EXPONENTIAL PERIODS AND O-MINIMALITY II

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Abstract. This paper is a sequel to [CHH20]. We complete the comparison between different definitions of exponential periods, and show that they all lead to the same notion. In [CHH20], we show that *naive* exponential periods are *absolutely convergent* exponential periods. We also show that naive exponential periods are up to signs volumes of definable sets in the o-minimal structure generated by $\mathbb{Q}$, the real exponential function and $\sin|_{[0,1]}$.

In this paper, we compare these definitions with *cohomological* exponential periods and periods of exponential Nori motives. In particular, naive exponential periods are the same as periods of exponential Nori motives, which justifies that the definition of naive exponential periods singles out the correct set of complex numbers to be called *exponential* periods.

Introduction

We strongly advise the reader to read the introduction of the companion paper [CHH20]. Let us now recall the definition of one of the main protagonists of that paper.

Let $k \subset \mathbb{C}$ be a subfield such that k is algebraic over $k_0 := k \cap \mathbb{R}$. See Section 9.1 for more on this condition on k. Recall from [CHH20, Definition 0.2] that a *naive exponential period* over k is a complex number of the form

$$\int_G \mathrm{e}^{-f}\omega$$

where $G \subset \mathbb{C}^n$ is an pseudo-oriented (not necessarily compact) closed k_0-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n_k$ that is regular on G and f is a rational function on $\mathbb{A}^n_k$ such that f is regular and proper on G and, moreover, $f(G)$ is contained in a strip

$$S_{r,s} = \{z \in \mathbb{C} \mid \Re(z) > r, |\Im(z)| < s\}.$$

The definition of *generalised naive exponential periods* and *absolutely convergent exponential periods* uses the same data, but with weaker conditions on f, ω and G. These definitions are repeated in detail in Definition 9.3.

There is an alternative approach of a very different flavour. As far as we understand, it is actually the original one: exponential periods appear as the entry of a period matrix in Hodge theory of vector bundles with irregular connections. We refer to the introduction of [CHH20] for more background. We call the elements in the image of the *period pairing*

$$H_n^{\mathrm{rd}}(X,Y;\mathbb{Q}) \times H^n_{\mathrm{dR}}(X,Y,f) \to \mathbb{C}$$

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cohomological exponential periods, see Definition 10.12.

Ordinary periods have an even more conceptual interpretation as a $\mathbb{C}$-valued point on the torsor of isomorphisms between the de Rham realisation and the Betti realisation, two fibre functors on the Tannaka category of mixed (Nori) motives, see [HMS17]. The same picture also applies in the case of exponential periods. Fresán and Jossen have developed a fully fledged theory of exponential motives in [FJ20].

In this sequel to [CHH20] we show that these three approaches yield the same set of exponential periods. Putting all the pieces of our two papers together, we get the following comparison theorem.

Theorem (Theorem 13.4)**.** *Let $k \subset \mathbb{C}$ be a field such that k/k_0 is algebraic. Then the following subsets of $\mathbb{C}$ agree:*

(1) $\mathcal{P}_{\mathrm{nv}}(k)$, i.e., naive exponential periods over k;
(2) $\mathcal{P}_{\mathrm{gnv}}(k)$, i.e., generalised naive exponential periods over k;
(3) $\mathcal{P}_{\mathrm{abs}}(k)$, i.e., absolutely convergent exponential periods over k;
(4) $\mathcal{P}_{\mathrm{mot}}(k)$, i.e., periods of all effective exponential motives over k;
(5) $\mathcal{P}_{\mathrm{coh}}(k)$, i.e., the set of periods of all (X, Y, f, n) with X a k-variety, $Y \subset X$ a subvariety, $f \in \mathcal{O}(X)$, $n \in \mathbb{N}_0$;
(6) $\mathcal{P}_{\log}(k)$, i.e., periods of all (X, Y, f, n) with (X, Y) a log pair, $f \in \mathcal{O}(X)$, $n \in \mathbb{N}_0$;
(7) $\mathcal{P}_{\mathrm{SmAff}}(k)$, i.e., periods of all $(X_\bullet, f_\bullet, n)$ for $(X_\bullet, f_\bullet) \in C_-(\mathrm{SmAff}/\mathbb{A}^1)$, $n \in \mathbb{N}_0$.

Moreover, the real and imaginary part of these numbers are up to sign volumes of bounded definable sets for the o-minimal structure $\mathbb{R}_{\sin,\exp,k_0}$ generated by $\exp$, $\sin|_{[0,1]}$ *and with paramaters in k_0, see* [CHH20, Definition 2.13]*.*

Global structure of the proof. We recall the following diagram from [CHH20]. It explains the global structure of the two papers, and how the different theorems contribute to the main comparison result.

$$\begin{tikzcd}
\mathrm{Vol} & & & & \\
\mathcal{P}_{\mathrm{nv}} \arrow[u, hook, "{\text{[CHH20, Theorem 5.12]}}"] \arrow[rr, hook, "{\text{[CHH20, Lemma 5.5]}}"] & & \mathcal{P}_{\mathrm{gnv}} \arrow[rr, equal, "{\text{[CHH20, Corollary 5.20]}}"] \arrow[dl, hook, "{\text{Proposition 12.1}}"] & & \mathcal{P}_{\mathrm{abs}} \\
& \mathcal{P}_{\log} \arrow[ul, hook, "{\text{Proposition 11.1}}"] \arrow[r, hook, "\text{triv}"] & \mathcal{P}_{\mathrm{coh}} \arrow[r, hook, "\text{triv}"] & \mathcal{P}_{\mathrm{SmAff}} \arrow[dl, hook, "{\text{Proposition 13.3}}"] & \\
& & \mathcal{P}_{\mathrm{mot}} \arrow[ul, hook, "{\text{Proposition 13.1}}"] & &
\end{tikzcd}$$

Structure of this paper. In Section 9 we recall notation and definitions that were introduced in [CHH20]. Section 10 is a technical section on the definition of cohomological exponential periods. For smooth affine varieties, we gave a definition in [CHH20, Section 6]. We now extend this definition to arbitrary pairs (X, Y) of a variety X and a closed subvariety $Y \subset X$.

As suggested by the diagram above, Section 11 is devoted to proving $\mathcal{P}_{\log}(k) \subset \mathcal{P}_{\mathrm{nv}}(k)$, whereas Section 12 shows the inclusion $\mathcal{P}_{\mathrm{gnv}}(k) \subset \mathcal{P}_{\log}(k)$. Finally, in Section 13 we prove the remaining parts, which are all very formal, and glue all the pieces together to obtain the main theorem.

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9. Recapitulations

We keep the notation from [CHH20]. We repeat them for the convenience of the reader.

9.1. **Fields of definition.** If z is a complex number, we write $\Re(z)$ and $\Im(z)$ for its real and imaginary part. Let $k \subset \mathbb{C}$ be a subfield. We denote by k_0 the intersection $k \cap \mathbb{R}$, by $\bar{k}$ the algebraic closure of k in $\mathbb{C}$, and by $\tilde{k}$ the real closure of k_0 in $\mathbb{R}$. The following conditions on k are equivalent:

$$k_0 \subset k \text{ is alg.} \iff k_0 \subset \bar{k} \text{ is alg.} \iff \tilde{k} \subset \bar{k} \text{ is alg.} \iff [\bar{k} : \tilde{k}] = 2.$$

If k satisfies these conditions, so does every intermediate extension $k \subset L \subset \mathbb{C}$ with $k \subset L$ algebraic.

9.2. **Categories of varieties.** Let $k \subset \mathbb{C}$ be a subfield. By variety we mean a quasi-projective reduced separated scheme of finite type over k. By X^{an} we denote the associated analytic space on $X(\mathbb{C})$.

9.3. **Good compactifications.** We say that a pair (X, Y) is a *log-pair*, if X is smooth variety of pure dimension d, and $Y \subset X$ a simple normal crossings divisor. A *good compactification* of (X, Y) is the choice of an open immersion $X \subset \bar{X}$ such that $\bar{X}$ is smooth projective, X is dense in $\bar{X}$ and $\bar{Y} + X_\infty$ is a simple normal crossings divisor where $\bar{Y}$ is the closure of Y in $\bar{X}$ and $X_\infty = \bar{X} \smallsetminus X$. If, in addition, we have a structure morphism $f\colon X \to \mathbb{A}^1$, we say that $\bar{X}$ is a good compactification relative to f if f extends to $\bar{f}\colon \bar{X} \to \mathbb{P}^1$. Good compactifications (relative to a structure morphism) exist by resolution of singularities, see also [CHH20, Section 1.3].

9.4. **Some semi-algebraic sets.** Let k be as in Section 9.1. Let X be a smooth variety, $f \in \mathcal{O}(X)$, $\bar{X}$ a good compactification, $X_\infty = \bar{X} \smallsetminus X$. We decompose $X_\infty = D_0 \cup D_\infty$ into simple normal crossings divisors such that $\bar{f}(D_\infty) = \{\infty\}$ and $\bar{f} : D_0 \to \mathbb{P}^1$ is dominant on all components, i.e., into vertical and horizontal components.

We denote by $B_{\bar{X}}(X)$ the oriented real blow-up of $\bar{X}^{\mathrm{an}}$ in X_∞^{an}, for details see [CHH20, Definition 4.2]. It is a k_0-semi-algebraic C^∞-manifold with corners, see [CHH20, Proposition 4.3].

In the case $X = \mathbb{A}^1$ and $\bar{X} = \mathbb{P}^1$, we write $\tilde{\mathbb{P}}^1 = B_{\mathbb{P}^1}(\mathbb{A}^1)$. This is a manifold with boundary: the compactification of $\mathbb{C} \cong \mathbb{R}^2$ by a circle at infinity, adding one point for each half ray emanating from the origin.

