EXPONENTIAL PERIODS AND O-MINIMALITY I

JOHAN COMMELIN, PHILIPP HABEGGER, AND ANNETTE HUBER

ABSTRACT. Let $\alpha \in \mathbb{C}$ be an exponential period. This is the first part of a pair of papers where we show that the real and imaginary part of α are up to signs volumes of sets definable in the o-minimal structure generated by $\mathbb{Q}$, the real exponential function and $\sin|_{[0,1]}$. This is a weaker analogue of the precise characterisation of ordinary periods as numbers whose real and imaginary part are up to signs volumes of $\mathbb{Q}$-semi-algebraic sets. Furthermore, we define a notion of naive exponential periods and compare it to the existing notions using cohomological methods. This points to a relation between the theory of periods and o-minimal structures.

INTRODUCTION

Exponential periods are, roughly speaking, complex numbers of the form

$$\int_\sigma \mathrm{e}^{-f}\omega \tag{1}$$

where ω is an algebraic differential form, f an algebraic function and σ a domain of integration of algebraic nature. They have a conceptual interpretation as entries of the period matrix between twisted de Rham cohomology and rapid decay homology; more on this later.

This paper is the first of two parts on exponential periods and o-minimality. The aim of these papers is to give several definitions that make (1) precise, and to compare these different definitions. Together, the papers prove the following main result.

Theorem 0.1 ([CH20, Theorem 13.4]). *Let $k \subset \mathbb{C}$ be a subfield such that k is algebraic over $k_0 = k \cap \mathbb{R}$. The following subsets of $\mathbb{C}$ agree:*

(1) naive exponential periods over k;
(2) cohomological exponential periods of triples (X, Y, f) where X is a smooth variety over k, $Y \subset X$ is a simple normal crossings divisor and $f \in \mathcal{O}(X)$ is a regular function;
(3) periods of effective exponential motives over k.

Additionally, for every such number its real and imaginary part are up to signs volumes of compact subsets of $\mathbb{R}^n$ definable over k_0 in the o-minimal structure $\mathbb{R}_{\sin,\exp}$.

The most interesting case from the number theoretic point of view is $k = \mathbb{Q}$, or equivalently, $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}} \cap \mathbb{R}$.

Let us now explain the notions appearing in this theorem.

Date: July 16, 2020.

0.1. **Naive exponential periods.** We propose the following very explicit definition as one way of making (1) precise.

Definition 0.2. Let $k \subset \mathbb{C}$ be a subfield such that k is algebraic over $k \cap \mathbb{R}$.

A *naive exponential period* over k is a complex number of the form

$$\int_G \mathrm{e}^{-f}\omega$$

where $G \subset \mathbb{C}^n$ is a pseudo-oriented (not necessarily compact) closed $(k \cap \mathbb{R})$-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n_k$ that is regular on G and f is a rational function on $\mathbb{A}^n_k$ such that f is regular and proper on G and, moreover, $f(G)$ is contained in a strip

$$S_{r,s} = \{z \in \mathbb{C} \mid \Re(z) > r, |\Im(z)| < s\}.$$

A pseudo-orientation on G is the choice of an orientation on a $(k \cap \mathbb{R})$-semi-algebraic open subset whose complement has positive codimension (and hence measure 0), see Definition 3.14.

We check that these integrals converge absolutely. In the case $f = 0$, we recover the notion of an (ordinary) naive period as introduced by Friedrich in [Fri04], see [HMS17, Definition 12.1.1]. The definition of a naive exponential period is not identical to the definition given by Kontsevich–Zagier in [KZ01, §4.3]. See Section 5.4 for more details about the difference.

0.2. **On o-minimality.** In his "Esquisse d'un Programme", Grothendieck set forth the need for, and the principles of, some form of "tame" topology. O-minimality provides a good theory of "tame" subsets of $\mathbb{R}^n$, avoiding Cantor sets, fractals, the graph of a space-filling curve and $\sin(1/x)$. In recent years, o-minimality has seen spectacular applications in algebraic geometry, most notably as an important tool in the proof of the André–Oort conjecture for $\mathcal{A}_g$ by Tsimerman following work of many people, see the survey [KUY18].

The 'o' in "o-minimality" stands for "order". The concept was first introduced by in work of Van den Dries in [vdD84] and Pillay–Steinhorn [PS84] at about the same time that Grothendieck was writing his "Esquisse d'un Programme". We recall the definition and some basic properties of o-minimal structures in Section 2.

By work of Wilkie [Wil96], Van den Dries and Miller [vdDM94], the structure of subsets of $\mathbb{R}^n$ defined using $+, \cdot, <$, the elements of $k \cap \mathbb{R}$, the real exponential function exp, and the restriction of the analytic function sin to the bounded interval $[0, 1]$ is an example of an o-minimal structure. We denote it by $\mathbb{R}_{\sin,\exp,k}$.

Theorem 0.3 (See Theorem 5.12)**.** *Let α be a naive exponential period over k. Then its real and imaginary part are up to signs volumes of compact subsets of $\mathbb{R}^n$ definable in $\mathbb{R}_{\sin,\exp,k}$.*

This generalises a result for ordinary periods: their real and imaginary part are volumes of compact semi-algebraic sets, see [HMS17, Proposition 12.1.6] together with [VS15]. There is a significant difference though: in the case of ordinary periods, we also have the converse implication. The volume of a compact $\mathbb{Q}$-semi-algebraic set is by definition a naive period. This is no longer clear or even expected in the exponential setting. The definable subsets appearing in the theorem are of a special shape. For example, we do not need to iterate the functions exp and $\sin|_{[0,1]}$. The number e^{e} is definable in the o-minimal structure and hence also appears as a volume. We do not expect it to be an exponential period.

Question 0.4. Is there a natural way to charaterise definable sets whose volumes are naive exponential periods?

0.3. **Exponential periods and cohomology.** The origins of the theory of exponential periods lie in a version of Hodge theory for vector bundles with irregular connections. To our knowledge such a theory was first considered by Deligne, see [DMR07, p. 17]. A systematic study of the period isomorphism was started by Bloch and Esnault in [BE00], and fully developed by Hien [Hie07]. He establishes a period isomorphism between de Rham cohomology of the connection and a suitable homology theory. The special and central case of exponential connections is treated by Hien and Roucairol [HR08]. If X is a smooth variety over a field $k \subset \mathbb{C}$, $f \in \mathcal{O}(X)$ a regular function, they consider the twisted de Rham complex Ω^*_f with differential $\omega \mapsto d\omega - df \wedge \omega$. Its hypercohomology is *twisted de Rham cohomology.* They define rapid decay homology of X^{an} (see Section 6.1) taking the role of singular cohomology in the classical case and a period pairing

$$H_n^{\mathrm{rd}}(X, \mathbb{Q}) \times H^n_{\mathrm{dR}}(X, f) \to \mathbb{C}$$

inducing a perfect pairing after extending scalars to $\mathbb{C}$. The numbers in the image of the pairing are the exponential periods. Their study in their own right was proposed by Kontsevich and Zagier in [KZ01].

As in the classical case, the theory can be extended to singular varieties and also relative cohomology. A full-fledged theory of exponential motives is being developed by Fresán–Jossen in [FJ20]. Their books also contains a very accessible account of the constructions and the proof of the period isomorphism. They also give many examples of interesting numbers that appear as exponential periods.

We prove:

Theorem 0.5 (Propositions 12.1 and 11.1 of [CH20])**.** *A complex number α is a naive exponential period over k if and only if there is a smooth variety X over k, a simple normal crossings divisor Y, and $f \in \mathcal{O}(X)$ such that α is in the image of the period pairing*

$$H_n^{\mathrm{rd}}(X, Y, \mathbb{Q}) \times H^n_{\mathrm{dR}}(X, Y, f) \to \mathbb{C}.$$

Again this generalises the result for ordinary periods, see [HMS17, Theorem 12.2.1]. Actually, the theorem also holds for general X and Y or even all periods of effective exponential Nori motives, see [CH20, Theorem 13.4]. The general proof is quite technical. In the present paper, we include the arguments in the curve case, see Section 8. It is more accessible, yet already contains the main ideas.

0.4. **Method of proof.** The strategy is very similar to the case of ordinary periods. Algebraic varieties admit triangulations by semi-algebraic simplices.

This allows us to represent homology classes by semi-algebraic sets. In the simplest case the period pairing on cohomology has the shape

$$(\sigma, \omega) \mapsto \int_\sigma \mathrm{e}^{-f}\omega,$$

suggesting the relation to naive periods. Conversely, the Zariski closure of a semi-algebraic set G is an algebraic variety X, and the Zariski closure of its boundary is a closed subvariety $Y \subset X$.

The main new tool compared to the classical case is the real oriented blow-up of a smooth analytic variety at some divisor. In the simplest case of $\mathbb{P}^1$ and the divisor ∞, it is the compactification of $\mathbb{C}$ by a circle at infinity. The points correspond to the directions of half rays. Its use is of long standing in the theory of irregular connections. Hien and Roucairol and also the exposition of Fresán–Jossen use it to establish the period isomorphism in the exponential case. Indeed, rapid decay homology of X can be computed as the homology of a certain partial compactification $B^\circ(X^{\mathrm{an}}, f)$ of X^{an} relative to its boundary, see Proposition 6.5. For details on $B^\circ(X^{\mathrm{an}}, f)$ see Definition 6.3 and Section 8.2. It is still semi-algebraic, more precisely, a semi-algebraic manifold with corners.

However, this is not yet enough to bound the imaginary part of $f(G)$, something that is crucial in showing that $\int_G \mathrm{e}^{-f}\omega$ is the volume of a definable set in the o-minimal structure $\mathbb{R}_{\exp,\sin,k}$. Recall that the complex exponential is *not* definable, only the real exponential and sin (or cos) restricted to bounded intervals. We introduce a smaller semi-algebraic subset $B^\sharp(X, f)$ of $B^\circ(X, f)$. The actual key step in the proof of our main theorem is the comparison between the homology of $B^\sharp(X, f)$ and $B^\circ(X, f)$ in [CH20, Proposition 11.4]. In the simplest case, they agree because a half-circle is contractible to a single point.

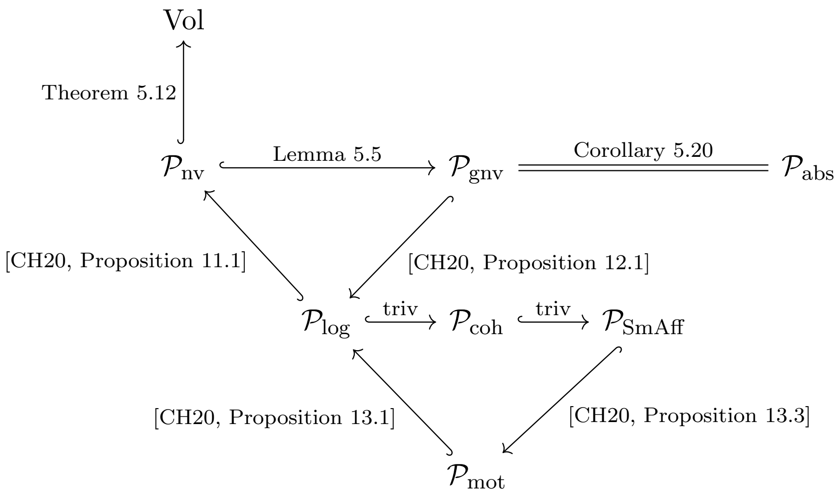
There are two reasons for the considerable length of the present paper and its companion: on the one hand, we aim for readers without a background in o-minimality and/or in the classical theory of periods and have chosen to reproduce definitions from the literature and to give detailed arguments and references. We have also added a section on the case of curves that is not needed for the proof of the main theorems, but should be more accessible and still uses all of the main ideas.

On the other hand, we ran into many technical problems. The first two are addressed in the present paper, the last in [CH20].

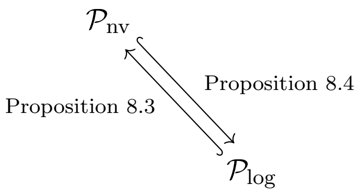
- For example, we do not know if the real oriented blow-up of a smooth variety can be embedded into $\mathbb{R}^n$ preserving both the semi-algebraic and differentiable structure. Instead we introduce the notion of a semi-algebraic (or more general: definable) manifold, at the price of having to extend results well-known for semi-algebraic subsets of $\mathbb{R}^n$ to the manifold setting.
- The standard triangulation results in semi-algebraic geometry or for sets definable in an o-minimal structure only give facewise differentiability of the simplices. This is not strong enough for a straightforward application of the Theorem of Stokes—something that we need for a well-defined period pairing depending only on homology classes. Our way out is by a result of Ohmoto–Shiota [OS17] who prove

the existence of C^1-parametrisations with applications to periods in mind. We can then use a subtle version of Stokes's theorem proved by Whitney in [Whi57] for "regular" differentials on C^1-manifolds.
- Finally, the period isomorphism has a simple description only in the case of a smooth affine variety. The general case is handled by hypercovers. This involves some checking of strict compatibilities between our real oriented blow-ups and their subspaces and a check that the abstract period pairing is still realised by integration.

0.5. **Structure of the papers.** The following diagram explains the global structure of the two papers, and how the different theorems contribute to the main comparison result.



Section 8 proves part of the central triangle in the case of curves. In this special case the main ideas of the proof are present, but several delicate problems are avoided.



0.6. **Outlook.** Our comparison results point to a deeper relation between periods and o-minimal theory. While the case of ordinary periods—with their incarnations as entries of periods matrices or as volumes of semi-algebraic sets—might be seen as a coincidence, this second instance suggests that this is not the case. Bakker, Brunebarbe, Klingler and Tsimerman have been pursuing a project of making a systematic use of tame geometry in Hodge theory and apply it successfully to questions related to the Hodge conjecture. A central tool was their GAGA theory merging complex spaces with o-minimal geometry. We hope that the period isomorphism can also be extended to the o-minimal setting, providing a new point of view on period numbers.

Acknowledgements. Many thanks to Amador Martin-Pizarro for teaching two of us (Commelin and Huber) not only the formalism but also the intuition of o-minimal theory. The review of o-minimal theory in Section 2 owes a lot to his talks. We also thank Fabrizio Barroero and Reid Barton for discussions on o-minimality and Lou van den Dries for carefully explaining aspects of C^p-cell decomposition and triangulation.

We thank Marco Hien, Ulf Persson and Claus Scheiderer for answering questions on the real oriented blow-up. Stefan Kebekus shared his insights on blow-ups in algebraic geometry. Finally, we appreciated the help of Nadine Große with the theory of integration.

We thank Amador Martin-Pizarro and Javier Frésan for their comments on our first draft.

1. Notation

1.1. **Fields of definition.** If z is a complex number, we write $\Re(z)$ and $\Im(z)$ for its real and imaginary part. Let $k \subset \mathbb{C}$ be a subfield. We denote by k_0 the intersection $k \cap \mathbb{R}$, by $\bar{k}$ the algebraic closure of k in $\mathbb{C}$, and by $\tilde{k}$ the real closure of k_0 in $\mathbb{R}$. Note that k is not automatically algebraic over k_0. (For example, let $a, b \in \mathbb{R}$ such that $\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(a,b)) = 2$, and consider $k = \mathbb{Q}(a+bi)$. Then $k_0 = \mathbb{Q}$.) The following conditions on k are equivalent:

$$k_0 \subset k \text{ is alg.} \iff k_0 \subset \bar{k} \text{ is alg.} \iff \tilde{k} \subset \bar{k} \text{ is alg.} \iff [\bar{k} : \tilde{k}] = 2.$$

If k satisfies these conditions, so does every intermediate extension $k \subset L \subset \mathbb{C}$ with $k \subset L$ algebraic.

1.2. **Categories of varieties.** Let $k \subset \mathbb{C}$ be a subfield. By variety we mean a quasi-projective reduced separated scheme of finite type over k. By X^{an} we denote the associated analytic space on $X(\mathbb{C})$.

1.3. **Good compactifications.** We say that a (X, Y) is a *log-pair* if X is smooth of pure dimension d, and Y a simple normal crossings divisor. A *good compactification* of (X, Y) is the choice of an open immersion $X \subset \bar{X}$ such that $\bar{X}$ is smooth projective, X is dense in $\bar{X}$ and $\bar{Y} + X_\infty$ is a simple normal crossings divisor where $\bar{Y}$ is the closure of Y in $\bar{X}$ and $X_\infty = \bar{X} \smallsetminus X$. If, in addition, we have a structure morphism $f : X \to \mathbb{A}^1$, we say that $\bar{X}$ is a good compactification relative to f if f extends to $\bar{f} : \bar{X} \to \mathbb{P}^1$. Let $f : X \to \mathbb{A}^1$ be a morphism. Consider the graph of f in $X \times \mathbb{A}^1$ and take its Zariski closure $\bar{X}''$ inside $\bar{X}' \times \mathbb{P}^1$, where $\bar{X}'$ is a projective variety containing X as a Zariski open and dense subset. We may consider X as a Zariski open and dense subset of $\bar{X}''$. The projection $\bar{X}'' \to \mathbb{P}^1$ is a morphism that extends f. By applying Hironaka's Theorem we see that a good compactification relative to f exists.

1.4. **Some semi-algebraic sets.** Let k be as in Section 1.1. Let X be a smooth variety, $\bar{X}$ a good compactification, $X_\infty = \bar{X} \smallsetminus X$. We denote by $B_{\bar{X}}(X)$ the oriented real blow-up of $\bar{X}^{\mathrm{an}}$ in X_∞^{an}, for details see Definition 4.2. It is a k_0-semi-algebraic C^∞-manifold with corners, see Proposition 4.3.

In the case $X = \mathbb{A}^1$, $\bar{X} = \mathbb{P}^1$, we write $\tilde{\mathbb{P}}^1 = B_{\mathbb{P}^1}(\mathbb{A}^1)$. This is a manifold with boundary: the compactification of $\mathbb{C} \cong \mathbb{R}^2$ by a circle at infinity, one for each half ray. For $s \in \mathbb{C} \smallsetminus \{0\}$, we write $s\infty$ for the point of $\partial\tilde{\mathbb{P}}^1$ corresponding to the half ray $s[0,\infty)$. We say $\Re(s\infty) > 0$ if $\Re(s) > 0$. We put

$$\begin{aligned} B^\circ &= \tilde{\mathbb{P}}^1 \smallsetminus \{s\infty \in \partial\tilde{\mathbb{P}}^1 \mid \Re(s\infty) \le 0\} = \mathbb{C} \cup \{s\infty \mid \Re(s) > 0\}, \\ \partial B^\circ &= B^\circ \smallsetminus \mathbb{C} = \{s\infty \mid \Re(s) > 0\}, \\ B^\sharp &= \tilde{\mathbb{P}}^1 \smallsetminus \{s\infty \in \partial\tilde{\mathbb{P}}^1 \mid s\infty \ne 1\infty\} = \mathbb{C} \cup \{1\infty\}, \\ \partial B^\sharp &= B^\sharp \smallsetminus \mathbb{C} = \{1\infty\}. \end{aligned}$$

If $G \subset \mathbb{R}^n$ is k_0-semi-algebraic, we will also denote by ∂G the complement $G \smallsetminus G^{\mathrm{int}}$ where G^{int} is the interior of G inside $X(\mathbb{R})$ where X is the Zariski-closure of G in $\mathbb{A}^n_{k_0}$. If G is of dimension d, then ∂G is of dimension at most $d-1$.

This agrees with the notation above. Note that we do not assume that G is closed.

1.5. **C^1-homology.** In this paper, we denote by Δ_n the standard simplex as normalised in [War83]:

$$\Delta_n = \left\{ (x_1, \dots, x_n) \mid x_i > 0 \text{ and } \sum_i x_i < 1 \right\} \subset \mathbb{R}^n.$$

It is open in the ambient space. We denote by $\bar{\Delta}_n$ its closure in $\mathbb{R}^n$. We fix the standard orientation. We define the face maps $k^i \colon \bar{\Delta}_{n-1} \to \bar{\Delta}_n$ as in [War83, (2) p.142]. Moreover, for any topological space X and subspace Y, we let $H_n(X, Y; R)$ denote n-th singular homology with coefficients in the ring R.

