by the universal property of $\mathcal{A}(T)$, there is also an exact faithful functor

$$\mathcal{A}(T) \rightarrow \mathcal{A}(T)^\otimes$$

We claim that it is an equivalence of abelian categories. The canonical additive functor $\mathbb{Z}D^+ \rightarrow \mathbb{Z}D^{\otimes,+}$ induces an exact functor $\pi : \text{Ab}(D) \rightarrow \text{Ab}(D)^\otimes$ such that $\widetilde{M}^\otimes \circ \pi = \widetilde{M}$ by the uniqueness in the universal property of Freyd's construction (see [BVP, Thm. 1.1]). The faithful exact functor $\bar{\pi} : \text{Ab}(D)/\text{Ker}\pi \xrightarrow{\cong} \text{Ab}(D)^\otimes$ is an equivalence by Proposition 2.5.

Thus, the composition $\text{Ab}(D) \xrightarrow{\pi} \text{Ab}(D)^\otimes \rightarrow \text{Ab}(D)^\otimes/\text{Ker}(\widetilde{M}^\otimes)$ is essentially surjective and is equivalent to the composition $\text{Ab}(D) \rightarrow \mathcal{A}(T) \rightarrow \mathcal{A}(T)^\otimes$ since they have equivalent compositions with the faithful functor $\mathcal{A}(T)^\otimes \rightarrow \mathcal{A}$. So $\mathcal{A}(T) \rightarrow \mathcal{A}(T)^\otimes$ also is essentially surjective hence an equivalence. \square

Signs. In many cases, notably in Nori's original application, we do not start with a tensor representation but with a tensor representation with signs. We explain the necessary modifications, following again the approach of [HMS, Def. 8.1.3].

2.11. Definition. A *graded quiver* is a quiver together with a function $|\cdot|$ assigning to each vertex a degree in $\mathbb{Z}/2\mathbb{Z}$. For an edge $e : v \rightarrow w$ we put $|e| = |w| - |v|$.

A *graded* \otimes -*quiver* is a graded quiver together with the data of \otimes -quiver such that $|v \otimes w| = |v| + |w|$ and $|\mathbf{1}| = 0$. The relations are the same as for a \otimes -quiver, except for relation (3) which is replaced by

$$(3') : (e \otimes \text{id}) \circ (\text{id} \otimes e') = (-1)^{|e||e'|} (\text{id} \otimes e') \circ (e \otimes \text{id}) \text{ for all pairs of edges } e, e';$$

The grading on D induces gradings on $\mathcal{P}(D)$, $\mathbb{Z}D$, and $\mathbb{Z}D^+$. In the case of the additive hull this means that every object is equipped with a decomposition into an even and an odd part. Note that morphisms are *not* required to preserve the degree. Recall that part of the data of a \otimes -quiver is the choice of edges $\alpha_{v,w} : v \otimes w \rightarrow w \otimes v$.

2.12. Definition. Let (D, \otimes) be a graded \otimes -quiver.

- (1) We define $\mathbb{Z}D^{\otimes, \text{sgn}}$ as the quotient of the category $\mathbb{Z}D$ modulo the relations of a graded \otimes -quiver. It is equipped with tensor product \otimes^{sgn} which agrees with \otimes on objects and for morphisms $\gamma : v \rightarrow v', \delta : w \rightarrow w'$

$$\gamma \otimes^{\text{sgn}} \delta = (-1)^{|\gamma||w|} \gamma \otimes \delta,$$

with associativity constraint $\beta_{u,vw}^{\text{sgn}} = \beta_{u,vw}$ and commutativity constraint given by

$$\alpha_{v,w}^{\text{sgn}} = (-1)^{|v||w|} \alpha_{v,w} : v \otimes w \rightarrow w \otimes v$$

for all objects v, w .

- (2) Let $\mathbb{Z}D^{\otimes, \text{sgn}, +}$ be the category $\mathbb{Z}D^{\otimes, +}$ with tensor structure given by the additive extension from $\mathbb{Z}D^{\otimes, \text{sgn}}$.
- (3) Set $\text{Ab}(D)^{\otimes, \text{sgn}} = \text{Ab}(\mathbb{Z}D^{\otimes, \text{sgn}, +})$ for the universal abelian category attached to $\mathbb{Z}D^{\otimes, \text{sgn}, +}$.