For $s \in \mathbb{C} \smallsetminus \{0\}$, we write $s\infty$ for the point of $\partial\tilde{\mathbb{P}}^1$ corresponding to the half ray $s[0,\infty)$. We say $\Re(s\infty) > 0$ if $\Re(s) > 0$, and analogously for $\leq$. We put

$$\begin{aligned} B^\circ &= \tilde{\mathbb{P}}^1 \smallsetminus \{s\infty \in \partial\tilde{\mathbb{P}}^1 \mid \Re(s\infty) \leq 0\} && = \mathbb{C} \cup \{s\infty \mid \Re(s) > 0\}, \\ \partial B^\circ &= B^\circ \smallsetminus \mathbb{C} && = \quad \{s\infty \mid \Re(s) > 0\}, \\ B^\sharp &= \tilde{\mathbb{P}}^1 \smallsetminus \{s\infty \in \partial\tilde{\mathbb{P}}^1 \mid s\infty \neq 1\infty\} && = \mathbb{C} \cup \{1\infty\}, \\ \partial B^\sharp &= B^\sharp \smallsetminus \mathbb{C} && = \quad \{1\infty\}. \end{aligned}$$

See [CHH20, Example 4.1] and [CHH20, Example 8.1] for illustrations.

We now return to the situation $f\colon X \to \mathbb{A}^1$ for general smooth X and f. Let $\tilde{f} : B_{\bar{X}}(X) \to \tilde{\mathbb{P}}^1$ be the induced map, see [CHH20, Lemma 4.4]. We also define

$$\begin{aligned} B^\circ_{\bar{X}}(X,f) &= B_{\bar{X}}(X) \smallsetminus \{x \in \partial(B_{\bar{X}}(X)) \mid \pi(x) \in D_0^{\mathrm{an}} \text{ or } \Re(\tilde{f}(x)) \leq 0\} \\ \partial B^\circ_{\bar{X}}(X,f) &= B^\circ_{\bar{X}}(X,f) \smallsetminus X^{\mathrm{an}} = B^\circ_{\bar{X}}(X,f) \cap \tilde{f}^{-1}(\{s\infty \in \tilde{\mathbb{P}}^1 \mid \Re(s) > 0\}). \end{aligned}$$

We are also going to need a variant (see Definition 11.3)

$$\begin{aligned} B^\sharp_{\bar{X}}(X,f) &= B_{\bar{X}}(X) \smallsetminus \{x \in \partial(B_{\bar{X}}(X)) \mid \pi(x) \in D_0^{\mathrm{an}} \text{ or } \tilde{f}(x) \neq 1\infty\} \\ \partial B^\sharp_{\bar{X}}(X,f) &= B^\sharp_{\bar{X}}(X) \smallsetminus X^{\mathrm{an}}. \end{aligned}$$

9.5. **C^1-homology.** In this paper, we denote by Δ_n the simplex

$$\{(x_1,\dots,x_n) \mid x_i > 0 \text{ and } \sum_i x_i < 1\} \subset \mathbb{R}^n.$$

It is open in the ambient space, from which it inherits the standard orientation. We denote by $\bar{\Delta}_n$ its closure in $\mathbb{R}^n$, and define the face maps $k^i\colon \bar{\Delta}_{n-1} \to \bar{\Delta}_n$ as in [War83, (2) p.142].

Let X be a C^1-manifold with corners, see [CHH20, Section 1.5] for more details. A *C^1-simplex* on X is a C^1-map

$$\sigma\colon \bar{\Delta}_n \to X$$

of C^1-manifolds with corners.

Let $S_n(X)$ be the space of formal $\mathbb{Q}$-linear combinations of C^1-simplices of dimension n. For $A \subset X$ closed, we denote $S_n(A) \subset S_n(X)$ the subspace spanned by simplices with image in A.

The restriction of σ to a face is again C^1, hence the usual boundary operator ∂ turns $S_*(X)$ into a complex. The barycentric subdivision of a C^1-simplex is again C^1.

As we argue in [CHH20, Theorem 1.3], given a C^1-manifold with corners, the complexes $S_*(X)$ and $S_*(X)/S_*(\partial X)$ compute singular homology of X and of $(X,\partial X)$, respectively.

9.6. **Semi-algebraic manifolds with corners.** Let k be as in Section 9.1. In [CHH20, Definition 3.1], we introduced the notions of a definable C^p-manifolds with corners and definable subsets $G \subset M$ with respect to a fixed o-minimal structure. In the present paper we restrict to the case of the o-minimal structure $\mathbb{R}_{\mathrm{alg},k_0}$ and call them *k_0-semi-algebraic.*

Definition 9.1 (See [CHH20, Definition 3.11])**.** Let $p \geq 1$. Let M be a k_0-semi-algebraic C^p-manifold with corners and G a k_0-semi-algebraic subset.

A *differential form* ω *of degree* d *on* G is a continuous section

$$\omega \colon G \to \Lambda^d T^* M.$$

In order to integrate differential forms, we need a notion of orientability.

Definition 9.2 (See [CHH20, Definition 3.14])**.** Fix an integer $p \geq 1$, let $d \geq 0$ be an integer, and let M be a k_0-semi-algebraic C^p-manifold with corners with $G \subset M$ a k_0-semi-algebraic subset of dimension d.

(1) A *pseudo-orientation* on G is the choice of an equivalence class of a definable open subset $U \subset \mathrm{Reg}_d(G)$ such that $\dim(G \smallsetminus U) < d$ and an orientation on U. Two such pairs are equivalent if the they agree on the intersection.
(2) Given a pseudo-orientation on G with U as in (1) and a differential form ω of degree d on G, we define

$$\int_G \omega := \int_U \omega$$

if the integral on the right converges absolutely.

By [CHH20, Theorem 3.22], the integral converges absolutely if G is compact.

9.7. **Periods.** Let k be as in Section 9.1.

Definition 9.3 (See [CHH20, Definition 0.2], [CHH20, Definition 5.4], [CHH20, Definition 5.17])**.** Let $k \subset \mathbb{C}$ be a subfield, such that k is algebraic over $k \cap \mathbb{R}$. A complex number

$$\alpha = \int_G \mathrm{e}^{-f} \omega$$

is called

(1) *naive exponential period* over k if $G \subset \mathbb{C}^n$ is a pseudo-oriented closed (not necessarily compact) k_0-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n_k$ that is regular on G and f is a rational function on $\mathbb{A}^n_k$ such that f is regular and proper on G and, moreover, $f(G)$ is contained in a strip

$$S_{r,s} = \{z \in \mathbb{C} \mid \Re(z) > r, |\Im(z)| < s\};$$

(2) *generalised naive exponential period* over k if $G \subset \mathbb{C}^n$ is a pseudo-oriented closed k_0-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n_k$ that is regular on G and f is a rational function on $\mathbb{A}^n_k$ such that f is regular and proper on G and, moreover, the closure of $f(G)$ in $\tilde{\mathbb{P}}^1$ is contained in $B^\circ = \mathbb{C} \cup \{s\infty \mid s \in S^1, \Re(s) > 0\}$;
(3) *absolutely convergent exponential period* over k if $G \subset \mathbb{C}^n$ is a pseudo-oriented (not necessarily closed) k_0-semi-algebraic subset, ω is a rational algebraic differential form on $\mathbb{A}^n$, f a rational function on $\mathbb{A}^n_k$ that is regular on G and the closure of $f(G)$ in $\tilde{\mathbb{P}}^1$ is contained in B°.

We denote $\mathcal{P}_{\mathrm{nv}}(k)$, $\mathcal{P}_{\mathrm{gnv}}(k)$ and $\mathcal{P}_{\mathrm{abs}}(k)$ the sets of all naive exponential periods over k, all generalised naive periods over k and all absolutely convergent exponential periods over k, respectively.

By definition, $\mathcal{P}_{\mathrm{nv}}(k) \subset \mathcal{P}_{\mathrm{gnv}}(k)$. By [CHH20, Corollary 5.20], we have $\mathcal{P}_{\mathrm{gnv}}(k) = \mathcal{P}_{\mathrm{abs}}(k)$. For properties and alternative descriptions of these sets we refer for [CHH20, Section 5].

10. Exponential periods: the general case

Throughout this section let $k \subset \mathbb{C}$ be a subfield such that k is algebraic over $k_0 = k \cap \mathbb{R}$. All varieties are defined over k.

We turn to the definition of exponential periods for general (X, Y), again following Fresán and Jossen in [FJ20]. Notation for the smooth affine case was set-up in [CHH20, Section 6].

10.1. **Complexes of varieties.** By $\mathrm{SmAff}/\mathbb{A}^1$ we denote the category of smooth affine varieties X together with a structure map $f\colon X \to \mathbb{A}^1$. Note that we do not require f to be smooth. Let $\mathbb{Z}[\mathrm{SmAff}/\mathbb{A}^1]$ be the additive hull of $\mathrm{SmAff}/\mathbb{A}^1$:

- the objects are the objects of $\mathrm{SmAff}/\mathbb{A}^1$;
- the morphisms are formal $\mathbb{Z}$-linear combinations of morphisms in $\mathrm{SmAff}/\mathbb{A}^1$, more precisely for connected X we have
$$\mathrm{Hom}_{\mathbb{Z}[\mathrm{SmAff}/\mathbb{A}^1]}(X, Y) = \mathbb{Z}[\mathrm{Mor}_{\mathrm{SmAff}/\mathbb{A}^1}(X, Y)];$$
- the disjoint union is the direct sum.

We denote by $C_+(\mathrm{SmAff}/\mathbb{A}^1)$ the category of bounded below homological complexes over $\mathbb{Z}[\mathrm{SmAff}/\mathbb{A}^1]$.

We denote by $\mathrm{SmProj}/\mathbb{P}^1$ the category of smooth projective varieties X together with a structure map $f\colon X \to \mathbb{P}^1$. As in the affine case we define $\mathbb{Z}[\mathrm{SmProj}/\mathbb{P}^1]$ and $C_+(\mathbb{Z}[\mathrm{SmProj}/\mathbb{P}^1])$.

