A p-ADIC ANALOGUE OF THE BOREL REGULATOR AND THE BLOCH-KATO EXPONENTIAL MAP

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Abstract. In this paper we define a p-adic analogue of the Borel regulator for the K-theory of p-adic fields. The van Est isomorphism in the construction of the classical Borel regulator is replaced by the Lazard isomorphism. The main result relates this p-adic regulator to the Bloch-Kato exponential and the Soulé regulator. On the way we give a new description of the Lazard isomorphism for certain formal groups. We also show that the Soulé regulator is induced by continuous and even analytic classes.

Contents

Date: February 5, 2010.

Introduction

The classical Borel regulator

$$b_\infty : K_{2n-1}(\mathbb{C}) \to \mathbb{C}$$

plays a decisive role in the study of algebraic number fields. Its simple description makes it possible to relate this regulator quite directly to special values of zeta functions of number fields, an insight we owe to Borel [Bor2]. Later, when Beilinson formulated his famous conjectures, he was able to relate his regulator r_∞ to Borel's regulator map, so that the computations by Borel implied one of the most impressive confirmations of Beilinson's conjectures.

One of the key features of Borel's computation is that he is able to compute the determinant of the regulator by considering suitable cup products inside group cohomology. The actual computation turns out to be just an application of the classical Tamagawa number formula for the general linear group.

It is frustrating that these beautiful ideas have not found further applications. In particular progress on the Tamagawa number conjecture for number fields is restricted to the abelian case and uses different methods.

One of the guiding ideas of this paper has been the analogue between Bloch and Kato's Tamagawa number conjecture and the classical Tamagawa number formula. In fact, it is our hope that one can reformulate the full Tamagawa number conjecture for number fields as a Tamagawa number formula. Pursuing this idea, we were led to a reformulation of the p-adic regulator for number fields which is completely parallel to Borel's definition of his regulator in the case of the reals.

Let us describe more precisely the main result of this paper. Let $K/\mathbb{Q}_p$ be a finite extension and $R \subset K$ its valuation ring. Soulé has defined a regulator map

$$r_p : K_{2n-1}(K) \to H^1_{\text{et}}(K, \mathbb{Q}_p(n))$$

between the K-theory of K and étale cohomology. We define in Section 1.2, in complete analogy with the classical Borel regulator for $\mathbb{C}$, a map

$$b_p : K_{2n-1}(R) \to K \ .$$

We then show (Theorem 1.3.2) that the diagram

(1)
$$\begin{array}{ccc} K_{2n-1}(R) & \xrightarrow{r_p} & H^1_{\text{et}}(K, \mathbb{Q}_p(n)) \\ & \searrow^{b_p} & \uparrow \scriptstyle{\exp_{\text{BK}}} \\ & & K = D_{\text{dR}}(\mathbb{Q}_p(n)) \end{array}$$

commutes for all $n \geq 1$. Here $\exp_{\text{BK}} : K \to H^1_{\text{et}}(K, \mathbb{Q}_p(n))$ is the Bloch-Kato exponential map, which is an isomorphism for $n > 1$.

The idea to define b_p follows closely Borel's construction (as reviewed in Section 1.1), only that we replace the van Est isomorphism

$$H^i(\mathfrak{g}, \mathfrak{k}, \mathbb{C}) \cong H^i_{\text{cont}}(G(\mathbb{C}), \mathbb{C})$$

between relative Lie algebra cohomology and continuous group cohomology, by the Lazard isomorphism

$$H^i(\mathfrak{g}, \mathbb{Q}_p) \cong H^i_{\text{cont}}(G(R), \mathbb{Q}_p)$$

(for details see Section 1.2). In the second paper [HK], we are able to relate the Lazard isomorphism to the local Tamagawa measure, given a second indication that this is indeed the correct construction to pursue.

The commutative diagram (1) above can be seen as a natural generalisation of the explicit reciprocity law for $\mathbb{G}_m$

$$\begin{array}{ccc} R^* & \xrightarrow{\partial} & H^1_{\text{et}}(K, \mathbb{Q}_p(1)) \\ & \searrow^{\log_p} & \uparrow \scriptstyle{\exp_{\text{BK}}} \\ & & K, \end{array}$$

(where ∂ is the Kummer map) established by Bloch and Kato in their work on the exponential map [BK] 3.10.1. The function $\log_p$ is continuous and even locally analytic. A key step in the proof of our Main Theorem is to establish continuity and even analyticity of the regulator map also for general n.