2.13. Remark. Note that $\mathbb{Z}D^{\otimes}$ is different from $\mathbb{Z}D^{\otimes, \text{sgn}}$ even as an additive category.

2.14. Lemma. $\mathbb{Z}D^{\otimes, \text{sgn}}$ and $\mathbb{Z}D^{\otimes, \text{sgn}, +}$ are well-defined tensor categories.

Proof. It suffices to consider $\mathbb{Z}D^{\otimes, \text{sgn}}$. We have to check that \otimes^{sgn} satisfies the axioms of a commutative tensor category. Condition (3)' ensures functoriality of \otimes^{sgn} . It is tedious but straightforward that β and α are functorial. E.g. for $\gamma : x \rightarrow x', \delta : y \rightarrow y'$ the diagram reads

$$\begin{array}{ccc} x \otimes y & \xrightarrow{(-1)^{|x||y|} \alpha} & y \otimes x \\ (-1)^{|\gamma||y|} \gamma \otimes \delta \downarrow & & \downarrow (-1)^{|\delta||x|} \delta \otimes \gamma \\ x' \otimes y' & \xrightarrow{(-1)^{|x'||y'|} \alpha} & y' \otimes x' \end{array}$$

It does not commute on the level of $\mathcal{P}(D)$. In order to check that it commutes in $\mathbb{Z}D^{\otimes, \text{sgn}}$, it is enough to treat the two special cases $\gamma = \text{id}$ or $\delta = \text{id}$ separately because $(\gamma, \delta) = (\gamma, \text{id}) \circ (\text{id}, \delta)$. In each of these cases the diagram commutes in $\mathcal{P}(D)$.

The pentagon axiom (concerning associativity) holds because it is a relation on D and no signs are involved. Unitality is preserved because $\mathbf{1}$ is of degree 0. The hexagon axiom reads

$$\begin{array}{ccccc} x \otimes (y \otimes z) & \xrightarrow{\beta} & (x \otimes y) \otimes z & \xrightarrow{(-1)^{(|x|+|y|)|z|} \alpha} & z \otimes (x \otimes y) \\ \text{id} \otimes (-1)^{|y||z|} \alpha \downarrow & & & & \downarrow \beta \\ x \otimes (z \otimes y) & \xrightarrow{\beta} & (x \otimes z) \otimes y & \xrightarrow{(-1)^{|x||z|} \alpha \otimes \text{id}} & (z \otimes x) \otimes y \end{array}$$

It commutes because the hexagon axiom holds for \otimes . \square

Again, we turn to representations. Following [HMS, Def. 8.1.3]:

2.15. Definition. Let (D, \otimes) be a graded \otimes -quiver. Let \mathcal{A} be an additive commutative tensor category. A *graded tensor representation* of (D, \otimes) is a representation $T : D \rightarrow \mathcal{A}$ of the underlying quiver together with a choice of natural isomorphisms

$$\kappa : T(u) \otimes T(v) \xrightarrow{\simeq} T(u \otimes v)$$

for all vertices $u, v \in D$, functorial in each variable and compatible with the associativity constraint and the unit in the obvious way and such that

- (1) for all vertices v, w

$$\begin{array}{ccc} T(v \otimes w) & \xrightarrow{T(\alpha)} & T(w \otimes v) \\ \kappa^{-1} \uparrow & & \downarrow \kappa \\ T(v) \otimes T(w) & \longrightarrow & T(w) \otimes T(v) \end{array}$$

commutes where the bottom arrow is $(-1)^{|v||w|}$ times the commutativity constraint in \mathcal{A} ;

- (2) for all edges $\gamma : v \rightarrow v'$ and vertices w

$$\begin{array}{ccc} T(v \otimes w) & \xrightarrow{T(\gamma \otimes \text{id})} & T(v' \otimes w) \\ \kappa \downarrow & & \downarrow \kappa \\ T(v) \otimes T(w) & \xrightarrow{T(\gamma) \otimes \text{id}} & T(v') \otimes T(w) \end{array}$$

commutes up to the factor $(-1)^{|\gamma||w|}$.