10.2. **Rapid decay homology for complexes.** Recall from [CHH20, Definition 6.6] the description of rapid decay homology for $(X, f) \in \mathrm{SmAff}/\mathbb{A}^1$. We put
$$S^{\mathrm{rd}}_*(X, f) = S_*(B^\circ(X, f))/S_*(\partial B^\circ(X, f))$$
where $S_*(-)$ is as in Section 9.5 the complex of C^1-simplices. By [CHH20, Theorem 1.3] it computes singular homology.

Note that the complex $S^{\mathrm{rd}}_*(X, f)$ depends on the choice of a good compactification $\bar{X}$ relative to f, but only in a weak way. We want to extend the construction to complexes of varieties.

Let X be a smooth variety, $f\colon X \to \mathbb{A}^1$. Recall from Section 9.3 that a *good compactification* of (X, f) is a pair $(\bar{X}, \bar{f})$ where $\bar{X}$ is smooth and projective, $\bar{f}\colon \bar{X} \to \mathbb{P}^1$ a morphism and $X \to \bar{X}$ is a dense open immersion such that the complement X_∞ is a simple divisor with normal crossing and $\bar{f}$ extends f.

Definition 10.1. Let $X_\bullet$ be a bounded below complex in $\mathbb{Z}[\mathrm{SmAff}/\mathbb{A}^1]$. A *good compactification of* $X_\bullet$ is a bounded below complex $\bar{X}_\bullet$ in $\mathbb{Z}[\mathrm{SmProj}/\mathbb{P}^1]$ together with a morphism of complexes $X_\bullet \to \bar{X}_\bullet$ such that for every n the map $X_n \to \bar{X}_n$ is a good compactification of (X_n, f).

Lemma 10.2. *Let X be a smooth variety, $f\colon X \to \mathbb{A}^1$.*

(1) The system of good compactifications of (X, f) is filtered.
(2) Given $g\colon Y \to X$ a morphism of smooth varieties and a good compactification of (X, f) there is a good compactification $\bar{Y}$ of $(Y, f \circ g)$ and morphism $\bar{Y} \to \bar{X}$ over g.

Proof. Let $X \to X_1$ and $X \to X_2$ be good compactifications of (X, f). Let X_3' be the closure of X in $X_1 \times_{\mathbb{P}^1} X_2$. Let $X_3 \to X_3'$ be a desingularisation making the boundary into a divisor with normal crossings. A morphism $h : X_1 \to X_2$ of good compactifications of (X, f) is uniquely determined if it exists because X is dense in X_1.

Let $g\colon Y \to X$ be a morphism of smooth varieties. Let $\bar{X}$ be a good compactification of X. Choose any compactification Y' of Y. Possibly after replacing Y' by a blow-up, the map g extends to Y'. Picking a desingularisation $\bar{Y}$ of Y' finishes the proof of this lemma. □

Corollary 10.3. *Let $(X_\bullet, f_\bullet)$ be a bounded below (homological) complex in $\mathbb{Z}[\mathrm{SmAff}/\mathbb{A}^1]$. Then the system of good compactifications of $(X_\bullet, f_\bullet)$ is non-empty, filtering and functorial.*

Proof. We construct $\bar{X}_n$ by induction on n. For $n \ll 0$ there is nothing to show. Suppose we have constructed good compactifications for $n < N$. Let $X_N = \bigcup X_N^j$ be the decomposition into connected components. The differential $d\colon X_N \to X_{N-1}$ is of the form $d = \sum_{i=1}^m a_i g_i$ for morphisms $g_i\colon X_N^{j(i)} \to X_{N-1}$ and $a_i \in \mathbb{Z}$. Let Y_i be a good compactification of X_N such that g_i lifts. Let $\bar{X}_N$ be a common refinement of $Y_1, \dots, Y_m$. By construction d lifts to $\bar{X}_N$. We need to check that the composition $\bar{X}_N \to \bar{X}_{N-1} \to \bar{X}_{N-2}$ vanishes. This is a combinatorial identity on the coefficients of the g_i. It can be checked on the dense open subsets $X_N \to X_{N-1} \to X_{N-2}$, where it holds because $X_\bullet$ is a complex. This finishes the proof of existence.

The same method also produces common refinements of two good compactifications and lifts of morphisms of complexes. □

Recall the functor S_*^{rd} computing rapid decay homology.

Definition 10.4. Let $(X_\bullet, f_\bullet)$ be in $C_+(\mathrm{SmAff}/\mathbb{A}^1)$. We define

$$S_*^{\mathrm{rd}}(X_\bullet, f_\bullet)$$

as the total complex of the double complex $(S_m^{\mathrm{rd}}(X_n, f_n))_{n,m}$ for some choice of good compactification $(\bar{X}_\bullet, \bar{f}_\bullet)$ of $(X_\bullet, f_\bullet)$.

Remark 10.5. By Corollary 10.3, this is well-defined up to canonical isomorphism in the derived category.

10.3. **Twisted de Rham cohomology and periods for complexes.** Recall from [FJ20], see also [CHH20, Section 6.2], that twisted de Rham cohomology of $(X, f) \in \mathrm{SmAff}/\mathbb{A}^1$ is defined as cohomology of the complex $\Omega^*(X)$ with differential $\Omega^p(X) \to \Omega^{p+1}(X)$ given by $d\omega - df \wedge \omega$.

Definition 10.6. Let $(X_\bullet, f_\bullet) \in C_+(\mathrm{SmAff}/\mathbb{A}^1)$. We define $H^n_{\mathrm{dR}}(X_\bullet, f_\bullet)$ to be the cohomology of the total complex $R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet)$ of the double complex $\Omega^*(X_\bullet, \mathcal{E}^{f_\bullet})$.

Lemma 10.7. *Let* $(X_\bullet, f_\bullet) \in C_+(\mathrm{SmAff}/\mathbb{A}^1)$. *Then the period map of* [CHH20, Definition 6.11] *extends to a pairing of complexes*

$$R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet) \times S_*^{\mathrm{rd}}(X_\bullet, f_\bullet) \to \mathbb{C},$$

i.e., a morphism of complexes

$$R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet) \to \mathrm{Hom}(S_*^{\mathrm{rd}}(X_\bullet, f_\bullet), \mathbb{C}).$$

Proof. We apply [CHH20, Lemma 6.12] to each X_n, then take total complexes. □

Definition 10.8. Let $(X_\bullet, f_\bullet) \in C_+(\mathrm{SmAff}/\mathbb{A}^1)$, $n \in \mathbb{N}$. The *period pairing* for $(X_\bullet, f_\bullet, n)$ is the induced map

$$H^n_{\mathrm{dR}}(X_\bullet, f_\bullet) \times H_n^{\mathrm{rd}}(X_\bullet, f_\bullet) \to \mathbb{C}.$$

The elements in the image of this pairing are called the *exponential periods* of $(X_\bullet, f_\bullet, n)$. We denote the set of these numbers for varying $(X_\bullet, f_\bullet, n)$ by $\mathcal{P}_{\mathrm{SmAff}}(k)$.

Remark 10.9. Fresán–Jossen interpret these periods as periods for a suitable category of effective exponential motives. We consider them in Section 13. The usual localisation amounts to inverting π. We do not consider the non-effective case in our paper.

10.4. **The relative case.** Let X be a variety over k, $Y \subset X$ a closed subvariety and $f \in \mathcal{O}(X)$. We want to define exponential periods for $H_n^{\mathrm{rd}}(X, Y, f)$ by reduction to the case $C_+(\mathrm{SmAff}/\mathbb{A}^1)$.

A simplicial or bisimplicial variety $X_\bullet \to X$ is called a *hypercover* of X, if it is a hypercover for the h-topology. We do not go into details about this topology, which is introduced and studied in [Voe96]. For our purposes it suffices to remark that in this case $H_n(X_\bullet^{\mathrm{an}}, \mathbb{Z}) \to H_n(X^{\mathrm{an}}, \mathbb{Z})$ is an isomorphism. The only examples that we are going to need are open and closed covers, Section 11.3. We say that a hypercover is smooth and/or affine, respectively, if all X_n are smooth and/or affine. By resolution of singularities, every hypercover can be refined by a smooth affine hypercover. If $g\colon Y \to X$ is a morphism of varieties, $X_\bullet \to X$ a smooth affine hypercover, then there is a smooth affine hypercover $Y_\bullet \to Y$ and a morphism $g_\bullet\colon Y_\bullet \to X_\bullet$ over g.

Lemma 10.10. *Let X be a variety over k, $Y \subset X$ a subvariety. Let $X_\bullet \to X$ be a smooth affine hypercover, $Y_\bullet \to Y$ a smooth affine hypercover with a morphism $Y_\bullet \to X_\bullet$ of simplicial schemes compatible with the inclusion. Let*

$$C(X, Y) = \mathrm{Cone}(Y_\bullet \to X_\bullet)$$

be the cone of the associated map of total complexes in $C_+(\mathrm{SmAff}/\mathbb{Z})$. *Then there is a natural isomorphism*

$$H_n^{\mathrm{rd}}(X, Y) \cong H_n^{\mathrm{rd}}(C(X, Y)).$$

Proof. Fix $r \in \mathbb{R}$. We put $T_r(X_n) = f_n^{-1}(S_r) \subset X_n^{\mathrm{an}}$ where $f_n\colon X_n \to X \to \mathbb{A}^1$ is the structure map of X_n and $S_r = \{z \in \mathbb{C} | \Re(z) \geq r\}$. By definition, $X_\bullet \to X$ is a universal homological cover, hence the base change $T_r(X_\bullet) \to T_r(X)$ is also a universal homological cover. This implies that the complex computing homology of X^{an} relative to $T_r(X)$ is quasi-isomorphic to the total complex of

$$S_*(X_\bullet^{\mathrm{an}})/S_*(T_r(X_\bullet)).$$

By [FJ20, Proposition 3.5.2] (see also [CHH20, Proposition 6.5]) and the fact that $S_*(-)$ computes singular homology (see [CHH20, Theorem 1.3]), we have for each n and sufficiently large r, a quasi-isomorphism

$$S_*(X_n^{\mathrm{an}})/S_*(T_r(X_n)) \to S_*(B_{\bar{X}_n}(X_n))/S_*(T_r(X_n)) \leftarrow S_*^{\mathrm{rd}}(X_n, f_n).$$

By taking total complexes this gives quasi-isomorphisms of the projective limit of the complexes computing rapid decay homology of X and $S_*^{\mathrm{rd}}(X_\bullet, f_\bullet)$. Note that projective limits are exact in our situation because all homology spaces are finite dimensional. The same arguments can be applied to Y. By taking cones we get the result for relative homology. □

Given this Lemma, we are led to define:

Definition 10.11 ([FJ20, Definition 7.1.6])**.** Let X be a variety over k, $f \in \mathcal{O}(X)$, $Y \subset X$ a closed subvariety. Choose $C(X,Y) \in C_+(\mathrm{SmAff}/\mathbb{A}^1)$ as in Lemma 10.10.