A *manifold with corners* is a second countable Hausdorff topological space for which every point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n \times \mathbb{R}^m_{\ge 0}$. Say $p \ge 1$. We will assume that each manifold with corners is equipped with a set of charts which need not be maximal; later on this set will be finite. A map defined on a subset A of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is called C^p if it extends to a C^p map on an open neighborhood of A in $\mathbb{R}^n$ with values in $\mathbb{R}^m$. A *C^p-manifold with corners* is a manifold with corners such that all transition maps betweens charts are C^p. A map between two C^p-manifolds with corners is called C^p if it is C^p on all charts.

Definition 1.1. A *C^1-simplex* on X is a continuous map

$$\sigma : \bar{\Delta}_n \to X$$

such that for any chart $\phi : U \to V \subset \mathbb{R}^n \times \mathbb{R}^m_{\ge 0}$ with U open in X the composition $\phi \circ \sigma|_{\sigma^{-1}(U)} : \sigma^{-1}(U) \to V$ extends to a C^1-map on an open neighbourhood of $\sigma^{-1}(U)$ in $\mathbb{R}^n$ with target $\mathbb{R}^{n+m}$.

Let $S_n(X)$ be the space of formal $\mathbb{Q}$-linear combinations of C^1-simplices of dimension n. For $A \subset X$ closed, we denote $S_n(A) \subset S_n(X)$ the subspace spanned by simplices with image in A.

The restriction of σ to a face is again C^1, hence the usual boundary operator ∂ turns $S_*(X)$ into a complex. The barycentric subdivision of a C^1-simplex is again C^1.

Remark 1.2. If ω is an n-form of class C^1, then $\sigma^*\omega = g\,\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$ for a C^0-function g on $\bar{\Delta}_n$. Hence the Lebesgue integral converges (absolutely).

Theorem 1.3. *Let X be a C^1-manifold with corners. Then the complex $S_*(X)$ of C^1-chains computes singular homology of X and $S_*(X)/S_*(\partial X)$ computes singular homology of X relative to its boundary ∂X.*

Proof. It is equivalent to prove the result in cohomology instead. The argument for the C^∞-case and smooth manifolds is given in [War83, Section 5.31]. It works without changes in the C^1-case, even for a manifold with corners.

The boundary ∂X is not a C^1-manifold itself, but only a closed subset in a C^1-manifold with corners. The constructions of loc. cit. still apply. E.g. the partition of unity needed on p. 193 is constructed on X, not on ∂X. On p. 194/196, U is not an open ball (the manifold case) or the intersection of an open ball with $\mathbb{R}^{n_1}_{\geq 0} \times \mathbb{R}^{n_2}$ (the case of a manifold with corners) but the boundary of the latter. If σ is a simplex with values in the boundary, then so does $\tilde{h}_p(\sigma)$ of Equation (21). □

Theorem 1.4 ([Whi57, Chapter III, §§16-17])**.** *Let X be a C^2-manifold with corners. Let ω be an n-form of class C^1 on X and let $\sigma : \bar{\Delta}_{n+1} \to X$ be a C^1-simplex. Then*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Proof. We first recall the notion of *regular* differential form in Euclidean space introduced by Cartan [Whi57, Section 16]. (Warning: this notion is unrelated to the usual concept of regularity in algebraic geometry.) A continuous r-form ω on an open subset $U \subset \mathbb{R}^n$ is regular if there exists a continuous and necessarily unique $(r+1)$-form ω' (then called $d\omega$) such that for every oriented linear $(r+1)$-simplex $\Delta \subset U$ we have

$$\int_\Delta \omega' = \int_{\partial\Delta} \omega.$$

Let ω and σ be as in the hypothesis. After passing to a barycentric subdivision we may assume that σ takes values in the domain of a chart $U \to V \subset \mathbb{R}^l \times \mathbb{R}^m_{\geq 0}$, with $U \subset X$ open. Working in this chart, σ extends to a C^1-map on a open neighbourhood $\Omega \subset \mathbb{R}^{n+2}$ of $\bar{\Delta}_{n+1}$ with target $\mathbb{R}^{l+m}$. We may identify ω with a C^1-form on an open subset $R \subset \mathbb{R}^{l+m}$ containing the image of Ω.

Note that Whitney's notion of smooth means C^1 in modern terms, see loc. cit. p. 15. Thus ω is regular on R and ω' is the usual $d\omega$ by Stokes's Theorem for linear simplices, see [Whi57, Theorem 14A].

By [Whi57, Theorem 16B], the pull-back of a regular form under a C^1-map is again regular and commutes with d. So $\sigma^*\omega$ is regular on Ω with $d\sigma^*\omega = \sigma^* d\omega$. By definition of regularity, we have $\int_{\bar{\Delta}_{n+1}} \sigma^* d\omega = \int_{\partial\bar{\Delta}_{n+1}} \sigma^*\omega$ when considering $\bar{\Delta}_{n+1}$ as a linear simplex in $\mathbb{R}^{n+2}$ with the usual orientation. So the formula of the Theorem holds. □

2. O-minimal structures

For the purposes of our paper it is helpful to think of o-minimal geometry as a generalisation of semi-algebraic geometry. The canonical reference for o-minimality is [vdD98]. Within the encyclopedia of mathematics, o-minimality is firmly rooted in the field of mathematical logic and more particularly model theory. In this section we briefly survey the essentials in a fashion that is geared towards geometers with no background in model theory. The reader is warned in advance that some of the definitions presented below are severe mutations of more general concepts in model theory.

Definition 2.1 (§2.1 of [vdD98])**.** A *structure* on a non-empty set R is a sequence $\mathcal{S} = (\mathcal{S}_m)_{m\in\mathbb{Z}_{\geq 0}}$ such that for each $m \geq 0$

(1) $\mathcal{S}_m$ is a boolean subalgebra of the power set $\mathcal{P}(R^m)$: that is, $\varnothing \in \mathcal{S}_m$, and $\mathcal{S}_m$ is closed under complements and binary unions and intersections;
(2) if $A \in \mathcal{S}_m$, then $R \times A$ and $A \times R$ belong to $\mathcal{S}_{m+1}$;
(3) $\{(x_1, \ldots, x_m) \in R^m \mid x_1 = x_m\} \in \mathcal{S}_m$;
(4) if $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$, where $\pi\colon R^{m+1} \to R^m$ is the projection onto the first m coordinates.

We are actually only going to need the case $R = \mathbb{R}$, but in this section we will present the definitions in the general setting.

Definition 2.2. Let $k \subset \mathbb{R}$ be a subfield. An example of a structure that is relevant to the topic of this paper is the structure of *k-semi-algebraic sets* over $\mathbb{R}$ consisting of those subsets of $\mathbb{R}^m$ that are of the form

$$\big\{x \in \mathbb{R}^m \mid f_1(x) = \ldots = f_k(x) = 0 \text{ and } g_1(x) > 0, \ldots, g_l(x) > 0\big\}$$

for some polynomials $f_i, g_j \in k[X_1, \ldots, X_m]$.

It is a non-trivial fact that the collection of semi-algebraic sets satisfies the final condition in Definition 2.1. This result is known as the Tarski–Seidenberg theorem. The structure does not change when we replace k by an algebraic subextension in $\mathbb{R}$, hence we may assume k to be real closed.

A structure can often be "generated" by a smaller collection of sets. This leads to the following concept (one that is more faithful to the model-theoretic point of view). We follow the terminology of [vdD98].

Definition 2.3 (Def 5.2 of [vdD98])**.** A *model theoretic structure* $\mathcal{R} = (R, (S_i)_{i\in I}, (f_j)_{j\in J})$ consists of a set R, called its *underlying set*, relations $S_i \subset R^{m(i)}$ $(i \in I, m(i) \in \mathbb{N}_0)$, and functions $f_j\colon R^{n(j)} \to R$ $(j \in J, n(j) \in \mathbb{N}_0)$. If $n(j) = 0$, we call f_j a *constant* and identify it with its unique value.

If $\mathcal{R} = (R, (S_i)_{i\in I}, (f_j)_{j\in J})$ is a model theoretic structure, and $C \subset R$ a subset, then we denote the model theoretic structure $(R, (S_i)_{i\in I}, (f_j)_{j\in J}, (c)_{c\in C})$ by $\mathcal{R}_C$. The elements of C are called *parameters*.

Definition 2.4 (§5.3 of [vdD98])**.** (1) Let $\mathcal{R} = (R, (S_i)_{i\in I}, (f_j)_{j\in J})$ be a model theoretic structure. We denote with $\mathrm{Def}(\mathcal{R})$ the smallest structure on R that contains the S_i, for $i \in I$, and the graphs of the functions f_j (for $j \in J$).
(2) A subset $A \subset R^m$ is called *definable* in $\mathcal{R}$ if $A \in \mathrm{Def}(\mathcal{R})_m$. A function $f\colon R^m \to R^n$ is definable in $\mathcal{R}$ if its graph $\Gamma(f) = \{(x, y) \mid y = f(x)\} \subset R^m \times R^n = R^{m+n}$ is definable in $\mathcal{R}$. A point $x \in R^m$ is definable in $\mathcal{R}$ if the singleton $\{x\} \subset R^m$ is definable in $\mathcal{R}$.
(3) Let $C \subset R$ be a subset. A subset/function/point is *definable in $\mathcal{R}$ with parameters from C* or *definable over C in $\mathcal{R}$* or *C-definable in $\mathcal{R}$* if it is definable in $\mathcal{R}_C$.

The following Proposition serves two purposes: it makes the relation of the previous definitions with logic apparent, and it is a useful result for showing that certain sets are definable.

Proposition 2.5. *If $\mathcal{R} = (R, (S_i)_{i\in I}, (f_j)_{j\in J})$ is a model theoretic structure, and $C \subset R$ a subset, then a subset $A \subset R^m$ is definable in $\mathcal{R}$ with parameters from C if and only if there exists a formula $\phi[x_1, \dots, x_m, y_1, \dots, y_n]$ in "the first-order language of $\mathcal{R}$" and elements $c_1, \dots, c_n \in C$ such that*

$$A = \{(a_1, \dots, a_m) \in R^m \mid \phi[a_1, \dots, a_m, c_1, \dots, c_n]\}.$$

Proof. See [vdD98, Exercise 1, Chapter 1.5]. □

Example 2.6.

(1) From now on, we will denote by $\mathbb{R}_{\text{alg}}$ the model theoretic structure $(\mathbb{R}, <, 0, 1, +, \cdot)$ and (consistent with Definition 2.4) for every subfield $k \subset \mathbb{R}$ we denote by $\mathbb{R}_{\text{alg},k}$ the model theoretic structure obtained from $\mathbb{R}_{\text{alg}}$ by adding elements in k as constants. This is justified by the fact that the structure $\text{Def}(\mathbb{R}_{\text{alg},k})$ consists precisely of the k-semi-algebraic sets introduced in Definition 2.2. Indeed, they are defined by first-order formulas in the language of $\mathbb{R}_{\text{alg}}$ with parameters from k.

(2) Let $A \subset \mathbb{R}^m$ be a k-semi-algebraic set. Using Proposition 2.5 it becomes straightforward to show that the topological closure $\bar{A} \subset \mathbb{R}^m$ is semi-algebraic. Indeed

$$\bar{A} = \{x \in \mathbb{R}^m \mid \forall \varepsilon \in \mathbb{R}, \exists y \in A, \varepsilon > 0 \to |x - y| < \varepsilon\},$$

which is clearly a first-order formula.

Remark 2.7. For our purposes it is essential to keep track of parameters. For example, π is $\mathbb{R}$-definable in $\mathbb{R}_{\text{alg}}$ but not $\mathbb{Q}$-definable in $\mathbb{R}_{\text{alg}}$. When dealing with definable sets we usually explicitly mention the scope of our parameters.

Definition 2.8. We say that a model theoretic structure $\mathcal{R}$ *expands* $(\mathbb{R}, <, 0, 1, +, \cdot)$ if its underlying set is $\mathbb{R}$, and if it contains the relation $<$, the constants $0, 1$, and the functions $+, \cdot$ with their usual interpretations.

Now we are finally ready for the central notion.

Definition 2.9 (See §3.2 and §5.7 of [vdD98])**.** A model theoretic structure $\mathcal{R}$ expanding $(\mathbb{R}, <, 0, 1, +, \cdot)$ is *o-minimal* if the $\mathbb{R}$-definable subsets of $\mathbb{R}$ are exactly the finite unions of points and (possibly unbounded) open intervals in $\mathbb{R}$.

Remark 2.10. Note that in this definition, Van den Dries considers $\mathbb{R}$-definable subsets of $\mathbb{R}$ in $\mathcal{R}$. In particular, it is not required that every interval is definable in $\mathcal{R}$ without introducing additional parameters.

Example 2.11. Since the $\mathbb{R}$-semi-algebraic subsets of the real line are exactly finite unions of points and (possibly unbounded) open intervals, we see that $\mathbb{R}_{\text{alg}}$ is an o-minimal structure.

Note that $\mathbb{N}$ and $\mathbb{Z}$ are not definable subsets in any o-minimal structure, because of the finiteness condition in the definition. In particular, the functions $\sin\colon \mathbb{R} \to \mathbb{R}$ and $\exp\colon \mathbb{C} \to \mathbb{C}$ (after identifying $\mathbb{C}$ with $\mathbb{R}^2$) cannot be definable in any o-minimal structure.

Definition 2.12. The model theoretic structure $(\mathbb{R}, <, 0, 1, +, \cdot, \exp)$ will be denoted by $\mathbb{R}_{\exp}$. Here $\exp\colon \mathbb{R} \to \mathbb{R}$ is the *real* exponential function (and not the complex one, this is important!).

Definition 2.13. Let $\mathcal{F}_{\mathrm{an}}$ be the collection of *restricted analytic* functions, that is, functions $f\colon \mathbb{R}^m \to \mathbb{R}$ that are zero outside $[0,1]^m$ and such that $f|_{[0,1]^m}$ can be extended to a real analytic function on an open neighbourhood of $[0,1]^m$.

We denote by $\mathbb{R}_{\mathrm{an}}$ the model theoretic structure $(\mathbb{R}, <, 0, 1, +, \cdot, \mathcal{F}_{\mathrm{an}})$ and by $\mathbb{R}_{\mathrm{an},\exp}$ the model theoretic structure $(\mathbb{R}, <, 0, 1, +, \cdot, \mathcal{F}_{\mathrm{an}}, \exp)$. Finally, we denote by $\mathbb{R}_{\sin,\exp}$ the model theoretic structure $(\mathbb{R}, <, 0, 1, +, \cdot, \sin|_{[0,1]}, \exp)$. For every subfield $k \subset \mathbb{R}$, we denote $\mathbb{R}_{\sin,\exp,k}$ the model theoretic structure where we adjoin all elements in k as parameters.

This is one of the protagonists in this paper.

Remark 2.14. The model theoretic structure $\mathbb{R}_{\sin,\exp}$ will be of most interest to us. Note that if the interval $I \subset \mathbb{R}$ is definable with parameters in C, then the functions $\sin|_I$ and $\cos|_I$ are definable in $\mathbb{R}_{\sin,\exp}$ with parameters in C. Indeed, one may use the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to define $\cos(\theta)$ for $\theta \in [0,1]$. After that, $\cos(\theta)$ can be arbitrarily extended using $\cos(-\theta) = \cos(\theta)$ and $\cos(2\theta) = 2\cos^2(\theta) - 1$. This allows one to define π: it is twice the smallest positive zero of cos. Finally, one can define sin on arbitrary bounded definable intervals by translating cos by $\pi/2$.

Theorem 2.15. *The model theoretic structures* $\mathbb{R}_{\exp}$*,* $\mathbb{R}_{\mathrm{an}}$*,* $\mathbb{R}_{\mathrm{an},\exp}$*, and* $\mathbb{R}_{\sin,\exp}$ *are o-minimal.*

Proof. For $\mathbb{R}_{\mathrm{an}}$, the result was proven by Van den Dries in [vdD86]. Wilkie proved that $\mathbb{R}_{\exp}$ is o-minimal in [Wil96]. Building on Wilkie's result (that was already announced in 1991), Van den Dries and Miller [vdDM94] showed that $\mathbb{R}_{\mathrm{an},\exp}$ is o-minimal. Finally, $\mathbb{R}_{\sin,\exp}$ is o-minimal because its definable sets are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an},\exp}$ and it expands $\mathbb{R}_{\mathrm{alg}}$. □

Remark 2.16. A fundamental fact about o-minimal structures is that each definable set is a finite disjoint union of basic building blocks called *cells*. If the set is defined over a subfield $k \subset \mathbb{R}$, then so are the cells. This follows from the Cell Decomposition Theorem [vdD98, Theorem 2.11, Chapter 3], see also [vdD98, Chapter 3, Section 2.19, Exc. 4]. Using this theorem one can introduce a good notion of dimension of definable sets that behaves as one expects intuitively. For example, if X is a nonempty definable set, then $\dim(\bar{X} \smallsetminus X) < \dim(X)$. See [vdD98, Chapter 4] for details and other properties of the dimension.

Remark 2.17. For the reader well versed in o-minimality we remark that for the remainder of this text, our o-minimal structures will always expand $(\mathbb{R}, <, 0, 1, +, \cdot)$. In particular, a definable subset of $\mathbb{R}^n$ is *connected* if and only if it is *definably connected*. Moreover, the word *compact* retains its meaning from point set topology.

3. Definable manifolds

Fix an arbitrary o-minimal structure $\mathcal{S}$ expanding $(\mathbb{R}, <, 0, 1, +, \cdot)$. and a subfield $k \subset \mathbb{R}$. In the remainder of this section, all definable sets are understood to be definable in $\mathcal{S}$ with parameters from k unless otherwise specified.

Definition 3.1. Let $0 \leq p \leq \infty$.

(1) A *definable C^p-manifold with corners* M is a C^p-manifold with corners together with the choice of a finite atlas $(\phi_i : U_i \to V_i \subset \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}_{\geq 0})_{i \in I}$ such that the V_i are open in $\mathbb{R}^{n_i} \times \mathbb{R}^{m_i}_{\geq 0}$ and definable and the transition maps $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ are definable and of class C^p on their domain. Its boundary ∂M is the union of the preimages of the boundaries of $\mathbb{R}^{n_i} \times \mathbb{R}^{m_i}_{\geq 0} \subset \mathbb{R}^{n_i + m_i}$ under the ϕ_i.

(2) A subset $G \subset M$ is called *definable* if $\phi_i(G \cap U_i)$ is definable in $\mathbb{R}^{n_i}$ for all i. We say G is an affine definable set if $M = \mathbb{R}^n$, i.e., if it is a definable set in the sense of Definition 2.4 and Proposition 2.5.

(3) A subset N of a definable C^p-manifold with corners M is called a *submanifold* if there is a C^p-manifold M and a C^p-immersion $M \to N$ that is a homeomorphism where N carries the subspace topology. I.e. our submanifolds are embedded and have no corners.

(4) Let $(M, \phi_i), (N, \psi_j)$ be definable C^p-manifolds with corners. A map of definable C^p-manifolds with corners is called a *definable C^p map* if all $\psi_j \circ f \circ \phi_i^{-1}$ are definable and C^p on their domain.

Remark 3.2. The definition of definable manifold includes the choice of a finite atlas. The finiteness condition is important, as, for example, we do not want manifolds with infinitely many connected components. So we cannot work with a maximal atlas. However, we could work with an equivalence class of finite atlases. Alternatively, one may rephrase the definition in the language of locally ringed sites, using the Grothendieck topology of definable open subsets and finite covers. The definition of a definable manifold is inspired by and related to the semialgebraic spaces of Delfs and Knebusch [DK81] and the complex analytic definable spaces of Bakker–Brunebarbe–Tsimerman [BBT18]. See Chapter 10 §1 [vdD98] for an introduction to general definable spaces.