- (3) for all edges $\gamma : v \rightarrow v'$ and vertices w

$$\begin{array}{ccc} T(w \otimes v) & \xrightarrow{T(\text{id} \otimes \gamma)} & T(w \otimes v') \\ \kappa \downarrow & & \downarrow \kappa \\ T(w) \otimes T(v) & \xrightarrow{\text{id} \otimes T(\gamma)} & T(w) \otimes T(v') \end{array}$$

commutes (without signs).

The following is a graded analogue of Proposition 2.8 (2).

2.16. Proposition. *Let (D, \otimes) be a graded \otimes -quiver. The natural map $D \rightarrow \mathbb{Z}D^{\otimes, \text{sgn}, +}$ is the universal graded \otimes -representation of (D, \otimes) . In detail: it is a graded \otimes -representation and if $T : D \rightarrow \mathcal{A}$ is a graded tensor representation in an additive commutative tensor category \mathcal{A} then T factors through an induced additive tensor functor as shown*

$$\begin{array}{ccc} D & \longrightarrow & \mathbb{Z}D^{\otimes, \text{sgn}, +} \\ & \searrow T & \downarrow M^{\otimes, \text{sgn}} \\ & & \mathcal{A} \end{array}$$

Proof. The argument is the same as in the ungraded case. Relation (3)' is forced by the signs in the graded tensor representation. \square

Now consider the category $\text{Ab}(D)^{\otimes, \text{sgn}}$ as in Definition 2.12 (3).

2.17. Theorem. *The category $\text{Ab}(D)^{\otimes, \text{sgn}}$ satisfies the graded analogue of Theorem 2.9.*

Proof. As in in the ungraded case. \square

Finally:

2.18. Theorem. *Let (D, \otimes) be a graded \otimes -quiver. Let $T : D \rightarrow \mathcal{A}$ be a graded tensor representation factoring via $\mathcal{A}^{\flat} \subseteq \mathcal{A}$ a \flat -subcategory (see Definition 1.7). Then $\mathcal{A}(T)$ is a commutative tensor category and $\mathcal{A}(T) \rightarrow \mathcal{A}$ is a faithful exact tensor functor. It is universal among such representations into an abelian tensor category \mathcal{B} compatible with a faithful exact tensor functor $\mathcal{A} \rightarrow \mathcal{B}$.*

Proof. Compare with the proof of Theorem 2.10. If T is such a graded tensor representation we get $M^{\otimes, \text{sgn}} : \mathbb{Z}D^{\otimes, \text{sgn}, +} \rightarrow \mathcal{A}$ and also an induced exact tensor functor $\widetilde{M}^{\otimes, \text{sgn}} : \text{Ab}(D)^{\otimes, \text{sgn}} \rightarrow \mathcal{A}$. Denote by $\mathcal{A}(T)^{\otimes, \text{sgn}}$ the quotient of $\text{Ab}(D)^{\otimes, \text{sgn}}$ by the kernel of $\widetilde{M}^{\otimes, \text{sgn}}$. We have that $\mathcal{A}(T) \rightarrow \mathcal{A}(T)^{\otimes, \text{sgn}}$ is an equivalence. \square

3. KÜNNETH COMPONENTS

We now consider the following situation modeled for the application to Nori motives. Let D be a triangulated category, \mathcal{A} an abelian category and $R : D \rightarrow D^b(\mathcal{A})$ an exact functor. We abbreviate $H_R^i := H^i \circ R$ and $H_R^* := \bigoplus H_R^i$. The latter is understood with values in $\text{gr}\mathcal{A}$. Let $\mathcal{A}(H_R^*)$ be the universal abelian category defined by H_R^* and $\mathcal{A}(H_R^0)$ that defined by H_R^0 . The commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{H_R^*} & \text{gr}\mathcal{A} \\ & \searrow^{H_R^0} & \downarrow \\ & & \mathcal{A} \end{array}$$

induces a functor $\mathcal{A}(H_R^*) \rightarrow \mathcal{A}(H_R^0)$. We also have $\widetilde{H}_R^* : \mathcal{A}(H_R^*) \rightarrow \text{gr}\mathcal{A}$.