(1) We define $H^n_{\mathrm{dR}}(X,Y,f)$ as cohomology of

$$R\Gamma_{\mathrm{dR}}(X_\bullet, Y_\bullet, f) = R\Gamma_{\mathrm{dR}}(C(X,Y)).$$

(2) We define the period pairing for (X,Y,f,n) as the period pairing

$$H_n^{\mathrm{rd}}(X,Y,f) \times H^n_{\mathrm{dR}}(X,Y,f) \to \mathbb{C}$$

for $C(X,Y)$.

We conclude this section by recalling the definition of a cohomological exponential period.

Definition 10.12 (See [CHH20, Definition 6.10])**.** Let X be a variety, $f \in \mathcal{O}(X)$, $Y \subset X$ a closed subvariety, $n \in \mathbb{N}_0$. The elements in the image of the period pairing for (X,Y,f,n) are called the *(cohomological) exponential periods* of (X,Y,f,n).

We denote $\mathcal{P}_{\mathrm{coh}}(k)$ the set of cohomological exponential periods for varying (X,Y,f,n) over k. We denote $\mathcal{P}_{\log}(k)$ the subset of cohomological exponential periods for varying (X,Y,f,n) such that (X,Y) is a log-pair.

Lemma 10.13. *Let K/k be an algebraic extension. Then*

$$\mathcal{P}_{\mathrm{coh}}(K) = \mathcal{P}_{\mathrm{coh}}(k).$$

Proof. The same argument as in the classical case, [HMS17, Corollary 11.3.5], also applies in the exponential case. □

11. Cohomological exponential periods are naive exponential periods

The aim of this section is to prove the key comparison in Proposition 11.1. See [CHH20, Proposition 8.4] for the corresponding statement in the special case where X is a curve. In that case, the main ideas of the proof are present, but several technicalities are avoided.

Proposition 11.1. *Let $k \subset \mathbb{C}$ be as in Section 9.1. Let (X, Y) be a log pair, i.e., X a smooth variety, $Y \subset X$ a simple normal crossings divisor. Let $f \in \mathcal{O}(X)$, and let α be a cohomological exponential period of (X, Y, f, n) (see Definition 10.12). Then α is a naive exponential period:*

$$\mathcal{P}_{\log}(k) \subset \mathcal{P}_{\mathrm{nv}}(k).$$

Remark 11.2. This justifies that our fairly restrictive definition of a naive exponential period was a reasonable choice.

The proof is technical and will take the rest of the section.

11.1. **Notation.** Throughout, let k be as in Section 9.1, $k_0 = k \cap \mathbb{R}$.

If X is a smooth variety, $f \in \mathcal{O}(X)$, $\bar{X}$ a good compactification relative to f, then we put $X_\infty = \bar{X} \smallsetminus X$. We decompose $X_\infty = D_0 \cup D_\infty$ where D_0 consists of the horizontal components and D_∞ of the vertical components mapping to ∞ in $\mathbb{P}^1$.

As before, we denote by $\tilde{f}\colon B_{\bar{X}}(X) \to \tilde{\mathbb{P}}^1$ the induced map on the oriented real blow-up of $\bar{X}^{\mathrm{an}}$ in X_∞^{an}.

Recall from Section 9.4 that

$$\begin{aligned} B^\circ_{\bar{X}}(X, f) &= B_{\bar{X}}(X) \smallsetminus \{x \in \partial(B_{\bar{X}}(X)) \mid \pi(x) \in D_0^{\mathrm{an}} \text{ or } \Re(\tilde{f}(x)) \leq 0\} \\ \partial B^\circ_{\bar{X}}(X, f) &= B^\circ_{\bar{X}}(X, f) \smallsetminus X^{\mathrm{an}} \end{aligned}$$

We introduce a variant.

Definition 11.3. We put

$$\begin{aligned} B^\sharp_{\bar{X}}(X, f) &= B_{\bar{X}}(X) \smallsetminus \{x \in \partial(B_{\bar{X}}(X)) \mid \pi(x) \in D_0^{\mathrm{an}} \text{ or } \tilde{f}(x) \neq 1\infty\} \\ \partial B^\sharp_{\bar{X}}(X, f) &= B^\sharp_{\bar{X}}(X) \smallsetminus X^{\mathrm{an}} \end{aligned}$$

The spaces $B_{\bar{X}}(X)$ and $B^\circ_{\bar{X}}(X, f)$ are k_0-semi-algebraic manifolds with corners by [CHH20, Proposition 4.3] and $B^\sharp_{\bar{X}}(X, f)$ is a k_0-semi-algebraic subset.

11.2. **A comparison of homology.** The first step in the argument is an alternative description of rapid decay homology using $B^\sharp(X, f)$ rather than $B^\circ(X, f)$. Let us motivate why this is needed. We are going to represent homology classes by k_0-semi-algebraic sets G such that $\bar{G} \subset B^\circ_{\bar{X}}(X, f)$. Hence $\overline{f(G)} \subset B^\circ$ as in the definition of a generalised naive exponential period. The proposition will allow us to even choose $\bar{G} \subset B^\sharp_{\bar{X}}(X, f)$. Hence $\overline{f(G)} \subset B^\sharp$ and the data defines a naive exponential period. Indeed, the closure of the strip

$$S_{r,s} = \{z \in \mathbb{C} \mid \Re(z) > r, |\Im(z)| < s\}$$

inside $\tilde{\mathbb{P}}^1$ is contained in $B^\sharp$. Actually, we can only apply this argument in the case of smooth X, but see Section 11.3 for the reduction.

Proposition 11.4. *Let V be a smooth variety, $f \in \mathcal{O}(V)$, $\bar{V}$ a good compactification, $n \geq 0$. Then the natural map*

$$H_n(B^\sharp_{\bar{V}}(V, f), \partial B^\sharp_{\bar{V}}(V, f); \mathbb{Z}) \to H_n(B^\circ_{\bar{V}}(V, f), \partial B^\circ_{\bar{V}}(V, f); \mathbb{Z})$$

is an isomorphism.

Proof. We are going to show the equivalent statement on cohomology. The spaces are paracompact Haussdorff and locally contractible, hence we may compute singular cohomology as sheaf cohomology. We abbreviate $B^\circ(V) = B^\circ_{\bar{V}}(V,f)$ and $B^\sharp(V) = B^\sharp_{\bar{V}}(V,f)$. Let $j^\circ\colon V^{\mathrm{an}} \to B^\circ(V)$ and $j^\sharp\colon V^{\mathrm{an}} \to B^\sharp(V)$ be the open immersions. Our relative cohomology is computed by applying $R\Gamma$ to $j^\circ_!\mathbb{Z}$ and $j^\sharp_!\mathbb{Z}$, respectively.

We compare their higher direct images on a subset of $\bar{V}^{\mathrm{an}}$. As in the definition of $B^\circ(V)$, let $\bar{V} \smallsetminus V = D_0 \cup D_\infty$ such that $D_\infty \subset \bar{f}^{-1}(\infty)$ and f is rational on D_0. Furthermore let $p^\circ : B^\circ(V) \to \bar{V}^{\mathrm{an}} \smallsetminus D_0^{\mathrm{an}}$, and $p^\sharp : B^\sharp(V) \to \bar{V}^{\mathrm{an}} \smallsetminus D_0^{\mathrm{an}}$ the projections. We consider the natural map

$$Rp^\circ_* j^\circ_! \mathbb{Z} \to Rp^\sharp_* j^\sharp_! \mathbb{Z}$$

and claim that it is a quasi-isomorphism.

We compute its stalks. For $x \in V^{\mathrm{an}}$, both sides are simply equal to $\mathbb{Z}$ concentrated in degree 0.

Let $x \in D_\infty^{\mathrm{an}} \smallsetminus D_0^{\mathrm{an}}$. The stalk of $R^i p^\sharp_* j^\sharp_! \mathbb{Z}$ in x is given by the limit of $H^i(p^{\sharp-1}(U), p^{\sharp-1}(U) \cap \partial B^\sharp(V); \mathbb{Z})$ for U running through the system of neighbourhoods of x. The analogous formula hold for p°. Hence it suffices to show that

$$H^i(p^{\sharp-1}(U), p^{\sharp-1}(U) \cap \partial B^\sharp(V); \mathbb{Z}) \to H^i(p^{\circ-1}(U), p^{\circ-1}(U) \cap \partial B^\circ(V); \mathbb{Z})$$

is an isomorphism for all U sufficiently small. This is a local question on $\bar{V}$. We choose local coordinates $z_1, \dots, z_n$ on $\bar{V}$ centered at x such that $\bar{f}(z_1, \dots, z_n) = z_1^{-d_1} \dots z_m^{-d_m}$, where m is the number of components of D_0 passing through x. Let U_ϵ be the polydisc of radius ϵ around the origin. On U_ϵ the real oriented blow-up is given by

$$\{(z_1, \dots, z_n, w_1, \dots, w_m) \in B_\epsilon(0)^n \times (S^1)^m \mid z_i w_i^{-1} \in \mathbb{R}_{\geq 0}\}.$$

We make a change of coordinates by writing $z_i = r_i w_i$ with $r_i \in [0, \epsilon)$. Hence over U_ϵ the real oriented blow-up is given by

$$(r_1, \dots, r_m, w_1, \dots, w_m, z_{m+1}, \dots, z_n) \in [0, \epsilon)^m \times (S^1)^m \times B_\epsilon(0)^{n-m}.$$

In it $\partial B^\sharp(V)$ is the subset of points with $r_1 \cdots r_m = 0$, $w_1^{d_1} \dots w_m^{d_m} = 1$ and $\partial B^\circ(V)$ is the subset of points with $r_1 \cdots r_m = 0$, $\Re(w_1^{d_1} \dots w_m^{d_m}) > 0$.