Remark 3.3. Robson (see [Rob83]) showed that all semi-algebraic spaces (the C^0-case of the above definition) are actually affine. However, it is not clear to us if this extends to the C^p-setting. The above notion is general enough for our needs.

Example 3.4. Let $\bar{\Delta} \subset \mathbb{R}^n$ be the closed simplex spanned by $v_0, \ldots, v_m \in k^n$. Then $\bar{\Delta}$ is a definable C^p-manifold with corners for all $p \geq 0$. As this example shows, the boundary of a manifold with corners does not have an induced structure of C^p-manifold for $p \neq 0$. We are particularly interested in the case $p = 1$ because every affine definable set G has a triangulation such that the maps $\bar{\Delta} \to G$ are maps of definable C^1-manifolds in the above sense. See [OS17] and [CP18], and also Proposition 7.4, where we quote this result.

Another well-known example are cells. We refer to Chapter 3 of [vdD98] for the definition and basic properties of C^0-cells. Chapter 7.3 [vdD98] introduces C^p-cells and proves the decomposition theorem for $p = 1$, the general case is similar.

Example 3.5. Let $C \subset \mathbb{R}^n$ be a definable C^p-cell of dimension d. It is easy to see that there is a of coordinates $\{x_{i_1}, \ldots, x_{i_d}\}$ on $\mathbb{R}^n$ inducing a definable homeomorphism $\phi = (x_{i_1}, \ldots, x_{i_d}) : C \to \phi(C) \subset \mathbb{R}^d$. We give C the structure of an affine definable C^p-manifold using the chart ϕ. Then the inclusion $C \to \mathbb{R}^n$ is a definable C^p-map of definable C^p-manifolds. In other words, cells are definable C^p-submanifolds of $\mathbb{R}^n$.

Definition 3.6. Fix an integer $p \geq 1$, let $d \geq 0$ be an integer, and let M be a definable C^p-manifold with corners with $G \subset M$ a definable subset. We define $\mathrm{Reg}_d(G)$ to be the set of $x \in G$ that admit an open neighbourhood U in M such that $G \cap U$ is a submanifold of M of dimension d.

Remark 3.7. The set $\mathrm{Reg}_d(G)$ is open in G, it is empty if $\dim G < d$. If $\dim(G) = d$, it is the maximal subset of G that is a submanifold of M having connected components of dimension d. If G and H are disjoint definable subsets of M, then in general there is no inclusion between the two sets $\mathrm{Reg}_d(G \cup H)$ and $\mathrm{Reg}_d(G) \cup \mathrm{Reg}_d(H)$.

The following lemma adapts to our situation the fact that the p-regular points of given dimension of a definable set constitute a definable set.

Lemma 3.8. *Let M, G, and $\mathrm{Reg}_d(G)$ be as in definition 3.6. Then $\mathrm{Reg}_d(G)$ is a definable subset of M and $\dim G \setminus \mathrm{Reg}_d(G) < d$ if $\dim G = d$.*

Proof. Assuming the first claim we begin by proving the last claim by contradiction. Suppose $H = G \setminus \mathrm{Reg}_d(G)$ has dimension $\dim G = d$. There is a chart of M on which $V \cap H$ becomes a definable set of dimension d. So we may assume $H \subset G \subset \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$. We fix a C^p-cell decomposition of G partitioning H and $G \setminus H$. One cell in H must have top dimension $\dim H = \dim G$ and this cell has a point not contained in the closure of any other cell. This point lies in $\mathrm{Reg}_d(G)$, which is a contradiction.

To show that $\mathrm{Reg}_d(G)$ is definable it suffices to work in a single chart. So without loss of generality G is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$ of dimension d. We use the classical theory of differential manifolds to characterize submanifolds locally as graphs of functions. I.e., $\mathrm{Reg}_d(G)$ is the set of points of G that have an open neighbourhood in M in which G is the graph of a C^p map defined on an open subset of a projection of $\mathbb{R}^{n+m}$ to d different coordinates. The argument laid out in [vdDM96, B.9] applies directly to our slightly more general situation, and implies the definability of $\mathrm{Reg}_d(G)$. □

Lemma 3.9. *Let $G \subset \mathbb{R}^n$ be a definable subset of dimension d. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^d$ denote the projection to the first d coordinates. There are pairwise disjoint definable open subsets $G_0, G_1, \ldots, G_N$ of $\mathrm{Reg}_d(G)$ with $\dim G \setminus (G_0 \cup \cdots \cup G_N) < d$ such that all fibres of $\pi|_{G_0}$ have positive dimension and such that $\pi|_{G_i}\colon G_i \to \pi(G_i)$ is a chart for all $i \in \{1, \ldots, N\}$.*

Proof. Without loss of generality $G = \mathrm{Reg}_d(G)$. Let G' be the set of points of G that are isolated in their fibre of $\pi|_G$. It is definable, see Corollary 1.6, Chapter 4 [vdD98]. Each fibre of $\pi|_{G'}$ is discrete and thus finite with uniformly bounded cardinality, see Corollary 3.7, Chapter 3 [vdD98]. Let N be largest cardinality of a fibre.

By definable choice, Proposition 1.2(i) Chapter 6 [vdD98], applied to the graph of $\pi|_{G'}$ there is a definable section $\psi_1\colon \pi(G') \to G'$ of $\pi|_{G'}$, i.e. $\pi \circ \psi_1$ is the identity. The image $\psi_1(\pi(G'))$ is a definable set. It lies in G' but possibly missing some branches. The set of missing points $G'_1 = G' \smallsetminus \psi_1(\pi(G'))$ is also definable. Now $\pi|_{G'_1}$ certainly still has finite fibres, but the maximal fibre count dropped to $N-1$. We repeat this step and find a section $\psi_2\colon \pi(G'_1) \to G'_1$ and again the fibre count of π on $G'_2 = G'_1 \smallsetminus \psi_2(\pi(G'_1))$ drops by one.

After N steps, all fibres are exhausted. We obtain definable maps $\psi_1, \dots, \psi_N$ defined on subsets of $\pi(G')$ whose images cover G' and are pairwise disjoint.

But the ψ_i may fail to be continuous. By the Cell Decomposition Theorem, [vdD98, Chapter 3, Theorem 2.11] applied to the domain of each ψ_i, we get, after adjusting N and renaming, finitely many continuous definable maps $\psi_i : C_i \to G'$ on cells $C_i \subset \mathbb{R}^d$ with $\bigcup_i \psi_i(C_i) = G'$ and with $\pi \circ \psi_i$ the identity for all $1 \le i \le N$. Observe that the $\psi_i(C_i)$ remain pairwise distinct.

Suppose $\dim C_i = d$, then C_i is open in $\mathbb{R}^d$. As G is a manifold, invariance of domain implies that $\psi_i(C_i)$ is open in G and $\psi_i : C_i \to \psi_i(C_i)$ is a homeomorphism. Thus $\pi|_{\psi_i(C_i)} : \psi_i(C_i) \to C_i$ is a chart. We can safely ignore cells C_i with $\dim C_i < d$; the union $H = \bigcup_{i:\dim C_i<d} \psi_i(C_i)$ is definable of dimension at most $d-1$. Fix a cell decomposition of $G \smallsetminus G'$ and let G_0 be the union of all d-dimensional cells; then G_0 is open, and possibly empty, in the submanifold G. We add the remaining cells to H. We retain $\dim H < d$ and the lemma follows from $G = G_0 \cup \bigcup_{i:\dim C_i=d} \psi(C_i) \cup H$. □

Lemma 3.10. *Let $p \ge 1$ and (M, ϕ_i), (N, ψ_j) be definable C^p-manifolds with corners. The the bundles TM, T^*M and their exterior powers have a natural structure of a definable C^{p-1}-manifold with corners. Moreover, a definable C^p-map $f\colon M \to N$ induces definable C^{p-1}-maps $df\colon TM \to TN$ and $d^*f\colon T^*N \to T^*M$.*

Proof. We only have to verify definability. This holds because the derivative of a definable differentiable function is definable. Indeed, in the 1-dimensional case the graph $\Gamma(f')$ of the derivative is given by the formula

$$\left\{(x,y) \,\middle|\, \forall\varepsilon > 0, \exists\delta > 0, \forall x', |x'-x| < \delta \to \left|\frac{f(x')-f(x)}{x'-x} - y\right| < \varepsilon \right\}. \quad \square$$

Many properties of affine definable sets extend immediately to the non-affine case. This is in particular the case for the notion of dimension and the stratification by submanifolds. We want to use these facts in order to integrate differential forms.

Definition 3.11. Let $p \ge 1$. Let (M, ϕ_i) be a definable C^p-manifold with corners and $G \subset M$ a definable subset. A *differential form ω of degree d on G* is a continuous section

$$\omega\colon G \to \Lambda^d T^*M.$$

It is called *definable*, if it is definable as a map in the sense of Definition 3.1 (4).

In the affine case, we can give an explicit description: Let $x_1, \dots, x_n$ be the standard coordinates on $\mathbb{R}^n$. For $I = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ a subset with $i_1 < i_2 < \cdots < i_d$ we write as usual

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_d}.$$

A differential form on G can be written uniquely as

$$\omega = \sum_I a_I \,\mathrm{d}x_I$$

with $a_I : G \to \mathbb{R}$ continuous. It is definable if and only if the a_I are definable.

Remark 3.12. Note that we do not put differentiability conditions or require that ω extends to a neighbourhood of G.

Lemma 3.13. *Let $p \geq 1$, and $f\colon M \to N$ be a definable C^p-map of definable manifolds with corners. Let $G \subset M$ and $H \subset N$ be definable subsets with $f(G) \subset H$. Then the pull-back of a differential form on H defines a differential form on G. If ω is definable, so is $f^*\omega|_G$.*

Proof. By definition, $f^*\omega|_G\colon G \to \Lambda^d T^*M$ is the composition

$$G \to H \to \Lambda^d T^* N \to \Lambda^d T^* M$$

of continuous maps. In particular, it is definable if ω is definable. □

As usual, we can only expect a well-defined integration theory for differential forms on oriented domains.

Definition 3.14. Fix an integer $p \geq 1$, let $d \geq 0$ be an integer, and let M be a definable C^p-manifold with corners with $G \subset M$ a definable subset of dimension d.

(1) A *pseudo-orientation* on G is the choice of an equivalence class of a definable open subset $U \subset \mathrm{Reg}_d(G)$ such that $\dim(G \smallsetminus U) < d$ and an orientation on U. Two such pairs are equivalent if they agree on the intersection. We thereby obtain an equivalence relation.
(2) Given a pseudo-orientation on G with U as in (1) and a differential form ω of degree d on G, we define

$$\int_G \omega := \int_U \omega$$

if the integral on the right converges absolutely.

Remark 3.15. The same definition also allows us to integrate a d-form ω over a G of dimension smaller than d: in this case $\mathrm{Reg}_d(G) = \emptyset$ and the integral is set to 0. Such integrals occur in our formulas and are to be read in this way.

Lemma 3.16. *Let $p \geq 1$. Let G be a definable subset of a definable C^p-manifold with corners M.*

(1) The integral is well-defined, i.e., independent of the choice of representative for the pseudo-orientation.
(2) By restriction a pseudo-orientation on G also induces the choice of a pseudo-orientation on every definable subset $G' \subset G$ with $\dim G = \dim G'$.
(3) The choice of a pseudo-orientation on G induces a choice of a pseudo-orientation on every definable superset $G \subset G''$ such that $\dim(G'' \smallsetminus G) < d$*, in particular on $\bar{G}$.*

(4) *Let $\pi\colon G' \to G$ be a definable modification, i.e., there is an open definable subset $U \subset \mathrm{Reg}_d(G)$ with $\dim(G \smallsetminus U) < d$ such that $\pi|_{U'}\colon U' = \pi^{-1}(U) \to U$ is an isomorphism of definable C^1-manifolds and $\dim(G' \smallsetminus U') < d$. Then the choice of a pseudo-orientation on G induces a pseudo-orientation on G'.*

Proof. If $U_1, U_2 \subset \mathrm{Reg}_d(G)$ are definable open such that $\dim(G \setminus U_i) < d$, then the same is true for $U_1 \cap U_2$. Hence it suffices to consider the case $U_1 \subset U_2$. By assumption the orientation on U_2 restricts to U_1. We have

$$\int_{U_2} \omega = \int_{U_1} \omega$$

because $U_2 \smallsetminus U_1$ has measure 0. The left hand side converges absolutely if and only if the right hand side does.

We fix a pseudo-orientation on G, i.e., an orientation on some $U \subset \mathrm{Reg}_d(G)$ such that $\dim(G \smallsetminus U) < d$.

Let $G' \subset G$, $U' = U \cap \mathrm{Reg}_d(G')$. The orientation on U restricts to an orientation on U'. We have $\dim(G' \smallsetminus U') < d$, hence this data defines the pseudo-orientation on G'.

Let $G \subset G''$, $U'' = U \cap \mathrm{Reg}_d(G'')$. The orientation on U restricts to an orientation on U''. As $\dim(G'' \smallsetminus G) < d$, we also have $\dim(G'' \smallsetminus U'') < d$, hence again this data defines a pseudo-orientation on G''.

The case of a modification combines the two operations. □

Corollary 3.17. *Let $G, H \subset M$ be definable subsets of dimension at most d of a definable manifold with corners, equipped with a pseudo-orientation on $G \cup H$. Let ω be a definable differential form of degree d on $G \cup H$. Then with the restricted pseudo-orientations*

$$\int_{G\cup H} \omega = \int_G \omega + \int_H \omega - \int_{G\cap H} \omega$$

and the left hand side is finite if and only if all terms on the right are.

Proof. We may assume $\dim G = \dim H = d$. We can decompose $G \cup H$ into the disjoint subsets $G \cap H, G \smallsetminus H, H \smallsetminus G$. Hence it suffices to check the formula in the case where the two sets are disjoint.

We start with an orientation on a definable open subset $U \subset \mathrm{Reg}_d(G \cup H)$ with $\dim(G \cup H) \smallsetminus U < d$. The pseudo-orientations on G and H are represented by the restricted orientations on $V = U \cap \mathrm{Reg}_d(G)$ and $W = U \cap \mathrm{Reg}_d(H)$, respectively. Then $V \cup W$ represents our pseudo-orientation on $G \cup H$. By definition and by the standard computation rules for integration on manifolds, we find

$$\int_{G\cup H} \omega = \int_{V\cup W} \omega = \int_V \omega + \int_W \omega = \int_G \omega + \int_H \omega. \qquad \square$$

Remark 3.18. (1) As in the case of ordinary orientations, the value of the integral depends on the choice of pseudo-orientation. Note that even a simple definable set like an interval U, admits infinitely many different pseudo-orientations. If U has n connected components, there are 2^n possible orientations and we can cut up U as much as we like.

(2) For each G the choice $U = \mathrm{Reg}_d(G)$ is canonical if it is possible to choose an orientation on this set. However, the behaviour of $\mathrm{Reg}_d(G)$ under standard topological operations is complicated. It is not true that the choice of an orientation on $\mathrm{Reg}_d(G)$ also induces an orientation on $\mathrm{Reg}_d(\bar{G})$ (take $G = \mathbb{R} \smallsetminus \{0\}$). Neither is it true that $\mathrm{Reg}_d(G') \subset \mathrm{Reg}_d(G)$ if $G' \subset G$ (take the x-axis in the union of the coordinate axes in $\mathbb{R}^2$). Our more flexible notion sidesteps these problems.
(3) Note also that every non-empty definable set G admits a pseudo-orientation because open cells are orientable and G admits a cell decomposition.
(4) The restriction operation described in the proof of Lemma 3.16(2) is well-defined in the following sense. Two representatives of a pseudo-orientation on G restrict to representatives of the same pseudo-orientation on G'. Moreover, the same holds true for the extension operation described in the proof of part (3). Finally, extending a pseudo-orientation from G to G'' and then restricting it back to G recovers the original pseudo-orientation. So the extension in part (3) of the lemma is unique.

Remark 3.19. If $G \subset \mathbb{R}^n$ is a definable open with the standard orientation and $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, then $\int_G \omega = \mathrm{vol}(G)$. This number is always finite if G is bounded.

We will see that the example of the volume form is really the general case, but before that we need to establish a technical lemma.

Lemma 3.20. *Let (M, ϕ_i) be a definable manifold with corners, $x \in M$. Then there is a definable open neighourhood $U_x \subset M$ with compact closure and such that $\bar{U}_x \subset U_i$ for some i.*

Proof. We fix i such that $x \in U_i$. Recall that $V_i = \phi_i(U_i)$ is open in $\mathbb{H} := \mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$. Hence there is a definable $0 < r < \infty$ such that the open ball $\mathbb{H} \cap B_r(\phi_i(x))$ is contained in V_i. Let $a \in \mathbb{H}$ be definable with distance at most $r/4$ from $\phi_i(x)$. Put $V_x = B_{r/2}(a) \cap \mathbb{H}$. Then $\bar{V}_x \subset B_r(\phi_i(x)) \cap \mathbb{H}$ is a compact set contained in V_i. We put $U_x = \phi_i^{-1}(V_x)$. □

Lemma 3.21. *A finite $\mathbb{Z}$-linear combination of volumes of definable bounded open subsets of $\mathbb{R}^d$ is up to sign the volume of a definable bounded open subset of $\mathbb{R}^d$.*

Proof. All contributions with a positive coefficient can be combined into a single one by taking the disjoint union of translates of the definable sets. In the same way all contributions with a negative coefficient can be combined into a single one. So it suffices to prove that the difference of the volumina of two definable bounded open subsets of $\mathbb{R}^d$ is up to sign the volume of a definable bounded open subset of $\mathbb{R}^d$. The argument of Viu-Sos, see [VS15, Section 4] in the semi-algebraic setting works identically in the definable case and provides what we want. □

Recall that we work with k-definable sets in a fixed o-minimal structure $\mathcal{S}$ expanding $(\mathbb{R}, <, 0, 1, +, \cdot)$.

Theorem 3.22. *Let $p \geq 1$, and (M, ϕ_i) be a definable C^p-manifold with corners, $G \subset M$ a pseudo-oriented compact definable subset of dimension d. Let ω be a differential form of degree d on G as in Definition 3.11. Then*

$$\int_G \omega$$

converges absolutely. If ω is definable, then the value is up to a sign the volume of a definable bounded open subset of $\mathbb{R}^{d+1}$.

Proof. We are going to rewrite our integral as a finite $\mathbb{Z}$-linear combination of other integrals. Eventually these summand will be absolutely convergent, proving absolute convergence of the original integral. In the definable case, every summand will be written as a difference between volumes of bounded definable open subsets of $\mathbb{R}^{d+1}$. By Lemma 3.21 this will imply that the original volume is up to a sign the volume of a single definable bounded open subset of $\mathbb{R}^{d+1}$ and hence finish the proof of the theorem.

We begin by showing how to reduce to the case $M = \mathbb{R}^n$. By Lemma 3.20 each point $x \in G$ has a definable open neighbourhood U_x in M such that $\bar{U}_x$ is compact and contained in one of the finitely many charts of M. By hypothesis G is compact, so it is covered by finitely many such neighborhoods, let us call them $U_1, \ldots, U_a$. The $\bar{U}_i$ and their multiple intersections inherit a pseudo-orientation from G. By the inclusion-exclusion principle, Corollary 3.17, we have

$$\int_G \omega = \sum_{i=1}^{a} \int_{\bar{U}_i} \omega - \sum_{i<j} \int_{\bar{U}_i \cap \bar{U}_j} \omega \pm \ldots .$$

We now replace G by one of the $\bar{U}_i$ (or multiple intersections), making it affine.