3.1. Definition. In the above situation let $\mathcal{A}_0(H_R^*) \subset \mathcal{A}(H_R^*)$ be the full subcategory of objects $X \in \mathcal{A}(H_R^*)$ with $\widetilde{H}_R^*(X) \in \text{gr}\mathcal{A}$ concentrated in degree 0.

The subcategory is abelian and closed under subquotients and extensions.

3.2. Remark. We are interested in the case where D is a triangulated tensor category, \mathcal{A} an abelian tensor category with an exact tensor product and R a tensor functor. Then H_R^* is a tensor functor, but H_R^0 is not. Hence while $\mathcal{A}(H_R^*)$ is a tensor category by the results of Section 1, this does not follow for $\mathcal{A}(H_R^0)$. It is, however, true for $\mathcal{A}_0(H_R^*)$. In good cases, it will be equivalent to $\mathcal{A}(H_R^0)$, giving the latter the tensor structure that we want.

3.3. Proposition. *Let D , \mathcal{A} and R be as above. Assume in addition that R can be lifted to an exact functor*

$$R : D \rightarrow D^b(\mathcal{A}_0(H_R^*)).$$

Then the natural functor

$$\mathcal{A}_0(H_R^*) \rightarrow \mathcal{A}(H_R^0)$$

is an equivalence of categories.

Proof. We abbreviate $\mathcal{A}' := \mathcal{A}_0(H_R^*)$. By assumption, there is a commutative diagram

$$\begin{array}{ccccc} D & \longrightarrow & D^b(\mathcal{A}') & \longrightarrow & D^b(\mathcal{A}) \\ & \searrow & \downarrow H^0 & & \downarrow H^0 \\ & & \mathcal{A}' & \longrightarrow & \mathcal{A} \end{array}$$

The functor $\widetilde{H}_R^0 : \mathcal{A}(H_R^0) \rightarrow \mathcal{A}$ is faithful and exact by construction. The same is true for $\widetilde{H}_R^* : \mathcal{A}' \rightarrow \text{gr}\mathcal{A}$. By definition, this functor takes values in degree 0, hence $\mathcal{A}' \rightarrow \mathcal{A}$ is also faithful and exact. This implies that the universal categories defined by $H^0 : D \rightarrow \mathcal{A}'$ and $H_R^0 : D \rightarrow \mathcal{A}$ agree. This gives $\mathcal{A}(H_R^0) \rightarrow \mathcal{A}'$ inverse to the inclusion. \square

3.4. Corollary. *Let D be a tensor triangulated category. Let \mathcal{A} be an abelian tensor category with an exact tensor product. Let $R : D \rightarrow D^b(\mathcal{A})$ be a tensor triangulated functor. Assume in addition, that R factors via $D^b(\mathcal{A}_0(H_R^*))$. Then $\mathcal{A}(H_R^0)$ carries a natural tensor structure such that $\mathcal{A}(H_R^0) \rightarrow \mathcal{A}$ is a tensor functor. If the tensor product on D is rigid and $\text{Hom}_{\mathcal{A}}$ exact in both variables, then the tensor product on $\mathcal{A}(H_R^0)$ is rigid as well.*

Proof. Combining Proposition 3.3 with the strategy of Remark 3.2 gives the tensor structure. If the tensor product on D is rigid and $\text{Hom}_{\mathcal{A}}$ exact, then by Proposition 1.11 tensor product on $\mathcal{A}(H_R^*)$ is rigid as well. Hence every object X of $\mathcal{A}_0(H_R^*)$ has a dual X^\vee in $\mathcal{A}(H_R^*)$. The object X^\vee is actually in $\mathcal{A}_0(H_R^*)$, as we can test by applying the forgetful functor to $\text{gr}\mathcal{A}$. \square

- 3.5. Remark.** (1) The use of the *bounded* derived category in the above argument is not very important. We can drop the assumption, if arbitrary direct sums exist in \mathcal{A} . This is needed in order to write down the Künneth formula or, equivalently, the tensor structure on $D(\mathcal{A})$.
- (2) We may also replace $D^b(\mathcal{A})$ by a tensor triangulated category equipped with a t -structure with heart \mathcal{A} without any change in the arguments.