We apply the long exact sequence for relative cohomology. Hence it suffices to compare cohomology of $p^{\sharp-1}(U_\epsilon)$ and $p^{\circ-1}(U_\epsilon)$ and their boundaries separately. Both $p^{\sharp-1}(U_\epsilon)$ and $p^{\circ-1}(U_\epsilon)$ are homotopy equivalent to their intersection with V^{an}, hence they have the same cohomology.

We now concentrate on the boundary. In both cases they are fibre bundles over

$$\{(r_1, \dots, r_m, w_1, \dots, w_{m-1}, z_{m+1}, \dots, z_n) \in \\ [0, \epsilon)^m \times (S^1)^{m-1} \times B_\epsilon(0)^{n-m} \mid r_1 \cdots r_m = 0\}.$$

In the case of $\partial B^\sharp(V)$, the fibre consists of d_m points, the solutions of $w_m^{d_m} = (w_1^{d_1} \dots w_{m-1}^{d_{m-1}})^{-1}$. In the case of $\partial B^\circ(V)$, the fibre consist of d_m open circle arcs centered around these points. In particular, the inclusion $p^{\sharp-1}(U_\epsilon) \cap \partial B^\sharp \to p^{\circ-1}(U_\epsilon) \cap \partial B^\circ$ is fibrewise a homotopy equivalence, hence it induces an isomorphism on cohomology. □

The goal of this whole section is to express α as a naive exponential period. In order to find the set G as in the definition of a naive exponential period, we are going to choose a k_0-semi-algebraic triangulation of $B^\sharp(V,f)$ that is globally of class C^1 (see [CHH20, Definition 7.1]), and with V as in the setting of the preceding proposition. Our next goal is therefore to construct a suitable smooth V from the log-pair (X,Y).

11.3. **Hypercovers.** By definition of cohomological exponential periods, we need to fix a smooth affine hypercover of our log-pair (X,Y). We do this explicitly.

Let $p\colon S\to T$ be a morphism. Its Čech-nerve is the simplicial scheme $S_\bullet\to T$ with

$$S_n = S\times_T\cdots\times_T S \quad (n+1 \text{ factors})$$

and the usual face and degeneracy maps. It is a hypercover, if p is a cover for the h-topology. We need two easy cases.

Let X be a smooth variety, $U^1,\dots,U^M$ an affine open cover. We put

$$U_0 = U^1 \amalg\cdots\amalg U^M \to U.$$

Let $U_\bullet$ be its Čech-nerve. Explicitly, we have

$$U_n = \coprod_{J\in\{1,\dots,M\}^{n+1}} U^J$$

with

$$U^{(j_0,\dots,j_n)} = \bigcap_{i=0}^n U^{j_i}.$$

Singular homology satisfies descent for open covers (the Mayer–Vietoris property), hence $U_\bullet\to X$ is a smooth affine hypercover, the Čech-complex defined by the open cover.

For the second special case, let X be a smooth variety, $Y\subset X$ a simple normal crossings divisor with irreducible components $Y^1,\dots,Y^N$. By assumption they are smooth. We put

$$Y_0 = Y^1\amalg\cdots\amalg Y^N\to Y.$$

Let $Y_\bullet$ be its Čech-nerve. Explicitly, we have

$$Y_n = \coprod_{J\in\{1,\dots,N\}^{n+1}} Y^J$$

with

$$Y^{(j_0,\dots,j_n)} = \bigcap_{i=0}^n Y^{j_i}.$$

Singular homology satisfies proper base change, hence $Y_\bullet\to Y$ is a smooth hypercover, the Čech-complex defined by the closed cover.

We can combine the two constructions. The bisimplicial scheme

$$Y_\bullet\cap U_\bullet\to Y$$

is a smooth affine hypercover. In the notation of Lemma 10.10

$$C(X,Y) = \mathrm{Cone}(Y_\bullet\cap U_\bullet\to U_\bullet).$$

We write $Y_{-1} = X$, then all terms of $C(X,Y)$ are direct sums of objects of the form $Y_n \cap U_m$ for $n \geq -1$ and $m \geq 0$.

The definition of the period pairing also requires the choice of a good compactification of $C(X,Y)$. We proceed as follows. Let $\bar{X}$ be a good compactification of (X,Y,f). We choose an open cover $U^1,\dots,U^M$ by affine subvarieties of X such that $\bar{Y} + \sum_{i=1}^M U^i_\infty$ is still a simple normal crossings divisor. This can be achieved by choosing U^i as the complement of a generic hyperplane U^i_∞ in $\bar{X}$. Note that $\bar{X}$ is a good compactification of each of the U^J. Hence the Čech-nerve of the map

$$\coprod_{i=1}^M \bar{X} \to \bar{X}$$

is a good compactification of $U_\bullet$. We denote it $\bar{U}_\bullet$. For each $I \subset \{1,\dots,N\}$ let $\bar{Y}^I$ be the closure of Y^I in $\bar{X}$. By the transversality assumption it is smooth and a good compactification. Hence

$$\bar{Y}_n = \coprod_{J\in\{1,\dots,N\}^{n+1}} \bar{Y}^J$$

defines a good compactification of Y_n and of $Y_n \cap U_m$ for all m. The complex

$$\mathrm{Cone}(\bar{U}_\bullet \cap \bar{Y}_\bullet \to \bar{U}_\bullet) \in C_+(\mathrm{SmProj}/\mathbb{A}^1)$$

is a good compactification of $C(X,Y)$.

Corollary 11.5. *Let (X,Y) be a log pair, $f\colon X \to \mathbb{A}^1$. With the notation above*

$$R\Gamma_{\mathrm{dR}}(X,Y,f) = R\Gamma_{\mathrm{dR}}(C(X,Y)) = \Omega^*(C(X,Y))$$

and rapid decay homology of (X,Y,f) is computed by

$$S^{\mathrm{rd}}_*(X,Y,f) := \mathrm{Cone}(S^{\mathrm{rd}}_*(\bar{U}_\bullet \cap \bar{Y}, f_\bullet) \to S^{\mathrm{rd}}(\bar{U}_\bullet, f_\bullet)).$$

Proof. The statement for de Rham cohomology is simply Definition 10.11. The claim for rapid decay homology is Lemma 10.10 in every degree. ☐

Our next aim is to get a clearer understanding of $B^\circ(-,f)$ and $B^\sharp(-,f)$ applied to $C(X,Y)$ and its good compactification $C(\bar{X},\bar{Y})$.

11.4. **Real oriented blow-up and closed Čech complexes.** Let X be smooth, $Y \subset X$ a simple normal crossings divisor, $f \in \mathcal{O}(X)$. Let $\bar{X}$ be a good compactification such that $Y + X_\infty$ is a simple normal crossing divisor and f extends to $\bar{X}$. Let $\bar{Y}$ be the closure of Y in $\bar{X}$. Denote by $B_{\bar{X}}(Y)$, $B^\circ_{\bar{X}}(Y,f)$ and $B^\sharp_{\bar{X}}(Y,f)$ the closure of Y^{an} in $B_{\bar{X}}(X)$, $B^\circ_{\bar{X}}(X,f)$ and $B^\sharp_{\bar{X}}(X,f)$, respectively. As in the last section let $Y_\bullet \to Y$ and $\bar{Y}_\bullet \to \bar{Y}$ be the Čech-complexes for the closed cover of Y and $\bar{Y}$ by their irreducible components.

Applying our oriented blow-ups, we get simplicial k_0-semi-algebraic manifolds with corners $B_{\bar{Y}_\bullet}(Y_\bullet)$ and $B^\circ_{\bar{Y}_m}(Y_\bullet, f_\bullet)$ and k_0-semi-algebraic subsets

$B^\sharp_{\bar{Y}_\bullet}(Y_\bullet)$. Note that

$$\begin{aligned} B_{\bar{Y}_m}(Y_m, f_m) &= \coprod_{J\in\{1,\dots,N\}^{n+1}} B_{\bar{Y}^J}(Y^J, f^J), \\ B^\circ_{\bar{Y}_m}(Y_m, f_m) &= \coprod_{J\in\{1,\dots,N\}^{n+1}} B^\circ_{\bar{Y}^J}(Y^J, f^J) \\ B^\sharp_{\bar{Y}_m}(Y_m, f_m) &= \coprod_{J\in\{1,\dots,N\}^{n+1}} B^\sharp_{\bar{Y}^J}(Y^J, f^J). \end{aligned}$$

Proposition 11.6. *The simplicial k_0-semi-algebraic sets*

$$B_{\bar{Y}_\bullet}(Y_\bullet) \to B_{\bar{X}}(Y), \quad B^\circ_{\bar{Y}_\bullet}(Y_\bullet, f_\bullet) \to B^\circ_{\bar{X}}(Y, f), \quad B^\sharp_{\bar{Y}_\bullet}(Y_\bullet, f_\bullet) \to B^\sharp_{\bar{X}}(Y, f),$$

are the Čech-nerves for the corresponding closed covers

$$B_{\bar{Y}_0}(Y_0) \to B_{\bar{X}}(Y), \quad B^\circ_{\bar{Y}_0}(Y_0, f_0) \to B^\circ_{\bar{X}}(Y, f), \quad B^\sharp_{\bar{Y}_0}(Y_0, f_0) \to B^\sharp_{\bar{X}}(Y, f),$$