We have $\omega = \sum_I a_I \, \mathrm{d}x_I$. Again, it suffices to treat the summands separately. After a coordinate permutation we may assume without loss of generality that $\omega = a \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$ where a is continuous on G. Recall that $\pi \colon \mathbb{R}^n \to \mathbb{R}^d$ denotes the projection onto the first d coordinates. Let $y_1, \ldots, y_d$ be the coordinates on $\mathbb{R}^d$. Hence

$$\pi^*(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d) = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d.$$

We let $G_0, G_1, \ldots, G_N$ be pairwise disjoint as in Lemma 3.9 applied to G. In particular, $G_0 \cup G_1 \cup \cdots \cup G_N$ equals G up to a subset of dimension at most $d-1$. All G_i inhert a pseudo-orientation from G and all $\pi|_{G_i}$ with $i \geq 1$ are charts. We may replace each such G_i by a finite union of open subsets, again up-to a subset of dimension $d-1$, and assume that all $G_1, \ldots, G_N$ carry an orientation in the classical sense and that $\pi|_{G_i} \colon G_i \to \pi(G_i)$ is orientation preserving. Thus

$$\int_G \omega = \sum_{i=0}^{N} \int_{G_i} \omega$$

by Corollary 3.17 if all integrals on the right converge absolutely. Thus it suffices again to treat a single $\int_{G_i} \omega$.

We begin with the easy case $i = 0$. By assumption, all fibres of $\pi|_{G_0}$ have positive dimension, hence $\pi|_{G_0}^* = 0$ on differential forms of degree d.

Thus the restriction of $\omega = a\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$ to G_0 vanishes. Hence $\int_{G_0} \omega$ converges absolutely with value 0, the volume of $\emptyset$.

Now we treat G_i with $i \geq 1$. Then $\pi|_{G_i} : G_i \to \pi(G_i) \subset \mathbb{R}^d$ is a chart and thus has an inverse $\psi_i : \pi(G_i) \to G_i$. Note that ψ_i is of C^p-class . Let $y_1, \ldots, y_d$ denote the coordinates of $\mathbb{R}^d$. The integral

$$\int_{\pi(G_i)} a \circ \psi_i \,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d$$

converges absolutely as a is continuous on the compact G and thus in particular bounded on G_i. Finally,

$$\psi_i^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d) = \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d$$

as $\pi \circ \psi_i$ is the identity. Thus

$$\int_{G_i} a\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d = \int_{\pi(G_i)} \psi_i^*(a\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d) = \int_{\pi(G_i)} a \circ \psi_i \,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d$$

converges absolutely.

Suppose that ω is definable, then a is definable. It remains to show that $\int_{\psi_i(G_i)} a \circ \psi \,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d$ is the volume of a definable bounded open subset of $\mathbb{R}^{d+1}$. This integral equals

$$\int_{C_+} a \circ \psi_i \,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d - \int_{C_-} |a \circ \psi_i| \,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d$$

with $C_\pm = \{y \in \psi_i(G_i) \mid \pm a(\psi_i(y)) > 0\}$ both definable bounded and open in $\mathbb{R}^d$. Hence it equals $\mathrm{vol}(U_+) - \mathrm{vol}(U_-)$ with $U_\pm = \{(y, z) \in C_\pm \times \mathbb{R} : 0 < z < |a(\psi_i(y))|\}$. Note that $U_\pm$ are both definable bounded and open in $\mathbb{R}^{d+1}$. This difference is the volume of a definable bounded open subset of $\mathbb{R}^{d+1}$ by Lemma 3.21. □

Remark 3.23. Let us explain why we cannot replace $\mathbb{R}^{d+1}$ by $\mathbb{R}^d$ in the theorem above. Consider the half-circle $G = \{(x, y) \in \mathbb{R} \times (0, \infty) \mid x^2 + y^2 = 1\}$. It is relatively compact, semi-algebraic and definable without parameters. Then $\mathrm{Reg}_1(G) = G$ for all $p \geq 1$. Now $\int_G y\,\mathrm{d}y = \pm \int_{-1}^{1} \sqrt{1 - y^2}\,\mathrm{d}y = \pm\pi/2$. As π is transcendental, $\int_G y\,\mathrm{d}y$ cannot be the volume of $\mathbb{Q}$-semi-algebraic subset of $\mathbb{R}$.

Remark 3.24. The natural way of computing the integral is to pull the differential form back to a chart (via the inverse of the chart map) and evaluate there. However, this pull-back involves a Jacobian matrix. Its entries are not bounded in general, hence convergence is not automatic.

Here is an explicit example: Let $M = \mathbb{R}^2$, $G = \{(y^2, y) | y \in [0, 1]\}$, $\omega = a\,\mathrm{d}x_1 + b\,\mathrm{d}x_2$ for continuous a, b on G. We have $\mathrm{Reg}_1(G) = \{(y^2, y) | y \in (0, 1)\}$. It is a submanifold. We can use the projections π_1 and π_2 to the first or second coordinate as a chart. In both cases the image in $\mathbb{R}$ is the open interval $I = (0, 1)$. The inverse $\psi_1 : I \to G$ of π_1 is $t \mapsto (t, \sqrt{t})$. Its Jacobian matrix is

$$\left(1, \frac{1}{2\sqrt{t}}\right).$$

The second entry is unbounded on I. We have

$$\psi_1^*\omega = (a \circ \phi_1)\,\mathrm{d}t + (b \circ \phi_1)\frac{1}{2\sqrt{t}}\,\mathrm{d}t.$$

The coefficient function is unbounded. (Note that $a \circ \phi_1$ and $b \circ \phi_1$ are bounded because a and b are. Note also that differentiability of a and b does not come into play. It suffices that they are continuous.) The solution is to treat the summands $a\,\mathrm{d}x_1$ and $b\,\mathrm{d}x_2$ separately and use the projection π_1 for the first summand and π_2 for the second summand. We then interpret

$$a\,\mathrm{d}x_1 = \pi_1^*((a \circ \phi_1)\,\mathrm{d}t), \quad b\,\mathrm{d}x_2 = \pi_2^*((b \circ \phi_2)\,\mathrm{d}t)$$

and the convergence issue disappears.

Remark 3.25. A similar convergence argument for integrals can also be found in [HKT15]. Alternatively, convergence also follows from the existence of triangulations that are strictly of class C^1, shown in [OS17] and [CP18]. These references treat explicitly the case of C^∞-forms, but actually this assumption is not needed.

4. Oriented real blowup

The oriented real blowup is a natural construction in the context of semi-algebraic geometry. Nevertheless, it seems that little is written about it from this point of view. The construction is discussed in §I.3 of [Maj84], §3.4 of [FJ20] and [Gil]. One of the main purposes of this section is to argue that the oriented real blowup is semi-algebraic (in other words, definable in $\mathbb{R}_{\mathrm{alg}}$) with suitable parameters. For a general discussion we refer to the aforementioned sources.

Let X be a topological space, let $\pi\colon L \to X$ be a complex (topological) line bundle on X, and let $s\colon X \to L$ be a section. Let L^* be the complement of the zero section. We put

$$B^*_{L,s} = \{l \in L^* \mid s(\pi(l)) \in \mathbb{R}_{\geq 0} l\}.$$

If $s(x) = 0$, then $B^*_{L,s}$ contains $L_x \smallsetminus \{0\}$, otherwise, it contains the unique open half-ray generated by $s(x)$. In particular, $B^*_{L,s}$ is stable under the fibrewise action of $\mathbb{R}_{>0}$.

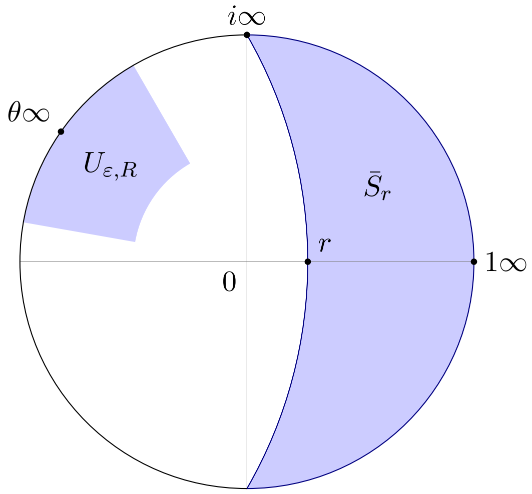
Following [Gil], we call the quotient the *simple oriented real blowup*:

$$\mathrm{Blo}_{L,s}(X) = B^*_{L,s}/\mathbb{R}_{>0}.$$

The simple oriented real blowup comes equiped with a natural projection map $\pi\colon \mathrm{Blo}_{L,s}(X) \to X$ that is an isomorphism outside the zero locus of s.

If X is a complex analytic space, and $D \subset X$ an effective Cartier divisor, and s the tautological section of $\mathcal{O}(D)$, then we will write B_D, and $\mathrm{Blo}_D(X)$ for $B_{\mathcal{O}(D),s}$ and $\mathrm{Blo}_{\mathcal{O}(D),s}(X)$ respectively.

Example 4.1. The blowup $\tilde{\mathbb{P}}^1 := \mathrm{Blo}_\infty(\mathbb{P}^1_{\mathbb{C}})$ is a compactification of $\mathbb{C}$ by a circle at infinity. The details of the following picture will be explained as we describe the general situation in local coordinates.



For every $z \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ there is a point $z\infty$ on the boundary: it is the point of intersection of the boundary and the half-ray $z \cdot \mathbb{R}_{\geq 0}$. A system of open neighbourhoods around $z\infty$ is given by the sets

$$U_{\varepsilon,R} = \{w \in \mathbb{C} \mid |w| > R \text{ and } |\arg(w) - \arg(z)| < \varepsilon\} \cup \{w\infty \mid |\arg(w) - \arg(z)| < \varepsilon\}$$

for small ϵ and positive real R.

The closure of the set $S_r = \{z \in \mathbb{C} \mid \Re(z) \geq r\}$ is given by the union of S_r and the half-circle $\{z\infty \mid \Re(z) \geq 0\}$.

Suppose that $L_1, \ldots, L_n$ are line bundles on X with respective sections $s_1, \ldots, s_n$, and put $L = L_1 \otimes \cdots \otimes L_n$ with section $s_1 \otimes \cdots \otimes s_n$. We may then form the fibre product

$$\mathrm{Blo}_{L_1,s_1}(X) \times_X \cdots \times_X \mathrm{Blo}_{L_n,s_n}(X)$$

which naturally maps to $\mathrm{Blo}_{L,s}(X)$.

Definition 4.2. Let X be a smooth analytic space, and let $D \subset X$ be a simple normal crossings divisor. Denote the (smooth) irreducible components of D by $D_1, \ldots, D_n$. The *oriented real blowup* of X in D, denoted by $\mathrm{OBl}_D(X)$ is the fibre product

$$\mathrm{Blo}_{D_1}(X) \times_X \cdots \times_X \mathrm{Blo}_{D_n}(X).$$

Note that $\mathrm{OBl}_D(X)$ comes with a natural projection map to X.

One topological intuition for $\mathrm{OBl}_D(X)$ is the complement of a tubular neighbourhood of D in X. We now make this picture precise by a description in local coordinates.

Consider a domain U in $\mathbb{C}^n$ and $D = D_1 \cup \ldots \cup D_m$ the union of the first m coordinate hyperplanes (intersected with U). In that case we have the following explicit description of $\mathrm{OBl}_D(U)$

$$\{(z_1, \ldots, z_n, w_1, \ldots, w_m) \in \mathbb{C}^n \times (S^1)^m \mid (z_1, \ldots, z_n) \in U, z_i w_i^{-1} \in \mathbb{R}_{\geq 0} \text{ for } 1 \leq i \leq m\} \tag{2}$$

and π is the projection $(z_1,\dots,z_n,w_1,\dots,w_m)\mapsto(z_1,\dots,z_n)$. In particular, it is a C^∞-manifold with corners. Local coordinates are defined by

$$\text{(3)}\qquad \mathrm{OBl}_D(U)\to\mathbb{R}^m_{\geq 0}\times(S^1)^m\times\mathbb{C}^{n-m}$$
$$(z_1,\dots,z_n,w_1,\dots,w_m)\mapsto\left(\frac{z_1}{w_1},\dots,\frac{z_m}{w_m},w_1,\dots,w_m,z_{m+1},\dots,z_n\right).$$

In particular, this gives the blow-up the the structure of a manifold with corners. As a consequence, we obtain the following result.

Proposition 4.3. *Let $k\subset\mathbb{C}$ be a field which is algebraic over $k_0=k\cap\mathbb{R}$. Let X be a smooth algebraic variety over k and let $D\subset X$ be a simple normal crossings divisor. Then the oriented real blowup $\mathrm{OBl}_D(X^{\mathrm{an}})$ can naturally be endowed with a structure of k_0-semi-algebraic C^∞-manifold with corners (see Definition 3.1) in such a way that the natural projection map $\pi\colon\mathrm{OBl}_D(X^{\mathrm{an}})\to X^{\mathrm{an}}$ is morphism of k_0-semi-algebraic C^∞-manifolds with corners.*

Proof. Without loss of generality $k_0=\tilde{k}$ is real closed and $k=\bar{k}$ algebraically closed. Let $(\bar{X},\bar{D})$ be a good compactification of the log pair (X,D). It suffices to prove the proposition for $(\bar{X},\bar{D})$ because $\mathrm{OBl}_D(X^{\mathrm{an}})$ is the preimage of X^{an} in $\mathrm{OBl}_{\bar{D}}(\bar{X})$. In other words, without loss of generality X^{an} is compact.

Without loss of generality X is connected. By definition, for every point $x\in X$, there is a Zariski-open neighbourhood U_x and an étale map $p\colon U_x\to\mathbb{A}^d$ (with $d=\dim X$) such that $p(x)=0$ and $D\cap U_x=p^{-1}(\{z_1\cdots z_m=0\})$. By the semi-algebraic implicit function theorem, the map p^{an} is invertible on an open ball B_x around 0 in $\mathbb{C}^d$. Let $V_x=p^{-1}(B_x)\subset X^{\mathrm{an}}$. The coordinate functions $z_1,\dots,z_m$ are both holomorphic and k_0-semi-algebraic. Hence the preimage

$$\pi^{-1}(V_x)\subset\mathrm{OBl}_D(X^{\mathrm{an}})$$

has the shape described after Definition 4.2. The map (3) defines a chart. More precisely, we also need to cover $S^1\subset\mathbb{R}^2$ by finitely many semi-algebraic charts. As X^{an} is compact, finitely many of the V_x suffice to cover X^{an}. The transition maps are C^∞ and k_0-semi-algebraic because the transition maps between the $p(V_x)$ are holomorphic and k_0-semi-algebraic. □

Lemma 4.4. *The construction of the oriented real blowup is functorial: Let X_1 and X_2 be analytic spaces, and let $D_i\subset X_i$ be a simple normal crossings divisor. Let $f\colon X_1\to X_2$ be a morphism, such that $f^{-1}(D_2)\subset D_1$. Then there is a natural morphism $\tilde{f}$ such that the following diagram commutes:*

$$\begin{array}{ccc}\mathrm{OBl}_{D_1}(X_1) & \xrightarrow{\tilde{f}} & \mathrm{OBl}_{D_2}(X_2)\\ \downarrow & & \downarrow\\ X_1 & \xrightarrow{f} & X_2\end{array}$$

If f is a morphism of smooth algebraic varities, then $\tilde{f}$ is a C^∞-morphism of semi-algebraic manifolds with corners.

Proof. Compute in local coordinates. □

Remark 4.5. In the future, it will often be the case that we start with a variety X that is not complete, and consider the oriented real blow-up of the boundary divisor X_∞ of a completion $\bar{X}$ of X. In such a situation, we will write $B_{\bar{X}}(X)$ instead of $\mathrm{OBl}_{X_\infty}(\bar{X})$.

Remark 4.6. It is not clear to us whether $\mathrm{OBl}_D(X)$ is affine as semi-algebraic C^1-manifold with corners. In other words, does there exist a semi-algebraic C^1-embedding of $\mathrm{OBl}_D(X)$ into $\mathbb{R}^n$? Compare with Remark 3.3.

5. Naive exponential periods

Let $k \subset \mathbb{C}$, $k_0 = k \cap \mathbb{R}$ and assume that k is algebraic over k_0, see the discussion in Section 1.1. Recall from Definition 0.2 the notion of a naive exponential period. We denote $\mathcal{P}_{\mathrm{nv}}(k)$ the set of naive exponential periods. Let $\tilde{\mathbb{P}}^1$ denote the real oriented blow-up of $\mathbb{P}^1$ at the point at infinity, see Example 4.1.

5.1. **Examples of integrals.**

We first consider some instructive examples.

Example 5.1. Let $G = [1, \infty) \subset \mathbb{C}$, $f = \frac{1}{z}$, $\omega = \mathrm{d}z$. Consider

$$\int_G \mathrm{e}^{-f}\omega = \int_1^\infty \mathrm{e}^{-\frac{1}{t}}\,\mathrm{d}t = \int_1^0 -\mathrm{e}^{-s}\frac{1}{s^2}\,\mathrm{d}s.$$

It does not converge. Indeed, the image $f(G) = (0, 1]$ is not closed, hence $f\colon G \to \mathbb{C}$ is not proper. The properness condition in the definition of a naive exponential period was added to exclude cases like this.

Example 5.2. Once again let $G = [1, \infty) \subset \mathbb{C}$, $f = \frac{1}{z}$, but $\omega = \frac{1}{z^2}$. As in the previous example, the data does not satisfy the definition of a naive exponential period because $f\colon G \to \mathbb{C}$ is not proper. However, this time

$$\int_G \mathrm{e}^{-f}\omega = \int_1^\infty \mathrm{e}^{-\frac{1}{t}}\frac{1}{t^2}\,\mathrm{d}t = \int_1^0 \mathrm{e}^{-s}\,\mathrm{d}s$$

converges. It can be understood as a naive exponential period with $G' = [0, 1]$, $f' = z$, $\omega' = \mathrm{d}z$.

Example 5.3. Let $s \in S^1$ with $\Re(s) > 0$. Consider the half ray $G_s = \{rs \mid r \geq 0\}$, $f = z$, $\omega = \mathrm{d}z$. If $s \neq 1$, this data does not satisfy the definition of a naive exponential period because $f(G_s) = G_s$ does not have bounded imaginary part. Nevertheless,

$$\int_{G_s} \mathrm{e}^{-f}\,\mathrm{d}z = \int_0^\infty \mathrm{e}^{-rs}s\,\mathrm{d}r = -\mathrm{e}^{-rs}\Big|_1^\infty = 1$$

converges and is obviously an exponential period. Note that it is independent of s. Actually, G_s defines a class in $H_1^{\mathrm{rd}}(\mathbb{A}^1, \{0\}; \mathbb{Z})$, see Section 6.1 below, because its closure in $\tilde{\mathbb{P}}^1$ is contained in $B^\circ = B^\circ_{\mathbb{P}^1}(\mathbb{A}^1, \mathrm{id})$. The homology class is independent of s (fill in the triangle between G_1 and G_s, the third edge is in ∂B°). The period integral only depends on the homology class, hence the independence follows from the abstract theory as well. We do not allow G_s in our definition of a naive exponential period, but the same number can be obtained as a naive exponential period for G_1. This is a general feature, see [CH20, Proposition 11.4]. In Definition 5.4 we will introduce the notion of a generalised naive exponential period which allows all G_s.

5.2. **General properties.**

Definition 5.4. A *generalised naive exponential period* over k is a complex number of the form

$$\int_G \mathrm{e}^{-f}\omega$$

where $G \subset \mathbb{C}^n$ is a pseudo-oriented closed k_0-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n_k$ that is regular on G and f is a rational function on $\mathbb{A}^n_k$ such that f is regular and proper on G and, moreover, the closure of $f(G)$ in $\tilde{\mathbb{P}}^1$ is contained in $B^\circ = \mathbb{C} \cup \{s\infty \mid s \in S^1, \Re(s) > 0\}$. We denote $\mathcal{P}_{\mathrm{gnv}}(k)$ the set of generalised naive exponential periods.

We are going to show in Corollary 5.11 that these generalised naive exponential periods converge absolutely. For the rest of this section we assume absolute convergence.