There is a version of the above criterion for integral coefficients. Let D be a triangulated category. Let \mathcal{A} be an abelian tensor category with a right exact tensor product such that its derivation on $D^b(\mathcal{A})$ exists. Let $\mathcal{A}^b \subset \mathcal{A}$ be a b -subcategory as in Definition 1.7. Let $R : D \rightarrow D^b(\mathcal{A})$ be a tensor functor. Note that $H_R^* : D \rightarrow \text{gr}\mathcal{A}$ is no longer a tensor functor because $H^* : D^b(\mathcal{A}) \rightarrow \text{gr}\mathcal{A}$ is not. However:

3.6. Lemma. *In this situation, let $D^b \subset D$ be the full subcategory of objects with H_R^* in $\text{gr}\mathcal{A}^b$. Then D^b is a tensor category and*

$$H_R^*|_{D^b} : D^b \rightarrow \text{gr}\mathcal{A}^b$$

is a tensor functor satisfying the assumptions of the universal property in Proposition 1.8.

Proof. Obviously $\text{gr}\mathcal{A}^b \subset \text{gr}\mathcal{A}$ consists of flat objects and is closed under kernels. It remains to check the claim on the tensor functor with $D = D^b(\mathcal{A})$. This amounts to the naive Künneth formula for these objects. The subcategory D^b is stable under the canonical truncation functor and shift. Hence it suffices to check the formula for objects of $\mathcal{A}^b \subset D^b$. They are flat, hence the derived tensor product agrees with the

tensor product in \mathcal{A}^b . As a byproduct of the formula we see that D^b is stable under the derived tensor product. \square

We now replace $\mathcal{A}(H_R^*)$ by $\mathcal{A}(H_R^*|_{D^b})$ and set as before $\mathcal{A}_0(H_R^*|_{D^b})$ to be the subobjects concentrated in degree 0.

3.7. Corollary. *Let D be a tensor triangulated category. Let \mathcal{A} be an abelian tensor category with a right exact tensor product. Let $\mathcal{A}^b \subset \mathcal{A}$ be a b -subcategory and assume that the derived tensor product exists on $D^b(\mathcal{A})$. Let $R : D \rightarrow D^b(\mathcal{A})$ be a tensor triangulated functor. Let D^b and $\mathcal{A}_0(H_R^*|_{D^b})$ be as above.*

Assume in addition, that R factors via $D^b(\mathcal{A}_0(H_R^|_{D^b}))$. Then $\mathcal{A}(H_R^0)$ carries a natural tensor structure such that $\mathcal{A}(H_R^0) \rightarrow \mathcal{A}$ is a tensor functor.*

4. NORI MOTIVES

Recall the original definition of Nori. Let k be field, $\sigma : k \rightarrow \mathbb{C}$ an embedding. Let Sch_k be the category of schemes which are separated and of finite type over the field k . Let D^{Nori} be Nori's quiver on Sch_k having vertices (X, Y, n) where $Y \subseteq X$ is a closed subscheme and $n \in \mathbb{Z}$ and edges $(X', Y', n) \rightarrow (X, Y, n)$ for each morphism $f : X \rightarrow X'$ in Sch_k such that $f(Y) \subseteq Y'$, and an additional edge $(Y, Z, n) \rightarrow (X, Y, n+1)$ for $Z \subseteq Y \subseteq X$ closed subschemes. Let

$$H_B : D^{\text{Nori}} \rightarrow \mathbb{Z}\text{-mod}$$

be the representation given by $(X, Y, n) \rightsquigarrow H_B^n(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$ the relative singular cohomology group after base change to the complex numbers.

4.1. Definition (Nori, see also [HMS, §9]). The abelian category

$$\text{ECM}_k := \mathcal{A}(H_B)$$

is the category of effective cohomological Nori motives. There is a non-effective version that we shall denote NM_k .