Proof. Let $Z = Y^J$ for $J \subset \{1, \dots, N\}^{m+1}$ for all m, J. Then $\bar{Z}$ is transverse to X_∞. The description of the real oriented blow-up in local coordinates immediately gives

$$\begin{aligned} B_{\bar{Z}}(Z) = \bar{Z}^{\mathrm{an}} \times_{\bar{X}} B_{\bar{X}}(X), \quad & B^\circ_{\bar{Z}}(Z, f) = \bar{Z}^{\mathrm{an}} \times_{\bar{X}^{\mathrm{an}}} B^\circ_{\bar{X}}(X, f), \\ & B^\sharp_{\bar{Z}}(Z, f) = \bar{Z}^{\mathrm{an}} \times_{\bar{X}^{\mathrm{an}}} B^\sharp_{\bar{X}}(X, f), \end{aligned}$$

In total we have

$$\begin{aligned} B_{\bar{Y}_\bullet}(Y_\bullet) &= \bar{Y}_\bullet \times_{\bar{X}} B(X), \\ B^\circ_{\bar{Y}_\bullet}(Y_\bullet, f_\bullet) &= \bar{Y}_\bullet \times_{\bar{X}} B^\circ(X, f), \\ B^\sharp_{\bar{Y}_\bullet}(Y_\bullet, f_\bullet) &= \bar{Y}_\bullet \times_{\bar{X}} B^\sharp(X, f). \end{aligned}$$

This gives the claim on Čech-nerves. □

Corollary 11.7. *Let X be a smooth variety, $f \in \mathcal{O}(X)$, $Y \subset X$ a simple normal crossings divisor. Choose a good compactification $\bar{X}$ of X such that $Y + X_\infty$ is a simple normal crossings divisor. Let $B^\circ_{\bar{X}}(Y, f)$ and $B^\sharp_{\bar{X}}(Y, f)$ be the closure of Y^{an} in $B^\circ_{\bar{X}}(X, f)$ and $B^\sharp_{\bar{X}}(X, f)$, respectively. Then*

$$H^{\mathrm{rd}}_n(Y, f) \cong H_n(B^\circ_{\bar{X}}(Y, f), \partial B^\circ_{\bar{X}}(Y, f); \mathbb{Q}) \cong H_n(B^\sharp_{\bar{X}}(Y, f), \partial B^\sharp_{\bar{X}}(Y, f); \mathbb{Q})$$

and

$$\begin{aligned} H^{\mathrm{rd}}_n(X, Y, f) &\cong H_n(B^\circ_{\bar{X}}(X, f), B^\circ_{\bar{X}}(Y, f) \cup \partial B^\circ_{\bar{X}}(X, f); \mathbb{Q}) \\ &\cong H_n(B^\sharp_{\bar{X}}(X, f), B^\sharp_{\bar{X}}(Y, f) \cup \partial B^\sharp_{\bar{X}}(X, f); \mathbb{Q}). \end{aligned}$$

Proof. Let $Y_\bullet \to Y$ be the Čech-nerve of the closed cover of Y by the disjoint union of its irreducible components. By Proposition 11.6, the natural map

$$B^\circ_{\bar{Y}_\bullet}(Y_\bullet, f_\bullet) \to B^\circ_{\bar{X}}(Y, f)$$

is a proper hypercover, hence it induces isomorphisms on singular homology. □

11.5. **Semi-algebraic triangulations of hypercovers.** We use the notation of Section 11.3.

Note that the natural map $B_{\bar{X}}(U^J) \to B_{\bar{X}}(X)$ induces an inclusion $B^\circ_{\bar{X}}(U^J, f) \subset B^\circ_{\bar{X}}(X, f)$.

Proposition 11.8. *There is a finite dimensional subcomplex*

$$S^\Delta_*(X, Y, f) \subset S^{\mathrm{rd}}_*(X, Y, f)$$

such that the inclusion is a quasi-isomorphism and every $S^\Delta_n(X, Y, f)$ has a finite basis consisting of k_0-semi-algebraic C^1-simplices of the form

$$\sigma : \bar{\Delta}_a \to B^\sharp(U_b) \quad a + b = n$$

or

$$\sigma : \bar{\Delta}_a \to B^\sharp(U_b \cap Y_c) \quad a + b + c = n - 1$$

such that σ is a homeomorphism onto its image.

Proof. By definition, $S^{\mathrm{rd}}_*(X, Y)$ is the total complex of

$$\mathrm{Cone}(S_*(B^\circ(U_\bullet \cap Y_\bullet, f), \partial) \to S_*(B^\circ(U_\bullet, f), \partial)$$

(where ∂ is an abbreviation for $\partial B^\circ(-, f)$ as applicable).

In order to unify notation, we write $Y_{-1} = X$ and also $Y^I = X$ for $|I| = -1$. We now want to choose compatible k_0-semi-algebraic triangulations in the sense of [CHH20, Section 7]. We first triangulate the base $B_{\bar{X}}(X)$ by applying [CHH20, Proposition 7.4]. We obtain a k_0-semi-algebraic triangulation of the compact k_0-semi-algebraic manifold with corners $B_{\bar{X}}(X)$ compatible with the finitely many k_0-semi-algebraic subsets $B^\sharp_{\bar{X}}(U^J \cap Y^I, f)$ and their boundaries.

In the next step, we want to triangulate the (bi)simplicial k_0-semi-algebraic sets $B^\sharp_{\bar{X}}(U_\bullet \cap Y_\bullet, f)$ and $B^\sharp_{\bar{X}}(U_\bullet, f)$ and their boundaries such that all structure maps and the maps between them are simplicial. We obtain this simply by pull-back of the triangulation of the base. By loc. cit. the simplices can be chosen to be C^1.

We apply [CHH20, Proposition 7.6] to these simplicial complexes and replace them by the closed core of their barycentric subdivisons. The subcomplexes

$$|\mathrm{cc}(\beta B^\sharp(U_a \cap Y_b, f))|$$

are deformation retracts, hence they have the same homology as $(B^\sharp(U_a \cap Y_b, f), \partial)$. By Proposition 11.4 their homology also agrees with homology of $(B^\circ(U_a \cap Y_b, f), \partial)$. The subcomplexes are compact.

We now consider the subcomplexes of $S_*(B^\circ(U_\bullet \cap Y_\bullet, f), \partial)$ and $S_*(B^\circ(U_\bullet, f), \partial)$ that compute the simplicial homology of $|\mathrm{cc}(\beta B^\sharp(U_\bullet \cap Y_\bullet, f))|$ and $|\mathrm{cc}(\beta B^\sharp(U_\bullet, f))|$ relative to their boundaries, respectively. By what we argued above, the inclusion of subcomplexes into the ambient complexes are quasi-isomorphisms. Let $S^\Delta_*(X, Y, f)$ be the total complex of the cone of the natural map between these subcomplexes. By construction it has a degreewise finite basis of the form given in the proposition. $\square$

11.6. **Proof of Proposition 11.1.**

Proof. Let $\alpha = \langle \Omega, \Sigma \rangle$ be an exponential period for the log-pair (X, Y, f). We want to express it as a naive exponential period.

We work with the hypercovers $U_\bullet$ and $Y_\bullet$ as in Section 11.3. By definition, $\Sigma \in H_n^{\mathrm{rd}}(X, Y; \mathbb{Z})$. We compute rapid decay homology via the complex $S_*^\Delta(X, Y, f)$ of Proposition 11.8. By definition this means that the cohomology class Σ is represented by a tuple $\sigma_{bc} \in S_a^\Delta(U_b \cap Y_c)$ with $a+b+c = n-1$, $b \geq 0$, $c \geq -1$.

Also by definition, Ω is represented by a cycle in $\Omega^*(\mathrm{Cone}(U_\bullet \cap Y_\bullet \to U_\bullet))$, i.e., a tuple $\omega_{bc} \in \Omega^a(U_b \cap Y_c)$ with $a + b + c = n - 1$, $b \geq 0$, $c \geq -1$ (again we use the convention that $Y_{-1} = X$). By definition of the period pairing $\langle \Omega, \Sigma \rangle$ is obtained by taking a linear combination of the integrals

$$\int_{\sigma_{bc}} \mathrm{e}^{-f} \omega_{bc}.$$

Each of the σ_{bc} is a linear combination of k_0-semi-algebraic ~~strictly~~ simplices globally of class C^1 with values in $B^\sharp(U_b \cap Y_c) \subset B^\circ(U_b \cap Y_c)$.

Recall that naive exponential periods form an algebra, hence it suffices to show that the integrals for the individual simplices define naive exponential periods.

Let $U = U_b \cap Y_c \subset \mathbb{A}^N$, $\omega = \omega_{bc} \in \Omega^a(U)$. Let $T \colon \bar{\Delta}_a \to B^\sharp(U, f)$ be a k_0-semi-algebraic C^1-simplex. Let $G = T(\bar{\Delta}_a) \cap U^{\mathrm{an}}$. We equip it with the pseudo-orientation induced from Δ_a. It is a closed k_0-semi-algebraic subset of $\mathbb{C}^N$ because U is affine and the inclusion $U^{\mathrm{an}} \to B^\sharp_{\bar{U}}(U, f)$ is k_0-semi-algebraic. Moreover, as U is affine, $f|_G$ is the restriction of a polynomial in $k[X_1, \ldots, X_N]$ to G and $\omega|_G$ the restriction of an algebraic differential form.

We need to check the condition on $f(G)$. The closure $\bar{G} = T(\bar{\Delta}_a) \subset B^\sharp(U, f)$ is compact, hence so is its image in $B^\sharp = \mathbb{C} \cup \{1\infty\}$. This implies that $f(G) \subset \mathbb{C}$ is contained in a strip $S_{r,s}$ as we want. Compactness of $\bar{G}$ also implies that the map $\bar{G} \to \tilde{\mathbb{P}}^1$ is proper. The preimage of the circle at infinity is precisely $\bar{G} \smallsetminus G$, hence $f \colon G \to \mathbb{C}$ is also proper.