Lemma 5.5. *Naive exponential periods are generalised naive exponential periods.*

Proof. The condition $f(G) \subset S_{r,s}$ implies $\overline{f(G)} \subset B^\circ$. □

Lemma 5.6. *The sets $\mathcal{P}_{\mathrm{nv}}(k)$ and $\mathcal{P}_{\mathrm{gnv}}(k)$ are $\bar{k}$-algebras. Moreover, $\mathcal{P}_{\mathrm{nv}}(k) = \mathcal{P}_{\mathrm{nv}}(\bar{k})$ and $\mathcal{P}_{\mathrm{gnv}}(k) = \mathcal{P}_{\mathrm{gnv}}(\bar{k})$.*

Proof. The arguments are the same for both notions. We formulate it for naive exponential periods.

For the first statement we use the same argument as for $f = 0$, see [HMS17, Proposition 12.1.5]:

We give the argument for the second. Let L/k be a finite subextension of $\bar{k}/k$. Since k is algebraic over k_0, the extension L/L_0 with $L_0 = L \cap \mathbb{R}$ is also algebraic, for every finite extension L/k. Hence, $\mathcal{P}_{\mathrm{nv}}(\bar{k}) = \bigcup_{L/k} \mathcal{P}_{\mathrm{nv}}(L)$ where L runs through all finite subextensions of $\bar{k}/k$. Thus it suffices to show that $\mathcal{P}_{\mathrm{nv}}(k) = \mathcal{P}_{\mathrm{nv}}(L)$ for L/k finite.

We view $\mathbb{A}^n_L \to \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ as an affine k-variety contained in $\mathbb{A}^{n+1}_k$. We call it $\tilde{A}$. Then

$$\tilde{A} \times_k \mathbb{C} = \bigcup_{\sigma: L \to \mathbb{C}} \mathbb{A}^n_{\mathbb{C}}$$

where σ runs through all embeddings of L into $\mathbb{C}$ fixing k. If $\int_G \mathrm{e}^{-f}\omega$ is a naive exponential period over L, then f and ω are defined over k when viewed on $\tilde{A} \subset \mathbb{A}^{n+1}$. The extension L_0/k_0 is algebraic, hence every L_0-semialgebraic set is also k_0-semialgebraic. □

In particular, we can move between k, $\bar{k}$, $\bar{k} \cap \mathbb{R}$ and $k_0 = k \cap \mathbb{R}$ without changing the set of naive exponential or generalised exponential naive periods.

Remark 5.7. The assumption $G \subset \mathbb{C}^n = (\mathbb{A}^n)^{\mathrm{an}}$ is surprising when comparing to the literature on ordinary periods. Most period references work with semi-algebraic $G \subset \mathbb{R}^n$. The two points of view are not equivalent even though of course $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as semi-algebraic manifolds. We work with $f \in k(z_1, \ldots, z_n)$ and $\omega \in \Omega^d_{k(z_1,\ldots,z_n)/k}$. Simply replacing $\mathbb{C}$ by $\mathbb{R}$ in the definition would eliminate all non-real periods (at least if we assume

$k \subset \mathbb{R}$ as we may by the above). In the case of ordinary periods, a complex number is a period if and only if its real and imaginary part can be written as $\int_G \omega$ with $G \subset \mathbb{R}^n$ and $\omega \in \Omega^d_{k(z_1,\dots,z_n)/k}$. We cannot show the same simple characterisation in the exponential case and it is very likely false.

Lemma 5.8. *Let $k = k_0 \subset \mathbb{R}$. The following are equivalent for $\alpha \in \mathbb{C}$:*

(1) The number α is a naive exponential period over k.

(2) It can be written as

$$\alpha = \int_G \mathrm{e}^{-f}\omega$$

with $G \subset \mathbb{R}^n$ a pseudo-oriented closed k-semi-algebraic subset of dimension d, $f \in k(i)(z_1,\dots,z_n)$ regular on G such that $f|_G \colon G \to \mathbb{C}$ is proper with image contained in $S_{r,s}$ and $\omega \in \Omega^d_{k(i)(z_1,\dots,z_n)/k(i)}$ is regular on G.

(3) Its real and imaginary part can be written as

$$\Re(\alpha) = \int_G \Big(\cos(f_2)\mathrm{e}^{-f_1}\omega_1 + \sin(f_2)\mathrm{e}^{-f_1}\omega_2\Big)$$
$$\Im(\alpha) = \int_G \Big(-\sin(f_2)\mathrm{e}^{-f_1}\omega_1 + \cos(f_2)\mathrm{e}^{-f_1}\omega_2\Big)$$

with $G \subset \mathbb{R}^n$ a pseudo-oriented closed k-semi-algebraic subset of dimension d, $f_1, f_2 \in k(z_1,\dots,z_n)$ regular on G such that $f_1|_G, f_2|_G \colon G \to \mathbb{R}$ are proper, $f_1(G)$ is bounded from below, $f_2(G)$ is bounded, and $\omega_1,\omega_2 \in \Omega^d_{k(z_1,\dots,z_n)/k}$ regular on G.

Moreover, f_1, f_2 in (3) are the real and imaginary parts of f in (2), respectively, and similarly for ω_1,ω_2. Finally, α is a generalised naive exponential period if and only if it can be written as in (2) with $\overline{f(G)} \subset B^\circ$.

Proof. Let G, f, ω as in the definition of a naive exponential period. By definition $G \subset \mathbb{C}^n$ with coordinates $z_1,\dots,z_n$. By sending a complex number to its real and imaginary part we view G as a real subset G' of $\mathbb{C}^{2n}$ with coordinates $x_1, y_1, x_2, y_2, \dots, x_n, y_n$. Let $\Sigma \colon \mathbb{C}^{2n} \to \mathbb{C}^n$ be given by $(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 + iy_1, \dots, x_n + iy_n)$. By definition $\Sigma(G') = G$, compatible with the pseudo-orientation. Put $f' = \Sigma^*(f)$ and $\omega' = \Sigma^*(\omega)$. Then by the transformation rule

$$\int_{G'} \mathrm{e}^{-f'}\omega' = \int_G \mathrm{e}^{-f}\omega.$$

Note that f' and ω' are defined over $k(i)$. This shows that (1) implies (2). Conversely, a number of the form in (2) is by definition a naive exponential period over $k(i)$. By Lemma 5.6 it is also a naive exponential period over k, so (2) implies (1). Let G, f, ω as in (1). We put $f = f_1 + if_2$ and $\omega = \omega_1 + i\omega_2$ and compute $\mathrm{e}^{-f}\omega$. The conditions on f and ω are equivalent to the conditions on f_1, f_2 and ω_1, ω_2. So properties (2) and (3) are equivalent.

The final claim follows as the equivalence proof of (1) and (2). □

5.3. **Convergence and definability.** The conditions on our domain of integration can be reformulated.

Lemma 5.9. *Let $f : \mathbb{A}^n \to \mathbb{P}^1$ be a rational function over k and let $G \subset \mathbb{C}^n$ be closed a semi-algebraic set such that f is regular and proper on G. Let ω be a rational differential form on $\mathbb{A}^n$ over k. Let $X \subset \mathbb{P}^n$ be the complement of the polar loci of f and ω, $\bar{X}$ a good compactification of X such that f extends to $\bar{f}\colon \bar{X} \to \mathbb{P}^1$. Let $\bar{G}$ be the closure of G in the real oriented blow-up $B_{\bar{X}}(X)$ of $\bar{X}$ at the divisor at infinity, see Remark 4.5, and $G_\infty = \bar{G} \smallsetminus G$. (The case $G_\infty = \emptyset$ is allowed.)*

Then f extends to a semi-algebraic C^∞-map $\tilde{f} : B_{\bar{X}}(X) \to \tilde{\mathbb{P}}^1$ of compact semi-algebraic C^∞-manifolds with corners with boundary mapping G_∞ to $\partial\tilde{\mathbb{P}}^1$. Moreover,

1. *(Naive exponential periods) $f(G) \subset S_{r,s}$ for some r, s if and only if $\tilde{f}(G_\infty) \subset \{1\infty\}$.*
2. *(Generalised naive exponential periods) $\overline{f(G)} \subset B^\circ$ if and only if $\tilde{f}(G_\infty) \subset B^\circ$.*

Proof. By definition of X, we have $\bar{f}^{-1}(\infty) \subset \bar{X} \smallsetminus X$. By Lemma 4.4 we get an induced C^∞-morphism of semi-algebraic manifolds with corners $\tilde{f}$.

Let $(g_i)_{i\geq 1}$ be a sequence in G converging to $g \in \bar{G}$. Assume $g \in G_\infty$. We have $g \in \partial B_{\bar{X}}(X)$ because $G \subset X^{\mathrm{an}}$ is closed. In particular, the image of g in $\bar{X}^{\mathrm{an}}$ is in the complement of X^{an}.

We claim that $\tilde{f}(g) \notin \mathbb{C}$. Assume $\tilde{f}(g) \in \mathbb{C} \subset \tilde{\mathbb{P}}^1$. Note that $\lim \tilde{f}(g_i) = \tilde{f}(g)$ by continuity. As f is proper, $f(G) \subset \mathbb{C}$ is closed. All $\tilde{f}(g_i)$ are in $f(G)$, hence so is $\tilde{f}(g)$. Let $D \subset \mathbb{C}$ be a closed disk around $\tilde{f}(g)$. It is compact, hence so is its preimage $E := \tilde{f}|_G^{-1}(D) \subset G$. There is $N \geq 1$ such that $\tilde{f}(g_i) \in D$ for all $i \geq N$. Hence their preimages g_i are in E. As E is compact, the limit point g is in E, in particular in G. This is a contradiction. We have shown that $\tilde{f}(G_\infty) \subset \partial\tilde{\mathbb{P}}^1$.

Note that $\overline{f(G)} = \tilde{f}(\bar{G})$. Hence (2) is obvious. For (1) note that $\bar{S}_{r,s} \cap \partial\tilde{\mathbb{P}}^1 = \{1\infty\}$. Hence $f(G) \subset S_{r,s}$ implies $\tilde{f}(G_\infty) \subset \{1\infty\}$. Conversely, consider a small open neighbourhood U of 1∞ in $\tilde{\mathbb{P}}^1$. It intersects $\mathbb{C}$ inside some strip of the form $S_{r,s}$. As $\bar{G}$ is compact, so is $G' = \bar{G} \smallsetminus \tilde{f}^{-1}(U)$. The image $f(G')$ is compact, so bounded in $\mathbb{C}$. By enlarging r and s, we ensure that both $f(G')$ and $f(G) \cap U$ are contained in the same strip. □

Lemma 5.10. *Let f and G be as in the definition of a generalised naive exponential period. Let $\bar{G}$ be the compactification of G as in Lemma 5.9 and $G_\infty = \bar{G} \smallsetminus G$. Let c be a rational function on $\mathbb{A}^n$ which is regular on G. The extension of $\mathrm{e}^{-f}c$ by 0 on G_∞ yields a continuous function on $\bar{G}$.*

Proof. Let $(g_i)_{i\geq 1}$ be a sequence in G converging to $g \in G_\infty$. Then

$$|\mathrm{e}^{-f(g_i)}| = \mathrm{e}^{-\Re(f(g_i))} \to 0$$

because $f(g_i)$ tends to $\tilde{f}(g) \in \partial B^\circ$. The function c has at worst a pole in g, but the exponential factors decays faster than $|c(g_i)|$ grows. In total

$$\lim_{i\to\infty} |\mathrm{e}^{-f(g_i)} c(g_i)| = 0.$$

□

Corollary 5.11. *Assume that G, f, ω define a generalised naive exponential period. Then*

$$\int_G \mathrm{e}^{-f}\omega$$

converges absolutely.

Proof. We apply Theorem 3.22 to $\bar{G} \subset B_{\bar{X}}(X)$ as in Lemma 5.9. It is compact. By Lemma 5.10, the C^∞-form $\mathrm{e}^{-f}\omega$ on G extends to a continuous form on $\bar{G}$. This is enough. □

Theorem 5.12. *If a number $\alpha \in \mathbb{C}$ is a naive exponential period over k, then its real and imaginary part are up to signs volumes of compact subsets $S \subset \mathbb{R}^n$ defined in the o-minimal structure $\mathbb{R}_{\sin,\exp} = (\mathbb{R}, <, 0, 1, +, \cdot, \sin|_{[0,1]}, \exp)$ with parameters from k_0.*

Proof. By Lemma 5.6, we may assume $k = k_0$. We use the characterisation of naive exponential periods given in parts (2) and (3) of Lemma 5.8. Thus $\alpha = \int_G \mathrm{e}^{-f}\omega$ with

$$\Re(\alpha) = \int_G \Big(\quad \cos(f_2)\mathrm{e}^{-f_1}\omega_1 + \sin(f_2)\mathrm{e}^{-f_1}\omega_2\Big),$$

$$\Im(\alpha) = \int_G \Big(-\sin(f_2)\mathrm{e}^{-f_1}\omega_1 + \cos(f_2)\mathrm{e}^{-f_1}\omega_2\Big)$$

where $G \subset \mathbb{R}^n$ is closed and k-semi-algebraic of dimension d carrying a pseudo-orientation, $f_1, f_2 \in k(z_1, \ldots, z_n)$ are regular and proper on G, $f_1(G)$ is bounded from below, $f_2(G)$ is bounded, and $\omega_1, \omega_2 \in \Omega^d_{k(z_1,\ldots,z_n)/k}$.

We want to apply Theorem 3.22. Again we apply it to the compact k_0-semialgebraic C^∞-manifold with corners $B_{\bar{X}}(X)$ of Lemma 5.9 and the closure $\bar{G}$ of G $B_{\bar{X}}(X)$. It is compact and a semi-algebraic subset of $B_{\bar{X}}(X)$, hence definable in $\mathbb{R}_{\sin,\exp}$. The forms $\Re(\mathrm{e}^{-f}\omega)$ and $\Im(\mathrm{e}^{-f}\omega)$ are definable on G_∞ because they vanish identically. Hence it remains to verify the definability on the affine G itself. The forms ω_1 and ω_2 are algebraic, in particular definable. By assumption f_2 is bounded, hence using Remark 2.14 the function $\sin(f_2)$ is definable in our o-minimal structure. The same is true for $\cos(f_2)$ because $\cos(f) = \sin(f + \pi/2)$, and π is definable in the o-minimal structure $\mathbb{R}_{\sin,\exp}$. □

Remark 5.13. The above argument does not work for generalised naive exponential periods. It is essential that the imaginary part of f is bounded on G. However, we are going to show (see [CH20, Theorem 13.4]) that every generalised naive exponential period is actually a naive exponential period, hence the consequence still applies.

Remark 5.14. In contrast to the case of ordinary periods, we do not expect that all volumes of definable sets in this o-minimal structure are naive exponential periods. The above argument only produces very special definable sets: there is no need of nesting exp or $\sin|_{[0,1]}$. The Euler number e is definable (as $\exp(1)$), hence also e^e (as $\exp(\mathrm{e})$). The number e is known to be an exponential period (e.g., $\int_0^1(\mathrm{e}^s + 1)\,\mathrm{d}s$). However, we do not see an obvious way to write e^e as an exponential period. It would be very interesting to give a characterisation of the sets that do occur.

5.4. **The definition of Kontsevich and Zagier.** In §4.3 of [KZ01], Kontsevich and Zagier give the following definition. An *exponential period in the sense of Kontsevich–Zagier* is "an absolutely convergent integral of the product of an algebraic function with the exponent of an algebraic function, over a real semi-algebraic set, where all polynomials entering the definition have algebraic coefficients". We take this to mean numbers of the form

$$\int_G \mathrm{e}^{-f}\omega$$

where $G \subset \mathbb{R}^n$ is semi-algebraic over $\tilde{\mathbb{Q}} = \overline{\mathbb{Q}} \cap \mathbb{R}$, $f \in \overline{\mathbb{Q}}(z_1, \dots, z_n)$, and ω a rational algebraic differential form defined over $\overline{\mathbb{Q}}$ such that the integral converges absolutely. It is not clear to us if they want $\dim(G) = n$. In this case, there is a prefered orientation from the orientation of $\mathbb{R}^n$, in the general case we have to orient G.

We have shown that naive and generalised naive exponential periods over $\overline{\mathbb{Q}}$ are absolutely convergent. In particular, a generalised naive exponential period over $\overline{\mathbb{Q}}$ is an exponential period in the sense of Kontsevich–Zagier.

What about the converse?

Example 5.15. Let $G = [1, \infty) \subset \mathbb{R}$, $f = iz$, $\omega = \frac{1}{z^2}\,\mathrm{d}z$. Then

$$\int_G \mathrm{e}^{-f}\omega = \int_1^\infty \frac{1}{t^2}\mathrm{e}^{-it}\,\mathrm{d}t = \int_1^\infty \frac{1}{t^2}\cos(-t)\,\mathrm{d}t + i\int_1^\infty \frac{1}{t^2}\sin(-t)\,\mathrm{d}t$$

converges absolutely because sin and cos are bounded by 1. However, the data does not define a generalised naive exponential period. The interval G is *not* a cycle for rapid decay homology of $(\mathbb{A}^1, \{1\})$. We do not have $\lim_{t\to\infty} \Re(f(t)) \to \infty$ on G.

Hence:

Conjecture 5.16. *There are exponential periods in the sense of Kontsevich–Zagier which are not (generalised) naive exponential periods.*

This is in the spirit of the period conjecture: if a number is not obviously a period, than it is not. As the example demonstrates, the condition on absolute convergence only implies that $\overline{f(G)} \subset \tilde{\mathbb{P}}^1$ is contained in $\mathbb{C} \cup \{s\infty | s \in S^1, \Re(s) \geq 0\}$. The above example uses the boundary point $i\infty$. For such f, the absolute convergence of the integral depends on the choice of ω.

We propose the following modification:

Definition 5.17. An *absolutely convergent exponential period* over k is a complex number obtained as the value of an absolutely convergent integral of the form

$$\int_G \mathrm{e}^{-f}\omega \tag{4}$$

where $G \subset \mathbb{C}^n$ is a pseudo-oriented (not necessarily closed) k_0-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n_k$ that is regular on G, f a rational function on $\mathbb{A}^n_k$ regular on G and the closure of $f(G)$ in $\tilde{\mathbb{P}}^1$ is contained in B°.

We denote $\mathcal{P}_{\mathrm{abs}}(k)$ the set of all absolutely convergent exponential periods over k.

Remark 5.18. The regularity condition for f and ω on G is harmless. We may replace G by the open subset G' of points in which f and ω are finite. The value of the integral only changes if $\dim(G-G')=\dim(G)$, i.e., if there is an open $U\subset G$ on which f or ω are infinite. The integral $\int_U \mathrm{e}^{-f}\omega$ does not make sense in this case, so we definitely want to exclude it. Note that the condition on $f(G)$ excludes Example 5.15 where we have $\overline{f(G)}=[i,i\infty]$ and $i\infty\notin B^\circ$.

We are going to show that every absolutely convergent exponential period is a generalised naive exponential period. Also for later use, let us be more precise.

Proposition 5.19. *Let α be an absolutely convergent exponential period over $k\subset\mathbb{R}$ with domain of integration as in (4) of dimension d. Then there are:*

- *a smooth affine variety X over k of dimension d,*
- *a simple normal crossings divisor $Y\subset X$,*
- *a closed k-semi-algebraic subset $G\subset X(\mathbb{R})$ of dimension d such that $\partial G = G\smallsetminus G^{\mathrm{int}}$ is contained in Y,*
- *a pseudo-orientation on G,*
- *a morphism $f\colon X_{k(i)}\to\mathbb{A}^1_{k(i)}$ such that $f|_G\colon G\to\mathbb{C}$ is proper and such that the closure $\overline{f(G)}\subset\tilde{\mathbb{P}}^1$ is contained in B°,*
- *a regular algebraic d-form ω on $X_{k(i)}$,*

such that

$$\alpha=\int_G \mathrm{e}^{-f}\omega.$$

Proof. We start with a presentation

$$\alpha=\int_G \mathrm{e}^{-f}\omega$$

with G of dimension d as in the definition of an absolutely convergent exponential period and modify the data without changing the value. In particular, G is equipped with a pseudo-orientation. With the same trick as in Lemma 5.8, we may assume that $G\subset\mathbb{R}^n=\mathbb{A}^n_k(\mathbb{R})$ is k-semi-algebraic with f,ω algebraic over $k(i)$.