4.2. Remark. The diagram D^{Nori} above agrees with the diagram $\text{Pairs}^{\text{eff}}$ of [HMS, Definition 9.1.1]. In loc.cit. the abelian categories are denoted by $\text{MM}_{\text{Nori}}^{\text{eff}}(k)$ and $\text{MM}_{\text{Nori}}(k)$, respectively. Non-effective motives are obtained either by localisation of the diagram or of the category with respect to the Lefschetz motive $\mathbf{1}(-1) = (\mathbb{G}_m, \{1\}, 1)$. This is somewhat premature at this point as it involves the tensor structure. We are going to concentrate on the effective case.

Tensor product via graded \otimes -quivers. Let $D^{\text{Nori}, \otimes}$ be the same quiver with, in addition, the following structure of a graded \otimes -quiver in the sense of Definition 2.11. The grading is given by

$$(X, Y, n) \mapsto \bar{n} \in \mathbb{Z}/2\mathbb{Z}$$

For vertices $(X, Y, n), (X', Y, n')$ we put

$$(X, Y, n) \otimes (X', Y', n') := (X \times_k X', X \times_k Y' \cup Y \times_k X', n + n')$$

making use of the product in Sch_k . We choose the vertex $\mathbf{1}$ and the edges $\text{id}, \alpha, \beta, \beta', u, u'$ in the canonical way, e.g., the unit $\mathbf{1} = (\text{Spec}(k), \emptyset, 0)$,

$$u : (X, Y, n) \rightarrow (\text{Spec}(k), \emptyset, 0) \otimes (X, Y, n)$$

and $u' : (\mathrm{Spec}(k), \emptyset, 0) \otimes (X, Y, n) \rightarrow (X, Y, n)$ the canonical maps. As relations we use the relations required by Definition 2.11. All this is completely parallel to [HMS, §9.3]. By construction we obtain a graded \otimes -quiver.

Recall that singular cohomology H_B^* is provided with a natural cross or external product

$$\kappa_{n,n'}^B : H_B^n(X, Y) \otimes H_B^{n'}(X', Y') \rightarrow H_B^{n+n'}(X \times_k X', X \times_k Y' \cup Y \times_k X')$$

Note that the representation H_B is *not* a \otimes -representation since $\kappa_{n,n'}^B$ fails to be an isomorphism, in general.

Following Nori, we set $D^{\mathrm{good}, \otimes}$ for the full sub- \otimes -quiver of vertices (X, Y, n) such that $H_B^*(X, Y)$ is concentrated in degree n and free as a \mathbb{Z} -module.

4.3. Lemma. *Betti cohomology $H_B^{\mathrm{good}} := H_B|_{D^{\mathrm{good}}}$: $D^{\mathrm{good}, \otimes} \rightarrow \mathbb{Z}\text{-mod}$ is a graded \otimes -representation with values in the subcategory $(\mathbb{Z}\text{-mod})^b$ of free \mathbb{Z} -modules of finite type.*

Proof. On good pairs, the map $\kappa_{n,n'}^B$ is indeed an isomorphism by the Künneth formula. The relations of the tensor quiver are all mapped to equalities in $\mathbb{Z}\text{-mod}$ by the standard properties of singular cohomology. Most are checked explicitly in [HMS, Proposition 9.3.1]. The remaining ones (e.g., concerning the inverse u' of u) are obvious. \square

Indeed, our definition of a graded \otimes -quiver was modeled on this case.

4.4. Corollary. *The abelian category $\mathcal{A}(H_B^{\mathrm{good}})$ carries a natural \otimes -structure compatible with the forgetful functor to $\mathbb{Z}\text{-mod}$.*

Proof. See Theorem 2.18. \square

Nori's Basic Lemma comes into play in comparing the universal categories for the two diagrams.

4.5. Theorem (Nori, see [HMS, Theorem 9.2.22]). *The quiver D^{Nori} can be represented in $\mathcal{A}(H_B^{\mathrm{good}})$ in a compatible way with H_B . In particular,*

$$\mathrm{ECM}_k \cong \mathcal{A}(H_B^{\mathrm{good}})$$

carries a natural tensor structure.