Therefore our α is a linear combination of numbers of the form

$$\int_{\bar{\Delta}_a} \mathrm{e}^{-f \circ T} T^* \omega = \int_G \mathrm{e}^{-f} \omega,$$

which are naive exponential periods. □

12. Generalised naive exponential periods are cohomological

Let $k \subset \mathbb{C}$ be a subfield, $k_0 = k \cap \mathbb{R}$ and assume that k is algebraic over k_0, see Section 9.1. Recall from Definition 9.3 the notion of a generalised naive exponential period. We denote by $\mathcal{P}_{\mathrm{gnv}}(k)$ the set of generalised naive exponential periods. Recall from Definition 10.12 the notion of an exponential period of a log pair and the set $\mathcal{P}_{\log}(k)$ of all such numbers.

The aim of this section is the proof of the following converse of Proposition 11.1:

Proposition 12.1. *Every generalised naive exponential period over k is an exponential period of a log-pair over k:*

$$\mathcal{P}_{\mathrm{gnv}}(k) \subset \mathcal{P}_{\log}(k).$$

More precisely, given

- *a pseudo-oriented k-semi-algebraic $G \subset \mathbb{C}^n$ of real dimension d,*
- *f a rational function and*
- *ω a rational algebraic d-form*

as in the definition of a generalised naive exponential period, there are

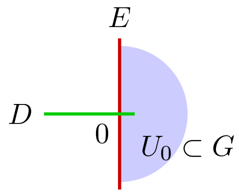
- *a smooth affine variety X of dimension d,*
- *a simple normal crossings divisors Y on X,*
- *a function $f \in \mathcal{O}(X)$ induced from the original f,*
- *a homology class $[G] \in H_d^{\mathrm{rd}}(X, Y; \mathbb{Z})$, and*
- *a cohomology class $[\omega] \in H^d_{\mathrm{dR}}(X, Y, f)$*

such that

$$\langle [\omega], [G] \rangle = \int_G \mathrm{e}^{-f} \omega.$$

12.1. **Horizontal divisors.** We will need to make the closure of G disjoint from the components of the divisor that are horizontal relative to f. We start with a local criterion.

Lemma 12.2. *Let $k \subset \mathbb{R}$ be a real closed field, so that $\bar{k} = k(i)$. Let $D, E \subset \mathbb{A}^n_k$ be unions of distinct coordinate hyperplanes. In other words, $D = \{\prod_{i \in I} x_i = 0\}$ and $E = \left\{\prod_{j \in J} x_j = 0\right\}$, with $I, J \subset \{1, \dots, n\}$ and $I \cap J = \varnothing$. Let G be a semialgebraic subset of $\mathbb{A}^n_k(\mathbb{R}) = \mathbb{R}^n$, such that G is disjoint from $D(\mathbb{R})$, and such that $\bar{G}$ contains the origin. Let $\partial\bar{G} = \bar{G} \smallsetminus \bar{G}^{\mathrm{int}}$ be its boundary in $\mathbb{R}^n$. Let $U \subset \mathbb{A}^n_k(\mathbb{R})$ be an open neighbourhood of the origin, and assume that $G \cap U$ is open in $\mathbb{R}^n$ and $\partial\bar{G} \cap U \subset E(\mathbb{R})$. Then D is empty.*



Proof. Without loss of generality, we may assume that U is an open ball. Note that $U \setminus E(\mathbb{R})$ has $2^{\#J}$ connected components. Since $G \cap U$ is open, and $\bar{G}$ contains the origin, we see that G intersects at least one of these components, say U_0. Since $\partial\bar{G} \cap U \subset E(\mathbb{R})$, we find that $U_0 \subset G$. On the other hand, for every $i \notin J$, it is clear that $\{x_i = 0\}$ intersects U_0. Hence D is empty. □

Setting 12.3. For the actual proof of Proposition 12.1, we are going to use the following data:

- a real closed field $k \subset \mathbb{R}$, hence $k(i) = \bar{k}$,
- a smooth affine variety X over k of dimension d,
- a simple normal crossings divisor $Y \subset X$,
- a closed k-semi-algebraic subset $G \subset X(\mathbb{R})$ of dimension d such that $\partial G \subset Y(\mathbb{R})$ (where $\partial G = G \smallsetminus G^{\mathrm{int}}$ inside $X(\mathbb{R})$),
- a pseudo-orientation on G,

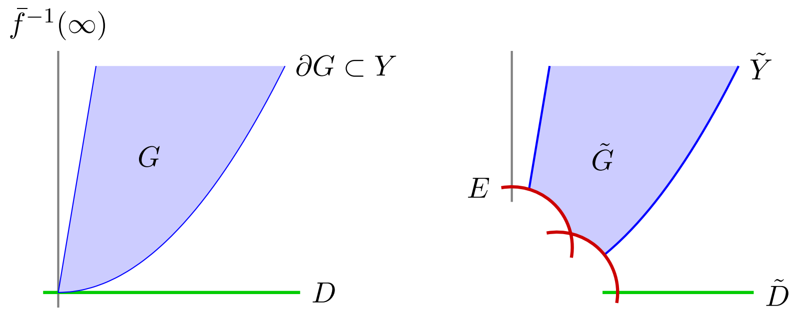
- a morphism $f\colon X_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ such that $f\colon G \to \mathbb{C}$ is proper and such that the closure $\overline{f(G)} \subset \tilde{\mathbb{P}}^1$ is contained in B°,
- a regular algebraic d-form ω on $X_{\bar{k}}$,
- a good compactification $\bar{X}$ of X such that f extends to $\bar{f}\colon \bar{X}_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$,
- and finally, we denote by $D \subset \bar{X}$ ~~is~~ the smallest subvariety of $\bar{X}$ containing all components of $(X_\infty)_{\bar{k}} = \bar{X}_{\bar{k}} - X_{\bar{k}}$ on which $\bar{f}$ is rational.

Lemma 12.4. *In this setting, we may choose $\bar{X}$ such that, in addition to being a good compactification, the closure of G in $\bar{X}^{\mathrm{an}}$ is disjoint from D^{an}.*

Proof. Without loss of generality, we may assume that X is connected. If D is empty, we are done. Hence assume that D is not empty. By the properness assumption on f, we see that $\bar{G} \cap D^{\mathrm{an}}$ lies in the preimage of $\infty \in \mathbb{P}^1$.

Let $\bar{Y}$ be the Zariski closure of $\partial\bar{G} \cup Y$ in $\bar{X}$, where $\partial\bar{G} = \bar{G} \smallsetminus (\bar{G})^{\mathrm{int}}$ viewed as subset of $\bar{X}$. It contains the closure of Y in $\bar{X}$, but possibly also additional components mapping to ∞. By resolution of singularities, we may find a modification $\pi\colon \tilde{X} \to \bar{X}$ such that $\pi^{-1}(X) \to X$ is an isomorphism, $\tilde{X}$ again smooth and such that $\tilde{D} \cup E \cup \tilde{Y}$ is a strict normal crossings divisor in $\tilde{X}$, where $\tilde{D}$ and $\tilde{Y}$ denote the strict transforms of D and Y respectively, and where E denotes the exceptional locus of π. In addition, we may assume that $\tilde{D}$ and $\tilde{Y}$ are disjoint.

Let $\tilde{G}$ denote the strict transform of G under π, i.e., the closure of $G \cong \pi^{-1}(G)$ in $\tilde{X}^{\mathrm{an}}$. It is contained in $\tilde{X}(\mathbb{R})$. Since π is proper, the closure of $\tilde{G}$ in $\tilde{X}(\mathbb{R})$ is contained in $\pi^{-1}(\bar{G})$. This means that $\partial\tilde{G} \subset E \cup \tilde{Y}$.



We will now show that $\tilde{D}(\mathbb{R})$ is disjoint from the closure of $\tilde{G}$ in $\tilde{X}(\mathbb{R})$. Suppose that x is contained in their intersection. Since $\tilde{Y}$ is disjoint from $\tilde{D}$, we conclude that $x \in E(\mathbb{R})$. As $\tilde{Y}$ is closed, there is even an open neighbourhood U of x in $\tilde{X}(\mathbb{R})$ such that U is disjoint from $\tilde{Y}(\mathbb{R})$. In particular we find that $G \cap U$ is open in $\tilde{X}(\mathbb{R})$, and that $\partial\tilde{G} \cap U \subset E(\mathbb{R})$. After a suitable choice of continuous semialgebraic coordinates, we see that this contradicts the conclusion of Lemma 12.2. Therefore the closure of $\tilde{G}$ is disjoint from $\tilde{D}(\mathbb{R})$.

Since $\tilde{f} = \bar{f} \circ \pi$ is not rational on E, we conclude that $\tilde{X}$ satisfies the conditions of the statement. □

12.2. **Proof of Proposition 12.1.**

Proof. Let α be a generalised naive exponential period. By [CHH20, Lemma 5.6] and Lemma 10.13, we may assume without loss of generality that $k \subset \mathbb{R}$ and that k is real closed and hence $k(i) = \bar{k}$. Generalised naive exponential periods are absolutely convergent, so we can use the characterisation of [CHH20, Proposition 5.19]. This brings us into Setting 12.3 with

$$\alpha = \int_G \mathrm{e}^{-f}\omega.$$

By Lemma 12.4, we may improve the good compactification $\bar{X}$ in such a way that the closure of G in $\bar{X}^{\mathrm{an}}$ is disjoint from the components of X_∞ on which $\bar{f}$ has a pole. This implies that the closure $\bar{G}$ of G in the real oriented blow-up $B_{\bar{X}}(X)$ is contained in $B^\circ_{\bar{X}}(X, f)$. Note that $\bar{G}$ is compact because $B_{\bar{X}}(X)$ is. We replace X by $X_{\bar{k}}$ from now on.