Let $X_0\subset\mathbb{P}^n_k$ be the Zariski-closure of G. It is an algebraic variety defined over k of dimension d, see the characterisation of dimension in [BCR98, Definition 2.8.1]. Moreover, $\dim X_0(\mathbb{R})=d$ as a real algebraic set. By assumption, f is a rational map on $X_{0,k(i)}$. After replacing X_0 by a blow-up centered in the smallest subvariety of X_0 defined over k containing the locus of indeterminancy of f, it extends to a morphism $f_0\colon X_{0,k(i)}\to\mathbb{P}^1_{k(i)}$. By construction, $G\subset X_0(\mathbb{R})$. Let $G_0:=\bar{G}\subset X_0^{\mathrm{an}}$ be the closure. It is contained in $X_0(\mathbb{R})$ and compact because X_0^{an} is. It inherits a pseudo-orientation from G. Let $Y_0\subset X$ be the union of $X_{0,\mathrm{sing}}$ and the Zariski closure of ∂G_0, where the boundary is taken inside $X_0(\mathbb{R})$. It has dimension less than d.

As the next step, let $\pi\colon X_1\to X_0$ be a resolution of singularities such that the preimage Y_1 of Y_0 is a divisor with normal crossings. The map π is an isomorphism outside Y_0. As $Y_0\subset X_0$ has codimension at least 1, the intersection $G_0\cap Y_0(\mathbb{R})$ has real codimension at least 1 in G_0. Let G_1

be the "strict transform" of G_0 in X_1^{an}, i.e., the closure of the preimage of $U = G_0 \smallsetminus (G_0 \cap Y_0(\mathbb{R}))$. By construction $\partial G_1 \subset G_1 \smallsetminus \pi^{-1}(U) \subset Y_1$. Let $\omega_1 = \pi^*\omega$. By Lemma 3.16 (4), the set G_1 inherits a pseudo-orientation. Moreover,

$$\int_{G_1} \mathrm{e}^{-f_1 \circ \pi} \pi^* \omega = \int_G \mathrm{e}^{-f} \omega,$$

where the left hand side converges absolutely because the right hand side does.

We claim that after further blow-ups, we can reach $X_2 \to X_1$ preserving the properties of X_1, Y_1, and G_1 such that, in addition, points of G_2 in the polar locus of ω_2 are contained in the polar locus of f_2.

We first prove the claim. Let $X_{1,\infty}$ be the polar locus of f and $X_{1,\omega}$ the polar locus of ω, i.e., the smallest closed subvarieties over k such that their base change to $k(i)$ contains the poles of f and ω, respectively. Note that G_1 is disjoint from $X_{1,\infty}$ because f_1 is regular on G_1 and G_1 is contained in the real points of X_1.

Let $x \in G_1$ be a point such that f_1 is regular, but ω_1 has a pole. Let U_1 be a small compact neighbourhood of x in G_1 in which f_1 is regular. By assumption,

$$\int_{U_1} \mathrm{e}^{-f_1} \omega_1$$

converges absolutely. As f_1 is regular on U_1, the factor e^{-f_1} and its inverse are bounded. Hence the absolute convergence of the integral is equivalent to absolute convergence of the integral

$$\int_{U_1'} \omega_1.$$

This case already shows up in the case of ordinary periods, see the proof of [HMS17, Lemma 12.2.4]. The argument is due to Belkale and Brosnan in [BB03]. After a blow-up $X_2 \to X_1$ we find holomorphic coordinates such that the pull-back ω_2 of ω_1 has the shape

$$\text{unit} \times \prod_{j=1}^{n} z_j^{e_j} \, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$$

with $e_j \in \mathbb{Z}$. Absolute convergence is only possible if $e_j \geq 0$ for all j, i.e., if ω_2 is regular on U_2. This finishes the proof of the claim.

Let X be the complement of the polar loci of f_2 and ω_2, $Y = X \cap Y_2$, f and ω the restrictions of f_2 and ω_2 to X, and $G = X^{\mathrm{an}} \cap G_2$. The map $f_2 \colon G_2 \to (\mathbb{P}^1)^{\mathrm{an}}$ is proper, and hence so is $f \colon G \to \mathbb{C}$. The data satisfies all properties stated in the proposition, with the exception that X is only quasi-projective rather than affine. We have $X \subset \mathbb{P}^N_k$ for some N. Let H be the hypersurface defined by the equation $X_0^2 + \cdots + X_N^2 = 0$. Then $\mathbb{P}^N_k \smallsetminus H$ is affine. Note that $G \cap H^{\mathrm{an}} = \emptyset$ because $G \subset \mathbb{P}^N(\mathbb{R})$ and $H(\mathbb{R}) = \emptyset$. Hence we may replace X by $X \smallsetminus (X \cap H)$, making it quasi-affine. Now X is of the form $X' \smallsetminus V(s_1, \ldots, s_m)$ for finitely many $s_i \in \mathcal{O}(X')$. Let $H' = V(s_1^2 + \cdots + s_m^2)$. Note that $X(\mathbb{R}) = (X \smallsetminus H')(\mathbb{R})$. Hence we may replace X by its open subset $X' \smallsetminus H'$, making it affine. $\square$

Corollary 5.20. *The set of absolutely convergent exponential period equals the set of generalised naive exponential periods:*

$$\mathcal{P}_{\mathrm{gnv}}(k) = \mathcal{P}_{\mathrm{abs}}(k).$$

Proof. By Corollary 5.11, every generalised naive exponential period is an absolutely convergent exponential period.

Let α be an absolutely convergent exponential period. By the same argument as for naive exponential periods (see Lemma 5.6), we may replace k by $k \cap \mathbb{R}$. We apply Proposition 5.19. Let $X' \subset X$ be a dense open affine subvariety, $G' = G \cap X'^{\mathrm{an}}$. As $G \subset X(\mathbb{R})$ is of full dimension, we have $\dim(G \smallsetminus G') < \dim(G)$, hence the integral does not change when restricting to the open subset G' of G. We replace X, G by X', G'. Now $X \subset \mathbb{A}^n$. The morphism $f\colon X_{k(i)} \to \mathbb{A}^1_{k(i)}$ extends to a rational morphism $\mathbb{A}^n_{k(i)} \to \mathbb{A}^1_{k(i)}$. The differential form ω on X extends to a rational differential form on $\mathbb{A}^n_{k(i)}$. This data satisfies the assumptions of the definition of a generalised naive exponential period. □

Remark 5.21. It is not clear to us if it is equivalent to restrict to $G \subset \mathbb{R}^n$ of dimension n in the definition of an absolutely convergent exponential period. We tend to expect that it fails to be true. The analogous statement for ordinary periods holds true because they turn out to be volumes of bounded semi-algebraic sets (see [HMS17, Section 12.2], also [VS15]). We have replaced this by our Theorem 3.22. Close inspection of the proof only shows that every naive exponential period (and hence by [CH20, Theorem 13.4] also all absolutely convergent exponential periods) can be written as a $\mathbb{Z}$-linear combinations of numbers of the form

$$\int_G \mathrm{e}^{-f}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

for $G \subset \mathbb{R}^n$ of dimension n, $f\colon G \to \mathbb{C}$ continuous with *semi-algebraic* real and imaginary part.

Remark 5.22. We pick up again on Example 5.15. As explained previously, the integral $\int_1^\infty \mathrm{e}^{-it} \frac{\mathrm{d}t}{t}$ converges absolutely, but does not obviously define a generalised naive period. We concentrate on the real part. Integration by parts gives

$$\begin{aligned} \int_1^\infty \frac{\cos(t)}{t^2}\, \mathrm{d}t &= \cos(1) - \int_1^\infty \frac{\sin(t)}{t}\, \mathrm{d}t \\ &= \cos(1) - \frac{\pi}{2} + \int_0^1 \frac{\sin(t)}{t}\, \mathrm{d}t \end{aligned}$$

because of the classical identity $\int_0^\infty \frac{\sin(t)}{t}\, \mathrm{d}t = \frac{\pi}{2}$. Note that the function $\frac{\sin(t)}{t}$ is entire, so there are no convergence issues with the last integral. The numbers $\cos(1)$ and $\pi/2$ are definable in the o-minimal structure $\mathbb{R}_{\sin,\exp}$ of Definition 2.13. The same is true for the function $\frac{\sin(t)}{t}$. Hence we have written our number as the volume of a set that is definable in $\mathbb{R}_{\sin,\exp}$. Note, however, that the formula does not give a presentation as an absolutely convergent exponential period. We have

$$\int_0^1 \frac{\sin(t)}{t}\,\mathrm{d}t = \int_0^1 \Im\left(\frac{\mathrm{e}^{it}}{t}\right)\mathrm{d}t,$$

but the real part does not converge for the choice $G = (0,1)$, $f = iz$, and $\omega = \frac{\mathrm{d}z}{z}$.

6. Review of cohomological exponential periods

Throughout this section let $k \subset \mathbb{C}$ be a subfield. All varieties are defined over k.

We give the definition of exponential periods following [FJ20] concentrating on the smooth affine case at the moment.

6.1. **Rapid decay homology.** [FJ20, 1.1.1] Given a real number r, let $S_r = \{z \in \mathbb{C} \mid \Re(z) \geq r\}$.

Definition 6.1. Let X be a complex algebraic variety, $Y \subset X$ a subvariety, $f \in \mathcal{O}(X)$. The *rapid decay homology* of (X, Y, f) is defined as

$$H_n^{\mathrm{rd}}(X, Y, f) = \lim_{r\to\infty} H_n(X^{\mathrm{an}}, Y^{\mathrm{an}} \cup f^{-1}(S_r); \mathbb{Q}).$$

For $r' \geq r$, there is a projection map on relative homology, so this really is a projective limit. A direct limit construction using singular cohomology yields rapid cohomology $H^n_{\mathrm{rd}}(X, Y, f)$. It is dual to rapid decay homology. By [FJ20, 3.1.2], these limits stabilise, so it suffices to work with a single, big enough r. Indeed:

Theorem 6.2 (Verdier [Ver76, Corollaire 5.1])**.** *There is a finite set $\Sigma \subset \mathbb{C}$ such that $f|_{f^{-1}(\mathbb{C}\smallsetminus\Sigma)} \colon f^{-1}(\mathbb{C} \smallsetminus \Sigma) \to \mathbb{C} \smallsetminus \Sigma$ is a fibre bundle.*

As S_r is contractible, this implies that all $f^{-1}(S_r)$ with r sufficiently large are homotopy equivalent to a fibre of f.

There is an alternative description of $H_n^{\mathrm{rd}}(X, f)$ which is better suited to the computation of periods. It is originally due to Hien and Roucairol, see [HR08]. We follow the presentation of Fresán and Jossen in [FJ20, Section 3.5].

We fix a smooth variety X and $f \in \mathcal{O}(X)$. Let $\bar{X}$ be a good compactification, i.e., such that $\bar{X}$ is smooth projective, $X_\infty = \bar{X} \smallsetminus X$ is a divisor with normal crossing and f extends to $\bar{f} \colon \bar{X} \to \mathbb{P}^1$. We decompose $X_\infty = D_0 \cup D_\infty$ into simple normal crossings divisors such that $\bar{f}(D_\infty) = \{\infty\}$ and $\bar{f} \colon D_0 \to \mathbb{P}^1$ is dominant on all components, i.e., into vertical and horizontal components.

Definition 6.3. We denote by $\pi \colon B_{\bar{X}}(X) \to \bar{X}^{\mathrm{an}}$ the real oriented blow-up $\mathrm{OBl}_{X_\infty}(\bar{X})$, see Definition 4.2. Let $\tilde{f} : B_{\bar{X}}(X) \to \tilde{\mathbb{P}}^1$ be the induced map, see Lemma 4.4. We also define

$$B^\circ_{\bar{X}}(X, f) = B_{\bar{X}}(X) \smallsetminus \left(\pi^{-1}(D_0^{\mathrm{an}}) \cup \tilde{f}^{-1}(\{s\infty \in \tilde{\mathbb{P}}^1 | \Re(s) \leq 0\})\right),$$

$$\partial B^\circ_{\bar{X}}(X, f) = B^\circ_{\bar{X}}(X, f) \smallsetminus X^{\mathrm{an}} = B^\circ_{\bar{X}}(X, f) \cap \tilde{f}^{-1}(\{s\infty \in \tilde{\mathbb{P}}^1 | \Re(s) > 0\}).$$

We are going to omit the subscript $\bar{X}$ as long as it does not cause confusion.

At this point we only consider them as topological spaces. In fact $B^\circ_{\bar{X}}(X, f)$ is a semi-algebraic C^∞-manifold with corners.

Remark 6.4. Our definition of $B^\circ(X,f)$ does not agree with

$$B^{\circ,\mathrm{FJ}} = X^{\mathrm{an}} \cup \tilde{f}^{-1}(\{s\infty \in \tilde{\mathbb{P}}^1 | \Re(s) > 0\})$$

as defined by Fresán–Jossen [FJ20, Section 3.5] and the earlier rapid decay literature. The two definitions differ if D_0 and D_∞ intersect. They agree in the curve case where $D_0 \cap D_\infty = \emptyset$ is automatic. If the intersection is non-empty, then $B^{\circ,\mathrm{FJ}}$ is not a manifold with corners whereas $B^\circ(X,f)$ always is. The issue is also addressed in [MH17, Section 2].

Proposition 6.5 ([FJ20, Proposition 3.5.2]). *Let X be a smooth variety over k. For sufficiently large r, the inclusion induces natural isomorphisms*

$$H_n(X^{\mathrm{an}}, f^{-1}(S_r);\mathbb{Q}) \cong H_n(B(X), \tilde{f}^{-1}(S_r);\mathbb{Q}) \cong H_n(B^\circ(X,f), \partial B^\circ(X,f);\mathbb{Q}).$$

In particular,

$$H_n^{\mathrm{rd}}(X,f) \cong H_n(B^\circ(X,f), \partial B^\circ(X,f);\mathbb{Q}).$$

Proof. Their proof is correct with the modified notion of $B^\circ(X,f)$. □

Recall that $S_*(M)$ denotes the complex of $\mathbb{Q}$-linear combinations of C^1-simplices for a C^p-manifold with corners M. Recall also Definition 1.1. It computes singular cohomology by Theorem 1.3.

Definition 6.6. Let X be a smooth variety, $f \in \mathcal{O}(X)$. Choose a good compactification $\bar{X}$. We put

$$S_*^{\mathrm{rd}}(X,f) = S_*(B^\circ_{\bar{X}}(X,f))/S_*(\partial B^\circ_{\bar{X}}(X,f)).$$

Remark 6.7. Fresán and Jossen work with piecewise C^∞-simplices instead, see [FJ20, Section 7.2.4]. We opt for the slightly more complicated notion of C^1-simplices as opposed to C^∞-simplices because they are well-suited for working with our semi-algebraic sets.

6.2. **Twisted de Rham cohomology: the smooth case.** Let X/k be a smooth variety, $f \in \mathcal{O}(X)$. We define a vector bundle with connection $\mathcal{E}^f = (\mathcal{O}_X, d_f)$ with $d_f(1) = -df$. The de Rham complex $\mathrm{DR}(\mathcal{E}^f)$ has the same entries as the standard de Rham complex for X, but with differential $\Omega^p \to \Omega^{p+1}$ given by $d\omega - df \wedge \omega$.

Definition 6.8. Let (X,f) be as above. We define *algebraic de Rham cohmology* $H^*_{\mathrm{dR}}(X,Y,f)$ of (X,f) as hypercohomology of $\mathrm{DR}(\mathcal{E}^f)$.

If X is affine, this is nothing but cohomology of the complex

$$R\Gamma_{\mathrm{dR}}(X) := [\mathcal{O}(X) \xrightarrow{d_f} \Omega^1(X) \xrightarrow{d_f} \dots].$$

The definition needs to be extended to the relative cohomology of singular varieties. We first consider a special case. Let X be smooth and $Y \subset X$ a simple divisor with normal crossings. Let $Y_\bullet \to Y$ be the Čech-nerve of the cover of Y by the disjoint union of its irreducible components, see Section 1. It is a smooth proper hypercover. In particular, $H_n(Y_\bullet^{\mathrm{an}},\mathbb{Z}) = H_n(Y^{\mathrm{an}},\mathbb{Z})$.

Definition 6.9. Let X be a smooth variety, $Y \subset X$ a divisor with simple normal crossings. We define *algebraic de Rham cohomology* $H^*_{\mathrm{dR}}(X,Y,f)$ of (X,Y,f) as hypercohomology of $\mathrm{Cone}\left(\pi_*\mathrm{DR}(\mathcal{E}^f|_{Y_\bullet}) \to \mathcal{E}^f\right)[-1]$.

6.3. **The period isomorphism.** Hien and Roucairol established the existence of a canonical isomorphism

$$H^n_{\mathrm{rd}}(X,f)\otimes_{\mathbb{Q}}\mathbb{C}\to H^n_{\mathrm{dR}}(X,f)\otimes_k\mathbb{C}$$

for smooth affine varieties X see [HR08, Theorem 2.7]. It is also explained and extended to the relative case for any variety X and subvariety Y by Fresán and Jossen, see [FJ20, Theorem 7.6.1]. We refer to it as the *period isomorphism.* It induces a *period pairing*

$$\langle -,-\rangle\colon H^n_{\mathrm{dR}}(X,Y,f)\times H_n^{\mathrm{rd}}(X,Y,f)\to\mathbb{C}. \tag{5}$$

Definition 6.10. Let X be a variety, $f\in\mathcal{O}(X)$, $Y\subset X$ a closed subvariety, $n\in\mathbb{N}_0$. The elements in the image of the period pairing (5) are called the *(cohomological) exponential periods* of (X,Y,f,n).

We denote $\mathcal{P}_{\mathrm{coh}}(k)$ the set of cohomological exponential periods for varying (X,Y,f,n) over k. We denote $\mathcal{P}_{\log}(k)$ the subset of cohomological exponential periods for varying (X,Y,f,n) such that (X,Y) is a log-pair.

The construction of the period map is non-trivial. Fortunately, we only need its explicit description in a special case.

Definition 6.11. Let X be smooth affine. We define a pairing

$$\Omega^n(X)\times S_n^{\mathrm{rd}}(X^{\mathrm{an}},f)\to\mathbb{C}$$

by mapping (ω,σ) to

$$\int_\sigma \mathrm{e}^{-f}\omega^{\mathrm{an}}.$$

Lemma 6.12. *The pairing is well-defined and induces a morphism of complexes*

$$\Omega^*(X,f)\to\mathrm{Hom}(S_*^{\mathrm{rd}}(X^{\mathrm{an}},f),\mathbb{C}).$$

On cohomology it induces the pairing (5).

Proof. Let $\omega\in\Omega^n(X)$, σ an n-dimensional C^1-simplex in $S_n^{\mathrm{rd}}(X^{\mathrm{an}},f)$. The smooth form ω^{an} on X^{an} defines a smooth form $\mathrm{e}^{-f}\omega^{\mathrm{an}}$ on $B^\circ(X,f)$. (Note that $\mathrm{e}^{-f}\omega^{\mathrm{an}}$ vanishes to any order on $\partial B^\circ(X,f)$, so it can be extended by 0 to a neighbourhood of the boundary). Hence the integral is well-defined.

The compatibility with the boundary map translates as

$$\int_\sigma \mathrm{e}^{-f}d_f\omega^{\mathrm{an}}=\int_{\partial\sigma}\mathrm{e}^{-f}\omega^{\mathrm{an}}$$

which holds by Stokes's formula (see Theorem 1.4) because $d_f\omega=d\omega-df\wedge\omega$.

The construction is the one of [FJ20, Chapter 7.2.7], only with our $S_*(X)$ instead of their complex, see Remark 6.7. □

By taking double complexes, this extends to general X and Y. We will discuss this in detail in [CH20, Section 10]. At this point, we handle the simplest case.