In the above, we are copying Nori's approach, but replace his approach to the universal abelian category and its tensor product with the one developed in this paper. We now turn to yet a different approach which does not mention D^{Nori} and D^{good} (at least not obviously so).

Tensor product via triangulated motives. Let $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ be Voevodsky's category of geometric motives over k with rational coefficients. Let

$$R_B : \mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \rightarrow D^b(\mathbb{Q}\text{-vsp.})$$

be the Betti-realisation. It maps the motive of an algebraic variety to its singular cochain complex.

4.6. *Remark.* The existence of the Betti-realisation is completely straightforward. The first reference with rational coefficients is [Hu1], [Hu2] as a byproduct of a functor into mixed realisations. With integral coefficients it is formulated in [Ha]. In the original literature on motives, realisation functors were usually contravariant. This is also the viewpoint taken in the above references.

More recently, Voevodsky and then Ayoub who, in [Ay] constructs Betti-realizations for motives over any base, has been using the covariant point of view.

For our application, it does not matter which point of view is taken. We fix on the contravariant one because we want to refer to [Ha] later on.

4.7. **Definition.** Let $\mathrm{MM}_k := \mathcal{A}(H_B^0)$ be the universal abelian category defined by the Betti-realisation.

Based on a sketch of Nori, Harrer (see [Ha]) was able to show:

4.8. **Theorem** (Harrer [Ha, Thm 7.3.1]). *The Betti-realisation factors naturally via the bounded derived category of MM_k and even that of $\mathcal{A}_0(H_B^*)$.*

4.9. *Remark.* The proof is based on Nori's basic lemma: for every affine variety X and subvariety Y , there is a subvariety $X \supset Z \supset Y$ such that the singular cohomology of the pair (X, Z) is concentrated in the degree equal to the dimension of X , i.e., $(X, Z, \dim X)$ is a good pair. As pointed out by Nori, this can be used in order to construct, for every affine variety X , a natural complex of motives. Using Čech-complexes, this extends to all varieties. Harrer's main effort was to establish functoriality of the construction with respect to finite correspondences. When working with rational coefficients (as we do), functoriality with respect to morphisms is enough, see [I2], [HMS]. Harrer's result is formulated for NM_k , but actually proved for $\mathcal{A}(H_B^{\mathrm{good}})$. The same proof also works without change for $\mathrm{MM}_k = \mathcal{A}(H_B^0)$ and even the refinement $\mathcal{A}_0(H_B^*)$.

4.10. **Theorem.** *The category MM_k carries a natural tensor structure such that $\mathrm{MM}_k \rightarrow \mathbb{Q}\text{-vsp.}$ is a tensor functor and $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q}) \rightarrow D^b(\mathrm{MM}_k)$ is a triangulated tensor functor.*

In particular, MM_k is Tannakian.

Proof. We apply Corollary 3.4 to the rigid tensor category $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ and the Betti-realisation. The assumption is satisfied by Theorem 4.8. This makes MM_k a rigid tensor category; the Betti-realisation $\mathrm{MM}_k \rightarrow \mathbb{Q}\text{-vsp.}$ is a fibre functor. \square

4.11. **Definition.** The *motivic Galois group* of k is defined as the Tannakian dual of the MM_k .

4.12. **Proposition.** *MM_k is naturally equivalent to Nori's original category, i.e., $\mathrm{NM}_k \cong \mathrm{MM}_k$. The motivic Galois group is naturally isomorphic to Nori's original motivic Galois group.*

Proof. For the abelian category, this is already shown in [HMS]. The tensor structures are based on the Künneth formula. In each case it is uniquely determined by its value for very good pairs, hence they are the same. The statement about the motivic Galois group follows. \square

4.13. *Remark.* The whole argument can also be done for motives with coefficients in any field (including finite fields) or Dedekind domain (in particular the integers).

Corollary 3.7 can be used instead of the more straightforward Corollary 3.4. Harrer’s work in see [Ha] handles the integral case. Note that Nori’s original construction gives a tensor category of motives with coefficients in any Noetherian ring R , so our result is actually weaker.

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