Let $Y_\bullet$ and their compactifications be as in Section 11.3. Note that we do not have to pass to an open Čech-cover because X is affine. By definition

$$R\Gamma_{\mathrm{dR}}(X, Y)^d = \Omega^d(X) \oplus \Omega^{d-1}(Y_0) \oplus \cdots \oplus \Omega^0(Y_{d-1})$$

The tuple $(\omega, 0, \ldots, 0)$ is a cocycle because $d\omega = 0$ and $\omega|_{Y_0} = 0$, both for dimension reasons. We denote the induced cohomology class by

$$[\omega] \in H^n_{\mathrm{dR}}(X, Y, f).$$

Recall that G is equipped with a pseudo-orientation. Let $G' \subset G$ be an oriented semi-algebraic subset with $\dim(G \smallsetminus G') < d$ that represents the pseudo-orientation. We apply [CHH20, Proposition 7.4] to the semi-algebraic manifold with corners $B^\circ_{\bar{X}}(X, f)$. Hence we may choose a semi-algebraic triangulation of $\bar{G}$ that is globally of class C^1 and that is compatible with he oriented subset G', and also compatible with the subsets $B^\circ_{\bar{X}}(Y^J) \cap \bar{G}$, and $\partial B^\circ_{\bar{X}}(Y^J, f) \cap \bar{G}$ for all J Here Y^J is the intersection of irreducible components of Y as in Section 11.3. The top dimensional simplices inherit an orientation from G'. We use the triangulation of $\bar{G}$ to define a cycle $(\sigma, \sigma_0, \ldots, \sigma_{d-1})$ in

$$\begin{aligned} &S^{\mathrm{rd}}_d(\mathrm{Cone}(Y_\bullet \to X)) = \\ &\qquad S_d(B^\circ(X, f), \partial) \oplus S_{d-1}(B^\circ(Y_0, f_0), \partial) \oplus \cdots \oplus S_0(B^\circ(Y_{d-1}, f_{d-1}, \partial) \end{aligned}$$

where we abbreviate $S_{n-1}(B^\circ(Y_i, f_i), \partial) = S_{n-1}(B^\circ_{\bar{Y}_i}(Y_i, f_i))/S_{n-1}(\partial B^\circ_{\bar{Y}_i}(Y_i, f_i))$.

In detail: We are given a simplicial complex K and a homeomorphism $h : |K| \to \bar{G}$ which extends to a C^1-map on a neighbourhood of $|K|$. For each closed top-dimensional simplex $a = [a_0, \ldots, a_d] \in K$, we choose a linear isomorphism $\bar{\Delta}_d \to [a_0, \ldots, a_d]$. By composition we obtain a C^1-map

$$T_a : \bar{\Delta}_d \to [a_0, \ldots, a_d] \xrightarrow{h|_{[a_0,\ldots,a_d]}} B^\circ_{\bar{X}}(X, f).$$

It is a homeomorphism onto its image. The image of T_a is oriented by the orientation on G'. We can arrange for T_a to respect this orientation. The formal linear combination

$$\sigma = \sum_{a \in K_d} T_a$$

is a chain on $B^\circ_{\bar{X}}(X, f)$. Its boundary $\check{\partial}\sigma$ is a linear combination of $(d-1)$-simplices with image contained in one of the components Y^i or in $\partial B^\circ_{\bar{X}}(X, f)$.

Let $\sigma_0 \in S_{d-1}(Y_0)$ be the chain defined by the simplices in the Y^i, ignoring the ones with image contained in $\partial B^\circ_{\bar{X}}(X,f)$. By construction, the simplices appearing in $\check{\partial}\sigma_0$ are contained in one of the Y^{ij}, hence they define $\sigma_1 \in S_{d-2}(Y_1)$. Recursively, we find all σ_a. By construction, $\check{\partial}(\sigma, \sigma_0, \dots, \sigma_{d-1})$ is a cycle. Let

$$[G] \in H^{\mathrm{rd}}_d(X,Y,f)$$

be its homology class. Because of the special shape of $[\omega]$, we have

$$\langle [\omega], [\sigma] \rangle = \sum_{a \in K_d} \int_{\bar{\Delta}_d} T_a^* \omega = \sum_{a \in K_d} \int_{T_a(\bar{\Delta}_d)} \omega = \int_G \omega.$$

We have written α as a cohomological period over $\bar{k}$. □

13. Conclusion

Fresán and Jossen develop a fully fledged theory of exponential motives in [FJ20]. It behaves very much like the theory of ordinary Nori motives. In particular, there is a so-called "basic lemma" for affine pairs (X,Y,f). We refer to their book for further details. We denote by $\mathcal{P}_{\mathrm{mot}}(k)$ the set of periods of effective exponential motives.

Proposition 13.1. *The periods of effective exponential motives are exponential periods in the sense of Definition 10.12 for a tuple (X,Y,f,n) with X smooth, Y a strict normal crossings divisor and $n = \dim X$. In other words,*

$$\mathcal{P}_{\mathrm{mot}}(k) \subset \mathcal{P}_{\log}(k).$$

Proof. By definition, every effective exponential motive is a subquotient of some exponential motive of the form $H^n(X,Y,f)$ for an affine k-variety X, $Y \subset X$ a subvariety, $f \in \mathcal{O}(X)$, and $X \smallsetminus Y$ smooth. Hence its periods are also periods of $H^n(X,Y,f)$.

There is a blow-up $\pi\colon \tilde{X} \to X$ such that $\tilde{X}$ is smooth and $\tilde{Y} = \pi^{-1}(Y)$ is a simple normal crossings divisor. By excision for rapid decay homology, we obtain an isomorphism

$$H^{\mathrm{rd}}_n(\tilde{X},\tilde{Y},f) \cong H^{\mathrm{rd}}_n(X,Y,f).$$

This isomorphism lifts to an isomorphism of motives. Hence they have the same periods. □

Remark 13.2. By Proposition 12.1 all exponential periods are even realised as cohomological exponential periods of *affine* log-pairs. This is not obvious from the purely motivic argument given above.

Proposition 13.3. *Periods of complexes of smooth affine varieties are periods of effective exponential Nori motives, i.e.,*

$$\mathcal{P}_{\mathrm{SmAff}}(k) \subset \mathcal{P}_{\mathrm{mot}}(k).$$

Proof. The argument is the same as in the case of ordinary Nori motives, see [HMS17, Theorem 11.4.2]. We give a sketch of the proof.

By [FJ20, Corollary 3.3.3], we may choose a good filtration $F_0X \subset F_1X \subset \dots F_nX = X$ of an affine variety X, i.e., one where in every step the relative homology is concentrated in a single degree equal to the dimension. By definition the exponential motives of X are computed as homology of the complex of exponential Nori motives

$$\dots H_{i+1}(F_{i+1}X, F_iX, f) \to H_i(F_iX, F_{i-1}X, f) \to \dots.$$

Given a complex $X_\bullet$ of affine varieties, we may choose compatible good filtrations on all entries of the complex. The exponential motives of $X_\bullet$ are defined as homology of the total complex of the double complex $H_i(F_iX_j, F_{i-1}X_j, f_j)$. This is compatible with the period computation, hence we have identified the periods of $X_\bullet$ with the periods of exponential period motives. $\square$

Theorem 13.4. *Let $k \subset \mathbb{C}$ be a field, $k_0 = k \cap \mathbb{R}$, and assume that k/k_0 is algebraic. Then the following subsets of $\mathbb{C}$ agree:*

(1) $\mathcal{P}_{\mathrm{nv}}(k)$, i.e., naive exponential periods over k;
(2) $\mathcal{P}_{\mathrm{gnv}}(k)$, i.e., generalised naive exponential periods over k;
(3) $\mathcal{P}_{\mathrm{abs}}(k)$, i.e., absolutely convergent exponential periods over k;
(4) $\mathcal{P}_{\mathrm{mot}}(k)$, i.e., periods of all effective exponential motives over k;
(5) $\mathcal{P}_{\mathrm{coh}}(k)$, i.e., the set of periods of all (X, Y, f, n) with X a k-variety, $Y \subset X$ a subvariety, $f \in \mathcal{O}(X)$, and $n \in \mathbb{N}_0$;
(6) $\mathcal{P}_{\log}(k)$, i.e., periods of all tuples (X, Y, f, n) with (X, Y) a log pair, $f \in \mathcal{O}(X)$, and $n \in \mathbb{N}_0$;
(7) $\mathcal{P}_{\mathrm{SmAff}}(k)$, i.e., periods of all tuples $(X_\bullet, f_\bullet, n)$ for $(X_\bullet, f_\bullet) \in C_-(\mathrm{SmAff}/\mathbb{A}^1)$ and $n \in \mathbb{N}_0$.

Moreover, the real and imaginary part of these numbers are up to sign volumes of bounded definable sets for the o-minimal structure $\mathbb{R}_{\sin,\exp,k_0}$ generated by $\exp$, $\sin|_{[0,1]}$ *and with paramaters in k_0, see* [CHH20, Definition 2.13].

Proof. The statement on volumes of definable sets is [CHH20, Theorem 5.12].

The following diagram shows all the inclusions that we have proved between the sets listed above.

$$\begin{tikzcd}
\mathrm{Vol} & & & & \\
\mathcal{P}_{\mathrm{nv}} \arrow[u, hook, "{[CHH20, Theorem 5.12]}"] \arrow[rr, hook, "{[CHH20, Lemma 5.5]}"] & & \mathcal{P}_{\mathrm{gnv}} \arrow[dl, hook, "\text{Proposition 12.1}"] \arrow[rr, equal, "{[CHH20, Corollary 5.20]}"] & & \mathcal{P}_{\mathrm{abs}} \\
& \mathcal{P}_{\log} \arrow[ul, hook, "\text{Proposition 11.1}"] \arrow[r, hook, "\text{triv}"] & \mathcal{P}_{\mathrm{coh}} \arrow[r, hook, "\text{triv}"] & \mathcal{P}_{\mathrm{SmAff}} \arrow[dl, hook, "\text{Proposition 13.3}"] & \\
& & \mathcal{P}_{\mathrm{mot}} \arrow[ul, hook, "\text{Proposition 13.1}"] & &
\end{tikzcd}$$

Therefore we have equality everywhere. $\square$

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