Example 6.13. Let X be a smooth affine variety, $Y \subset X$ a smooth closed subvariety, $f \in \mathcal{O}(X)$. Then relative twisted de Rham cohomology is computed by the complex

$$\begin{aligned}R\Gamma_{\mathrm{dR}}(X,Y,f) &= \mathrm{Cone}\left(\Omega^*(X) \to \Omega^*(Y)\right)[-1]\\ &= \left[\Omega^0(X) \to \Omega^1(X) \oplus \Omega^0(Y) \to \Omega^2(X) \oplus \Omega^1(Y) \to \dots\right]\end{aligned}$$

with differential induced by d_f and restriction. Its rapid decay homology is computed by the complex

$$S_*^{\mathrm{rd}}(X,Y,f) = \mathrm{Cone}\left(S_*^{\mathrm{rd}}(Y,f) \to S_*^{\mathrm{rd}}(X,f)\right).$$

Explicitly: let $\bar{X}$ be a good compactification of X such that f extends to a morphism on $\bar{X}$ with target $\mathbb{P}^1_k$ and such that the closure $\bar{Y}$ of Y in $\bar{X}$ is a good compactification as well. Then

$$\begin{aligned}&\mathrm{Cone}(S_*^{\mathrm{rd}}(Y,f) \to S_*^{\mathrm{rd}}(X,f)) =\\ &\quad [S_0^{\mathrm{rd}}(X,f) \leftarrow S_1^{\mathrm{rd}}(X,f) \oplus S_0^{\mathrm{rd}}(Y,f) \leftarrow S_2^{\mathrm{rd}}(X,f) \oplus S_1^{\mathrm{rd}}(Y,f) \leftarrow \dots].\end{aligned}$$

Let σ be a cycle in $S_n^{\mathrm{rd}}(X,Y)$, i.e., a chain σ_X on X such that $\partial\sigma_X = \sigma_Y$ is supported on Y. In the second incarnation, we identify it with $(\sigma_X, -\sigma_Y)$. Let ω be cocycle in $R\Gamma_{\mathrm{dR}}(X,Y,f)$, i.e., a pair of differential forms $(\omega_X, \omega_Y) \in \Omega^n(X) \oplus \Omega^{n-1}(Y)$ such that $d\omega_X = 0$, $d\omega_Y = \omega_X|_Y$. Their period is

$$\langle[\omega],[\sigma]\rangle = \int_{\sigma_X} \omega_X - \int_{\sigma_Y} \omega_Y.$$

7. Triangulations

We fix a real closed field $\tilde{k} \subset \mathbb{R}$ and work with semi-algebraic sets of $\mathbb{R}^N$ defined over $\tilde{k}$. We expect that everything holds in general for o-minimal structures, but we do not need this for our application. We use the set-up of [vdD98, Chapter 8] for complexes. It is not completely standard, but very convenient for us.

Let $n \in \mathbb{N}_0$. Let $a_0, \dots, a_n \in \tilde{k}^N$ be affine independent. The *open n-simplex* defined by these vectors is the set

$$\begin{aligned}\sigma &= (a_0, \dots, a_n)\\ &= \left\{ \sum_{i=0}^n \lambda_i a_i \in \mathbb{R}^N : \text{for all } i \text{ we have } \lambda_i > 0 \text{ and } \lambda_0 + \dots + \lambda_n = 1 \right\}.\end{aligned}$$

We fix the orientation given by $d\lambda_1 \wedge \dots \wedge d\lambda_n$. The closure of σ is denoted by $[a_0, \dots, a_n]$ and obtained by relaxing to $\lambda_i \geq 0$ in the definition above. We call $[a_0, \dots, a_n]$ a *closed k-simplex*. The points $a_0, \dots, a_n$ are uniquely determined by $[a_0, \dots, a_n]$ and thus by σ. As usual, a face of σ is a simplex spanned by a non-empty subset of $\{a_0, \dots, a_n\}$. Then $[a_0, \dots, a_n]$ is a disjoint union of faces of σ. We write $\tau < \sigma$ if τ is a face of σ and $\tau \neq \sigma$.

A finite set K of simplices in $\mathbb{R}^N$ is called a *complex* if for all $\sigma_1, \sigma_2 \in K$ the intersection $\overline{\sigma_1} \cap \overline{\sigma_2}$ is either empty or the closure of common face τ of

σ_1 and σ_2. Van den Dries's definition does not ask for τ to lie in K. So the *polyhedron* spanned by K

$$|K| = \bigcup_{\sigma \in K} \sigma$$

may not be a closed subset of $\mathbb{R}^N$. We call K a closed complex if $|K|$ is closed or equivalently, if for all $\sigma \in K$ and all faces τ of σ, we have $\tau \in K$. Note that $\bigcup_{\sigma\in K} \sigma$ is a disjoint union, this is an advantage of working with "open" simplices. We write $\overline{K}$ for the complex obtained by taking all faces of all simplices in K. Note that K is $\tilde{k}$-semi-algebraic.

Definition 7.1. Let M be a $\tilde{k}$-semi-algebraic C^1-manifold with corners, $A \subset M$ be a $\tilde{k}$-semi-algebraic subset. A *semi-algebraic triangulation* of A is a pair (h, K) where K is a complex and where $h : |K| \to A$ is a $\tilde{k}$-semi-algebraic homeomorphism. We say that it is *globally of class* C^1, if h extends to a C^1-map on an open neighbourhood of $|K|$.

Let $B \subset A$ be a $\tilde{k}$-semi-algebraic subset. We say that (h, K) is compatible with B if $\Phi(B)$ is a union of members of K.

Remark 7.2. Note that there are weaker definitions of C^1-triangulations in the literature, see for example Remark 9.2.3(a) [BCR98] or [Shi97, Chapter II]. However, Ohmoto-Shiota have shown the existence of semi-algebraic triangulations globally of class C^1 for locally closed semi-algebraic subsets of $\mathbb{R}^N$, see [OS17]. Czapla-Pawłucki [CP18] show even stronger regularity properties (that we do not need) in the o-minimimal setting.

7.1. **Existence of triangulations.** Our aim is to triangulate semi-algebraic manifolds with corners, see Definition 3.1.

Lemma 7.3. *Let X a compact $\tilde{k}$-semi-algebraic C^1-manifold with corners with atlas $(\phi_i : U_i \to V_i | i = 1, \dots, N)$. Then there are $\tilde{k}$-semi-algebraic functions of class C^1*

$$f_1, \dots, f_m : X \to [0, 1]$$

such that for every j there is i

- *such that the support of f_j is contained in U_i,*
- *there is an open subset $W_j \subset U_i$ on which f_j is identically 1,*

and, moreover, the W_j are a cover of X.

Proof. For each $P \in X$ we fix a chart U_i containing P. Each V_i is an open subset of some $\mathbb{R}^{n_i} \times \mathbb{R}^{m_i}_{\geq 0}$. There is an open ball B in $\mathbb{R}^{n_i+m_i}$ centered at $\phi_i(P)$ of radius $r > 0$ such that $\phi_i(P) \in B \cap \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}_{\geq 0} \subset V_i$. Let $f'_P : \mathbb{R}^{n_i+m_i} \to [0, 1]$ be a $\tilde{k}$-semi algebraic C^1-function that is identically 1 on the open ball of radius $r/2$ centered at $\phi_i(P)$ and with support contained completely in B. We denote by W_P the preimage in U_i of the said ball of radius $r/2$ and by f_P the composition $f'_P \circ \phi_i$ extended by zero on $X \smallsetminus V_i$. As X is compact, there are finitely many $P_1, \dots, P_m$ such that $W_{P_1} \cup \dots \cup W_{P_m} = X$. The lemma follows with $W_j = W_{P_j}$ and $f_j = f_{P_j}$. □

Proposition 7.4. *Let X be a compact $\tilde{k}$-semi-algebraic C^1-manifold with corners, $A_1, \dots, A_M$ semi-algebraic subsets of X. Then there is a $\tilde{k}$-semi-algebraic triangulation of X compatible with $A_1, \dots, A_M$ that is globally of class C^1.*

Proof. As a first step, we ignore the regularity issue and consider X as a compact $\tilde{k}$-semi-algebraic space. By [Rob83, Theorem 1] it is affine. By [BCR98, Theorem 9.2.1] or [vdD98, Theorem 8.2.9] it admits a $\tilde{k}$-semi-algebraic triangulation (h, K) of X compatible with $A_1, \ldots, A_M$. Using [OS17], we are going to construct a $\tilde{k}$-semi-algebraic homeomorphism $\Phi : |K| \to |K|$ which respects all simplices and such that $h \circ \Phi$ is C^1. This new triangulation has the required properties.

In detail: Let $\phi_1, \ldots, \phi_N$ and $f_1, \ldots, f_m$ be as in Lemma 7.3. The map $f_j\phi_i : X \to \mathbb{R}^d$ is well-defined and $\tilde{k}$-semi-algebraic. We apply the "panel beating" of [OS17, Corollary 3.3] to the maps

$$g_j = (f_j\phi_i) \circ h : |K| \to X \to \mathbb{R}^d.$$

Note that they formulate the results in the semi-algebraic setting, but they point out that the proof is written in a way that it also applies in other settings such as ours.

This gives us a $\tilde{k}$-semi-algebraic homeomorphism $\Phi : |K| \to |K|$ respecting all simplices such that all $g_j \circ \Phi$ are C^1. We claim that $h \circ \Phi$ is C^1. As $W_1, \ldots, W_m$ cover X, it suffices to check the claim after restricting to the preimage of some W_j. By definition, a map is C^1 if its composition with ϕ_i is. This is the case because $f_j\phi_i = \phi_i$ on W_j and g_j is C^1. □

Remark 7.5. We briefly sketch what the C^1-triangulation result of Ohmoto–Shiota [OS17] boils down to in the 1-dimensional setting. This suffices for the context considered in Section 8. Say $\gamma\colon [0,1] \to \mathbb{R}^N$ is a continuous $\tilde{k}$-semi-algebraic map, then there exist $0 = t_1 < \cdots < t_m = 1$ in $\tilde{k}$ such that all $\gamma|_{(t_i,t_{i+1})}$ are C^1. Thus γ is represented in homology by a chain of paths that are C^1 on $(0,1)$. So assume that γ is such a path. For $\ell \in \mathbb{Z}$ suffiently large the right-sided derivative of $t \mapsto \gamma(t^\ell)$ at $t = 0$ exists and vanishes; this follows from asymptotic behavoir of semi-algebraic functions see [vdDM96, 4.12]. Extending $t \mapsto \gamma(t^\ell)$ to the left with value $\gamma(0)$ yields a C^1-function on $(-\infty, 1)$. The same procedure works at $t = 1$ by reparametrizing with $1 - (1-t)^\ell$ and extending to the right with value $\gamma(1)$.

7.2. **A deformation retract.** We are going to show that, up to deformation, a complex K can be identified with a closed complex. The arguments are similar to the ones in [vdD98, Chapter 8 (3.5)]. Compare also Friedrich's [HMS17, Proposition 2.6.8] and its proof.

If σ is a simplex in $\mathbb{R}^N$, then $b(\sigma)$ denotes its barycenter. Let $K \subset \mathbb{R}^N$ be a complex. We denote by $\beta(K)$ its barycentric subdivision as defined in [vdD98, Chapter 8 (1.8)]. Note that $|K| = |\beta(K)|$.

We define the *closed core* of a complex K as

$$\mathrm{cc}(K) = \{\sigma \in K \mid K \text{ contains all faces of } \sigma\}.$$

Then $\mathrm{cc}(K)$ is a subcomplex of K. It is a closed complex by definition. But it can be empty: consider a complex consisting of a single simplex of positive dimension. This problem is remedied by passing to the barycentric subdivision. More precisely, if K is non-empty, then $\mathrm{cc}(\beta(K))$ is non-empty. Indeed, the barycenter $b(\sigma)$ of $\sigma \in K$ defines a face $(b(\sigma))$ of $\beta(K)$; it must lie in $\mathrm{cc}(\beta(K))$.

Finally, note that if L is a subcomplex of K, then $\beta(L) \subset \beta(K)$ and $\mathrm{cc}(L) \subset \mathrm{cc}(K)$, so we have $\mathrm{cc}(\beta(L)) \subset \mathrm{cc}(\beta(K))$.

Proposition 7.6. *Let K be a complex. There exists a $\tilde{k}$-semi-algebraic retraction $r\colon |K| = |\beta(K)| \to |\mathrm{cc}(\beta(K))|$ with the following properties.*

(i) For each $x \in |K|$ the half open line segment $[x, r(x))$ is contained in the simplex of $\beta(K)$ containing x.

(ii) The map

$$H(x,t) = (1-t)x + tr(x)$$

is a $\tilde{k}$-semi-algebraic strong deformation retraction $H\colon |K| \times [0,1] \to |K|$ onto $|\mathrm{cc}(\beta(K))|$.

We use a variation of the arguments found in §3, Chapter 8 [vdD98].

Proof. Let $b = b(\sigma)$ be a vertex of $\beta(K)$. As in loc. cit. we define a continuous semi-algebraic function

$$\lambda_\sigma\colon |K| = |\beta(K)| \to [0,1]$$

which vanishes on $|\tau|$ if b is not a vertex of $\tau \in \beta(K)$ and equals the barycentric coordinate with respect to b if it is.

Let us define furthermore

$$\Lambda(x) = \sum_{\sigma \in K} \lambda_\sigma(x).$$

We claim that $\Lambda(x) > 0$ for all $x \in |K|$. Indeed, x is contained in a simplex $(b(\sigma_0), \ldots, b(\sigma_n))$ of $\beta(K)$; here $\sigma_0 < \cdots < \sigma_n$ are open simplices of $\overline{K}$ and $\sigma_n \in K$. In particular, $\lambda_{\sigma_n}(x) > 0$. Thus the contribution coming from σ_n to the sum $\Lambda(x)$ is strictly positive. As all other contributions are non-negative we find $\Lambda(x) > 0$, as desired.

We are ready to define $r(x)$ for $x \in |K|$ as

$$r(x) = \frac{\sum_{\sigma \in K} \lambda_\sigma(x) b(\sigma)}{\Lambda(x)}.$$

Thus $r\colon |K| \to \mathbb{R}^m$ is $\tilde{k}$-semi-algebraic and continuous.

Let us verify that $r(|K|) \subset |\mathrm{cc}(\beta(K))|$. Say $x \in |K|$ and let $\sigma_0, \ldots, \sigma_n$ be as before. Say $\sigma \in K$. We recall that $\lambda_\sigma(x) > 0$ if and only if σ is among $\{\sigma_0, \ldots, \sigma_n\}$. Let $\sigma_{i_0} < \cdots < \sigma_{i_k} = \sigma_n$ be those among the $\sigma_0, \ldots, \sigma_n$ that lie in K. So $r(x) = \sum_{j=0}^k \alpha_j b(\sigma_{i_j})$ with coefficients $\alpha_j \in [0,1]$ such that $\sum_{j=0}^k \alpha_j = 1$. Observe that $\alpha_j > 0$ since $x \in (b(\sigma_0), \ldots, b(\sigma_n))$. Thus $r(x) \in (b(\sigma_{i_0}), \ldots, b(\sigma_{i_k}))$. Finally, $b(\sigma_{i_j}) \in \sigma_{i_j} \in K$ for all j. Therefore, $\beta(K)$ contains all faces of the simplex $(b(\sigma_{i_0}), \ldots, b(\sigma_{i_k}))$ which must thus be an element of $\mathrm{cc}(\beta(K))$. We conclude $r(x) \in |\mathrm{cc}(\beta(K))|$. So the target of r is $\mathrm{cc}(\beta(K))$, as claimed.

Moreover, $(b(\sigma_{i_0}), \ldots, b(\sigma_{i_k}))$ is a face of $(b(\sigma_0), \ldots, b(\sigma_n)) \in \beta(K)$, hence by convexity the ray $[x, r(x))$ is in the simplex of $\beta(K)$ containing x.

We now verify that r is a retraction. We still assume $x \in (b(\sigma_0), \ldots, b(\sigma_n))$ as above. Note that $x = \sum_{\sigma \in \overline{K}} \lambda_\sigma(x) b(\sigma)$ and $\sum_{\sigma \in \overline{K}} \lambda_\sigma(x) = 1$. If $\lambda_\sigma(x) > 0$

for some $\sigma \in \overline{K}$, then σ is among $\sigma_0, \dots, \sigma_n$. Hence

$$\sum_{i=0}^{n} \lambda_{\sigma_i}(x) b(\sigma_i) = x \quad \text{and} \quad \sum_{i=0}^{n} \lambda_{\sigma_i}(x) = 1.$$

Now suppose $x \in |\mathrm{cc}(\beta(K))|$. By definition, $\beta(K)$ contains all faces of $(b(\sigma_0), \dots, b(\sigma_n))$. In particular, $b(\sigma_i) \in |K|$ and hence $\sigma_i \in K$ for all i. So $\Lambda(x) = 1$ and $r(x) = x$. In particular, r is a retraction.

Keeping the notation above for $x \in (b(\sigma_0), \dots, b(\sigma_n)) \in \beta(K)$, we find for all $t \in (0, 1)$ that

$$(1-t)x + tr(x) = \frac{1}{\Lambda(x)} \sum_{i=0}^{n} \left((1-t)\Lambda(x) + t w_{\sigma_i}(x)\right) \lambda_{\sigma_i}(x) b(\sigma_i),$$

here w_{σ_i} is constant 1 if $\sigma_i \in K$ and constant 0 else wise. Each factor in the sum on the right is strictly positive, which implies $(1-t)x + tr(x) \in (b(\sigma_0), \dots, b(\sigma_n))$. As we have seen before, $r(x) = x$ for $x \in |\mathrm{cc}(\beta(K))|$. Altogether, this proves claim (ii). □

8. The case of curves

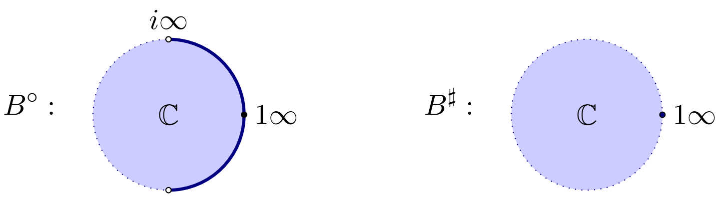
Let $k \subset \mathbb{C}$ be a subfield which is algebraic over $k_0 = k \cap \mathbb{R}$. For simplicity, we assume that k is algebraically closed. In this section we will show that naive exponential periods of the form $\int_G \mathrm{e}^{-f}\omega$ where G is 1-dimensional are the same as cohomological exponential periods of smooth marked curves. This comparison is a special case of the general result in [CH20, Theorem 13.4], but we include it to illustrate the key ideas of the general proof, while avoiding several technical problems.

This section is organised as follows: first we give some elementary examples of cohomological exponential periods and explain why they are naive exponential periods. This is followed by an intermezzo in which we describe the oriented real blow-up of a marked curve, because it features several times in the remainder of the section. Finally, we prove the inclusions announced above.

8.1. **Examples of cohomological exponential periods.**

In Section 5.1 we saw explicit examples of naive exponential periods. We will now look at some examples of cohomological exponential periods, before considering the case for general curves.

Example 8.1. We start with the simplest non-trivial case: $X = \mathbb{A}^1$, $Y = \{0\}$, $f = \mathrm{id}$. Then $H_1^{\mathrm{rd}}(\mathbb{A}^1, \{0\}, \mathrm{id}) = H_1(B^\circ(\mathbb{A}^1, \mathrm{id}), \{0\} \cup \partial B^\circ(\mathbb{A}^1, \mathrm{id}); \mathbb{Q})$. Both $B^\circ = B^\circ(\mathbb{A}^1, \mathrm{id})$ and its boundary are contractible, hence $H_1^{\mathrm{rd}}(\mathbb{A}^1, \{0\}, \mathrm{id})$ is of dimension 1. The generator is the path from 0 to a point on ∂B°, i.e., one of the G_s of Example 5.3. We use $G_1 = [0, \infty)$ because it is in the subspace $B^\sharp$ as defined in Section 1.4.



The boundary in singular homology maps it to the class of the point 0 with multiplicity -1.

The relative de Rham complex has the shape

$$k[z] \xrightarrow{P\mapsto(dP-P\,\mathrm{d}z,P(0))} k[z]\,\mathrm{d}z \oplus k.$$

As in Example 6.13, the periods of $(Q\,\mathrm{d}z, a)$ are computed as

$$\int_{G_1} \mathrm{e}^{-z} Q\,\mathrm{d}z - a.$$

The general theory tells us that $H^1_{\mathrm{dR}}(\mathbb{A}^1, \{0\}, \mathrm{id})$ also has dimension 1. It is easy to see that $(\mathrm{d}z, 0)$ is a not in the image of the differential: Indeed $P \mapsto dP - P\,\mathrm{d}z$ is injective, and the preimage of $\mathrm{d}z$ under this injection is the constant polynomial -1, which does not have constant coefficient 0. Hence $(\mathrm{d}z, 0)$ generates our cohomology. The periods of $(\mathbb{A}^1, \{0\}, \mathrm{id}, 1)$ are precisely the elements k as

$$\int_{G_1} \mathrm{e}^{-z}\,\mathrm{d}z = 1.$$

Unsurprisingly, these elements are naive exponential periods as explained in Example 5.3. We now turn to $X = \mathbb{A}^1, Y = \{0\}$ and $f = z^n$. In this case the boundary of $B^\circ(\mathbb{A}^1, f)$ has n components, hence $H^{\mathrm{rd}}_1(\mathbb{A}^1, \{0\}, f)$ is of dimension n. As generators for homology we can use the n different preimages of $[0, \infty)$ under $z \mapsto z^n$. They are of the form G_{s^m} for $m = 0, \dots, n-1$ with $s = \mathrm{e}^{2\pi i/n}$. The boundary map in singular homology maps each of them to the point 0 with multiplicity -1.

In this case the de Rham complex has the shape

$$k[z] \xrightarrow{P\mapsto(dP-nz^{n-1}P\,\mathrm{d}z,P(0))} k[z]\,\mathrm{d}z \oplus k.$$

All elements in $H^1_{\mathrm{dR}}(\mathbb{A}^1, \{0\}, f)$ are represented by pairs (QdZ, a). Their periods are computed as

$$\int_{G_s} \mathrm{e}^{-z^n} Q\,\mathrm{d}z - a.$$

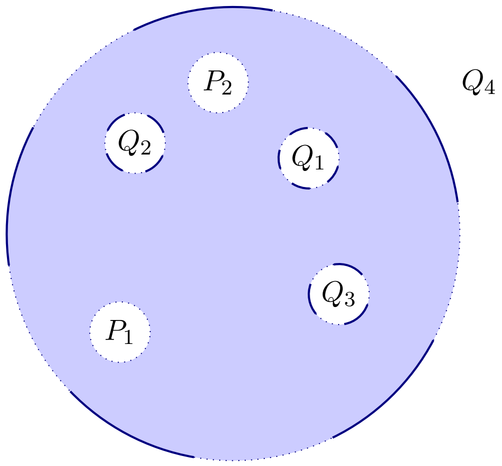
These are naive exponential periods.

Remark 8.2. The preceding example provides an explicit instance of [CH20, Proposition 11.4] which is an important ingredient in the final comparison theorem: rapid decay homology is not only computed by $B^\circ(\mathbb{A}^1, f)$, but also by $B^\sharp(\mathbb{A}^1, f) = \mathbb{C} \cup \tilde{f}^{-1}(1\infty)$ so we can choose intervals with end points E in $\{0\} \cup \tilde{f}^{-1}(1\infty)$.

8.2. **The oriented real blow-up of a marked curve.** Let $\bar{C}$ be a smooth projective complex curve, or in other words, a compact Riemann surface. Let $\bar{f}\colon \bar{C} \to \mathbb{P}^1$ be a non-constant meromorphic function. Let $Q_1, \dots, Q_n \in \bar{C}$ denote the poles of $\bar{f}$, let $P_1, \dots, P_m$ be some points on $\bar{C}$ distinct from the Q_i, and denote by $C \subset \bar{C}$ the complement of $\{P_1, \dots, P_m, Q_1 \dots Q_n\}$. Denote by $f\colon C \to \mathbb{A}^1$ the restriction of $\bar{f}$ to C.

We now consider the real oriented blow-up $B(C) = B_{\bar{C}}(C)$ and the map $\tilde{f}\colon B(C) \to \tilde{\mathbb{P}}^1$ induced by f. It adds a circle to $\bar{C}^{\mathrm{an}}$ in each of the points P_i and Q_j. The algebraic map $f\colon C \to \mathbb{A}^1$ induces a semi-algebraic map of manifolds with boundary $\tilde{f}\colon B(C) \to \tilde{\mathbb{P}}^1$. The circles around the P_i are mapped to $f(P_i) \in \mathbb{C}$. The circles around the Q_i are mapped to the circle at infinity of $\tilde{\mathbb{P}}^1$. As in Definition 6.3 let $B^\circ(C, f) \subset B(C)$ be the open subset of points either in C^{an} or mapping to $\Re(s\infty) > 0$ on the boundary of $\tilde{\mathbb{P}}^1$. So it removes the circles around the P_i's and some circle segments from the circles around the the Q_j's.

The following figure illustrates the case $\bar{C} = \mathbb{P}^1$.



8.3. **A 1-dimensional comparison.** We now show that generalised naive exponential periods are cohomological exponential periods.

Proposition 8.3. *Let $\alpha = \int_G \mathrm{e}^{-f}\omega$ be a generalised naive exponential period over k_0 as in Definition 5.4. Assume that $\dim(G) = 1$. Then α is a cohomological exponential period for a tuple $(C, Y, f, 1)$, where C is a smooth curve defined over k, $Y \subset C$ is finite set of points, and $f\colon C \to \mathbb{A}^1_k$ is a regular function.*

This is a special case of [CH20, Proposition 12.1].

Proof. By Corollary 5.11, every generalised naive exponential period is absolutely convergent. Hence we may apply Proposition 5.19 to obtain a smooth affine curve C over k_0, a finite set of points $Y \subset C(k_0)$, a pseudo-oriented 1-dimensional k-semi-algebraic subset G of $C(\mathbb{R})$ with endpoints in Y, a function $f\colon C_k \to \mathbb{A}^1_k$ that is proper on G and such that $\overline{f(G)} \subset B^\circ$, and a regular 1-form ω on C_k, such that $\alpha = \int_G \mathrm{e}^{-f}\omega$. By abuse of notation we replace C and Y by C_k and Y_k from now on.

Certainly, the form ω defines a class $[\omega] \in H^1_{\mathrm{dR}}(C, Y, f)$.

The semi-algebraic set $\mathrm{Reg}_1(G)$ is semi-algebraically homeomorphic to a finite union of open intervals and circles. We may consider connected components separately. Thus, without loss of generality, $\mathrm{Reg}_1(G)$ is homeomorphic to an open interval and G its closure in C^{an}. The semi-algebraic set G is homeomorphic to either a circle, or to an interval with 0, 1 or 2 end points in C^{an}. By assumption, we are given an orientation on the complement of finitely many points of G. We may consider these intervals separately, enlarging Y if necessary.

Let $\bar{C}$ be a smooth compactification of C, and $\bar{G}$ the closure of G in $B(C)$. It is compact because $B(C)$ is. Lemma 5.9 implies $\bar{G} \subset B^\circ(C, g)$.

By construction, the boundary of $\bar{G}$ is contained in $Y \cup \partial B^\circ_{\bar{C}}(C,f)$. It defines a class $[G] \in H_1^{\mathrm{rd}}(C,Y,f) = H_1(B^\circ(C,g), Y \cup \partial B^\circ(C,f); \mathbb{Q})$. Finally, as in Example 6.13, the period pairing of these classes is computed as

$$\langle [\omega], [G] \rangle = \int_{\bar{G}} \mathrm{e}^{-f} \omega = \alpha.$$

This proves the result: α is indeed a cohomological exponential period. □

8.4. **Converse direction.** We now want to express cohomological exponential periods as naive exponential periods. This means that we start with a marked curve $Y \subset C$, and cohomology classes $\gamma \in H_1^{\mathrm{rd}}(C,Y)$ and $\omega \in H^1_{\mathrm{dR}}(C,Y,f)$. We want to show that the period pairing $\langle \omega, \gamma \rangle$ is a naive exponential period. Let us sketch the ingredients of the proof:

(*i*) The first step is the observation that rapid decay homology $H_1^{\mathrm{rd}}(C,Y)$ is computed as the ordinary homology of the space $B^\circ(C,f)$.

(*ii*) We then note that $B^\circ(C,f)$ is homotopic to a certain subset $B^\sharp(C,f)$. We will give an ad hoc definition of this subset here, for the general definition see [CH20, Definition 11.3].

This step is crucial, because in the next step it will allow us to obtain semi-algebraic sets G whose image is contained in a suitable strip: $f(G) \subset S_{r,s}$. See also Remark 8.2.

(*iii*) Finally, we use semi-algebraic triangulation results and the delicate Proposition 7.6 to realise γ as a linear combination of homology classes of semi-algebraic sets. This will allow us to realise $\langle \omega, \gamma \rangle$ as naive exponential period.

Proposition 8.4 ([CH20, Proposition 11.1]). *Let $C \subset \mathbb{A}^n$ be a smooth affine curve over k, $f \in \mathcal{O}(C)$, and $Y \subset C$ a proper closed subvariety. Then every cohomological exponential period of $(C,Y,f,1)$ is a naive exponential period.*

Proof. By definition, $f \in k[C]$. We also write f for a polynomial in $k[z_1,\dots,z_n]$ representing it. As C is affine, the twisted de Rham cohomology $H^1_{\mathrm{dR}}(C,Y,f)$ is a quotient of $\Omega^1(C) \oplus \bigoplus_{y \in Y} k$ hence every element is represented by a tuple (ω, a_y). We also write ω for the element of $\Omega^1(\mathbb{A}^n)$ representing $\omega \in \Omega^1(C)$.

Step 1. Let $\bar{C}$ be a smooth compactification of C and let $Z = \bar{C} \smallsetminus C$ be the points at infinity. By Proposition 6.5,

$$H_1^{\mathrm{rd}}(C,Y;\mathbb{Z}) = H_1(B^\circ(C,f), Y^{\mathrm{an}} \cup \partial B^\circ(C,f); \mathbb{Z}).$$

We decompose $Z = Z_f \cup Z_\infty$ such that f is regular in the points of Z_f and has a pole in the points of Z_∞. Let $d_z \geq 1$ be the multiplicity of $\bar{f}$ at $z \in Z$. The oriented real blow-up of $\bar{C}$ in Z replaces each point $z \in Z^{\mathrm{an}}$ by a circle S_z. It is compact. The boundary is a disjoint union of circles. The map $\tilde{f}\colon B(C) \to \tilde{\mathbb{P}}^1$ maps these circles either to $\mathbb{C}$ (the case $z \in Z_f$) or to the circle at infinity (the case $z \in Z_\infty$). In the latter case, the map on the circle is a d_z to 1 cover.

By definition the subset $B^\circ(C,f)$ is the union of the preimage of C^{an} and those points P in the circles S_z above $z \in Z^{\mathrm{an}}_\infty$ that are in the preimage of the half circle $\{w\infty \mid \Re(w) > 0\}$. Hence the boundary of $B^\circ(C,f)$ consists of d_z many circle segments for every $z \in Z^{\mathrm{an}}_\infty$.

Step 2. Now consider the smaller subset $B^\sharp(C,f)$ defined as the union of the preimage of C^{an} and the points P in the circles S_z above $z \in Z_\infty^{\mathrm{an}}$ that are in the preimage of 1∞. Hence the boundary $\partial B^\sharp(C,f)$ of $B^\sharp(C,f)$ consists of d_z many disjoint points for every $z \in Z_\infty^{\mathrm{an}}$. In particular the boundaries of $B^\circ(C,f)$ and $B^\sharp(C,f)$ are homotopy equivalent. Both $B^\circ(C,f)$ and $B^\sharp(C,f)$ are homotopy equivalent to C^{an}. Thus

$$H_1^{\mathrm{rd}}(C,Y;\mathbb{Z}) = H_1(B^\sharp(C,f), Y^{\mathrm{an}} \cup \partial B^\sharp(C,f);\mathbb{Z}).$$

Note that $B^\sharp(C,f)$ is not a manifold with corners, hence we are not able to interpret the right hand side in the sense of C^1-homology as defined in Section 1.5. However, it is a topological space so ordinary singular homology is perfectly well-defined and this is how we interpret the right-hand side.

Step 3. The space $B^\circ(C,f)$ is a k_0-semi-algebraic C^∞-manifold with boundary. By Proposition 7.4, it has a k_0-semi-algebraic triangulation compatible with $B^\sharp(C,f)$, Y and $\partial B^\sharp(C,f)$ that is globally of class C^1. In particular, the points in $Y^{\mathrm{an}} \cup \partial B^\sharp(C,f)$ are vertices. By Proposition 7.6, the closed core of its barycentric subdivision is a strong deformation retraction of $B^\sharp(C,f)$. We denote the closed core by A. Hence

$$H_1(B^\sharp(C,f), Y^{\mathrm{an}} \cup \partial B^\sharp(C,f);\mathbb{Z}) = H_1(A, Y^{\mathrm{an}} \cup \partial B^\sharp(C,f);\mathbb{Z}).$$

The subcomplex A is compact, hence simplicial and singular homology of A agree. Therefore every homology class is represented by a linear combination of closed semi-algebraic 1-simplices in A. The triangulation is C^1, hence the closed 1-simplices in the triangulation of C define elements of $S_1(C,f)$. In all, each homology class in $H_1^{\mathrm{rd}}(C,Y;\mathbb{Z})$ is represented by linear combination of C^1-paths in $B^\sharp(C,f)$ with boundary in $Y^{\mathrm{an}} \cup \partial B^\sharp(C,f)$. The period integral is defined by integrating $\mathrm{e}^{-f}\omega$ on these paths, see Definition 6.11 and Lemma 6.12.

Let $\gamma\colon [0,1] \to A$ be one these simplices. We put $G = \gamma([0,1]) \cap C^{\mathrm{an}}$. We need to check that it satisfies the conditions needed for naive exponential periods. The closure $\bar{G} = \gamma([0,1])$ differs from G by at most two points, the end points. The image $\tilde{f}(\bar{G})$ in $\tilde{\mathbb{P}}^1$ is compact and contained in $B^\sharp(C,f)$, hence $f(G)$ is contained in a suitable strip $S_{r,s}$ for $r,s > 0$. The map $\tilde{f}\colon \bar{G} \to \tilde{\mathbb{P}}^1$ is proper because $\bar{G}$ is compact. By definition, the preimage $\tilde{f}^{-1}(1\infty)$ does not contain any points of G. Hence $f\colon G \to \mathbb{C}$ is also proper. We conclude that $\int_G \mathrm{e}^{-f}\omega$ is a naive exponential period.

Our cohomological period was a linear combination of such. The same arguments as in the case of ordinary periods (see [HMS17, Proposition 12.1.5]) show that a linear combination of naive exponential periods is a naive exponential period. □

References

[BB03] P. Belkale and P. Brosnan. Periods and Igusa local zeta functions. *Int. Math. Res. Not.*, 49:2655–2670, 2003.

[BBT18] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. o-minimal GAGA and a conjecture of Griffiths, 2018.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors.

[BE00] Spencer Bloch and Hélène Esnault. Gauß-Manin determinant connections and periods for irregular connections. Number Special Volume, Part I, pages 1–31. 2000. GAFA 2000 (Tel Aviv, 1999).

[CH20] Johan Commelin and Annette Huber. Exponential periods and o-minimality II, 2020.

[CP18] Małgorzata Czapla and Wiesław Pawłucki. Strict C^1-triangulations in o-minimal structures. *Topol. Methods Nonlinear Anal.*, 52(2):739–747, 2018.

[DK81] Hans Delfs and Manfred Knebusch. Semialgebraic topology over a real closed field. II. Basic theory of semialgebraic spaces. *Math. Z.*, 178(2):175–213, 1981.

[DMR07] Pierre Deligne, Bernard Malgrange, and Jean-Pierre Ramis. *Singularités irrégulières*, volume 5 of *Documents Mathématiques (Paris) [Mathematical Documents (Paris)]*. Société Mathématique de France, Paris, 2007. Correspondance et documents. [Correspondence and documents].

[FJ20] J. Fresán and P. Jossen. Exponential motives, 2020. version July/20, manuscript available at http://javier.fresan.perso.math.cnrs.fr/expmot.pdf.

[Fri04] B. Friedrich. Periods and algebraic de Rham cohomology, 2004. Diplomarbeit Leipzig, arXiv:math.AG/0506113v1.

[Gil] William D. Gillam. Oriented real blowup. Unpublished notes available at `http://www.math.boun.edu.tr/instructors/wdgillam/orb.pdf`.

[Hie07] Marco Hien. Periods for irregular singular connections on surfaces. *Math. Ann.*, 337(3):631–669, 2007.

[HKT15] Masaki Hanamura, Kenichiro Kimura, and Tomahide Terasoma. Integrals of logarithmic forms on semi-algebraic sets and a generalized Cauchy formula, Part I: convergence theorems. Preprint arXiv:1509.06950, 2015.

[HMS17] Annette Huber and Stefan Müller-Stach. *Periods and Nori motives*, volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2017. With contributions by Benjamin Friedrich and Jonas von Wangenheim.

[HR08] Marco Hien and Céline Roucairol. Integral representations for solutions of exponential Gauss-Manin systems. *Bull. Soc. Math. France*, 136(4):505–532, 2008.

[KUY18] B. Klingler, E. Ullmo, and A. Yafaev. Bi-algebraic geometry and the André-Oort conjecture. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 319–359. Amer. Math. Soc., Providence, RI, 2018.

[KZ01] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[Maj84] Hideyuki Majima. *Asymptotic analysis for integrable connections with irregular singular points*, volume 1075 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[MH17] Saiei-Jaeyeong Matsubara-Heo. On the rapid decay homology of F.Pham, 2017. arXiv:1705.06052.

[OS17] Toru Ohmoto and Masahiro Shiota. C^1-triangulations of semialgebraic sets. *J. Topol.*, 10(3):765–775, 2017.

[PS84] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. *Bull. Amer. Math. Soc. (N.S.)*, 11(1):159–162, 1984.

[Rob83] Robert Robson. Embedding semi-algebraic spaces. *Math. Z.*, 183(3):365–370, 1983.

[Shi97] Masahiro Shiota. *Geometry of subanalytic and semialgebraic sets*, volume 150 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997.

[vdD84] Lou van den Dries. Remarks on Tarski's problem concerning ($\mathbf{R}$, $+$, $\cdot$, exp). In *Logic colloquium '82 (Florence, 1982)*, volume 112 of *Stud. Logic Found. Math.*, pages 97–121. North-Holland, Amsterdam, 1984.

[vdD86] Lou van den Dries. A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bull. Amer. Math. Soc. (N.S.)*, 15(2):189–193, 1986.

[vdD98] L. van den Dries. *Tame Topology and O-minimal Structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[vdDM94] Lou van den Dries and Chris Miller. On the real exponential field with restricted analytic functions. *Israel J. Math.*, 85(1-3):19–56, 1994.

[vdDM96] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84(2):497–540, 1996.

[Ver76] J.-L. Verdier. Stratifications de Whitney et théorème de Bertini-Sard. *Invent. Math.*, 36:295–312, 1976.

[VS15] Juan Viu-Sos. A semi-canonical reduction for periods of Kontsevich-Zagier, 2015. arXiv:1509.01097, to appear: Int. J. of Number Theory.

[War83] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Whi57] Hassler Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.

[Wil96] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.