Karoubi has defined a p-adic regulator for p-adic Banach algebras in [Kar1], [Kar2], which was studied in more detail in the thesis of Hamida [Ha]. As he pointed out to us, our construction should be directly related to his regulator in the case of p-adic fields. In his preprint [Ta], Tamme has made this relation explicit and calculated that the Karoubi regulator on $K_{2n-1}(R)$ is in fact equal to $\frac{(-1)^{n-1}}{(n-1)!(2n-2)!}b_p$.

On the way to establishing the diagram (1), we also get a new description of the Lazard isomorphism, which should be of indepent interest. Namely, we show that for a smooth algebraic group scheme H/R with formal group $\mathcal{H}$ the Lazard isomorphism $H^i_{\mathrm{la}}(\mathcal{H},\mathbb{Q}_p)\cong H^i(\mathrm{Lie}H,\mathbb{Q}_p)$ is induced by the map

$$\begin{aligned}\Phi : \mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\otimes n} &\to \bigwedge\nolimits^n \mathrm{Lie}H\\ f_1\otimes\ldots\otimes f_n &\mapsto df_1\wedge\ldots\wedge df_n\end{aligned}$$

familiar from cyclic homology. In the sequel [HKN] together with Niko Naumann we use this description to establish the same isomorphism also for K-analytic groups induced by smooth group schemes. In the same paper, we also establish a $\mathbb{Z}_p$-integral version of Lazard's isomorphism for certain p-adic Lie groups.

Unfortunately, the proof of diagram (1) is quite involved and technical. The idea is to follow Beilinson's proof, which leads to a comparison between the classical Borel regulator and Beilinson's regulator in Deligne cohomology. Our strategy adapts Beilinson's ideas to the p-adic case replacing Deligne cohomology by syntomic cohomology.

The paper is organised as follows: in the first section we construct the p-adic analogue of the Borel regulator and formulate our main results. We also draw some immediate consequences.

The second section recalls what we need about rigid syntomic cohomology and introduces the evaluation map 2.3.2, which is the main tool in the comparison with the locally analytic group cohomology.

The third section relates this evaluation map to locally analytic group cohomology via the suspension map and Lie algebra cohomology. Here we follow quite closely Beilinson's ideas as explained by Rapoport [Ra] and Burgos [Bu].

The fourth section shows finally that the map between locally analytic group cohomology and Lie algebra cohomology is indeed the Lazard isomorphism.

The final section collects the loose ends and finishes the proof.

It is a great pleasure to thank José Burgos for answering some questions about Beilinson's proof of his comparison results and Elmar Große-Klönne for his help with rigid cohomology. We also thank Max Karoubi for his comments on an earlier version of this manuscript. Of course the whole article owes its very existence to the beautiful ideas of Borel and Beilinson.

Notation

In this section we collect various notations, which will be needed later.

0.1. **Rings of functions.** Let p be a fixed prime. Let R be a discrete valuation ring finite over $\mathbb{Z}_p$ with uniformiser π, residue field k and field of fractions K. Throughout let G be GL_N, the general linear group over R and $\mathfrak{g}$ its K-Lie algebra, ie. $\mathfrak{gl}_N = M_N(K)$ the $N \times N$-matrices.

To a smooth R-scheme X, we attach a number of spaces. The set $X(R)$ is denoted X^δ. It carries a natural structure of (locally) analytic manifold over K, which is denoted X^{la}. The underlying topological space of X^{la} is denoted X^{cont}. In Section 2 we are also going to consider its structure as overconvergent rigid analytic manifold, denoted $X^\dagger$ (see Example 2.1.2). In the case $X = G$, the set G^δ is a group, G^{cont} a topological group and G^{la} is a K-Lie group.

We also denote

- $\mathcal{O}(X)$ the global sections of the R-scheme X,
- $\mathcal{O}(X^{\mathrm{alg}})$ the global sections of the K-scheme $X \times_R K$,
- $\mathcal{O}(X^\delta)$ the ring of set-theoretic K-valued functions on $X(R)$,
- $\mathcal{O}(X^{\mathrm{cont}})$ the ring of continuous K-valued functions on $X(R)$,
- $\mathcal{O}(X^{\mathrm{la}})$ the ring of K-valued locally analytic functions on X^{la}, i.e. functions which are locally representable by convergent power series with coefficients in K,
- $\mathcal{O}(X^\dagger)$ the ring of overconvergent rigid analytic functions on $X^\dagger$.

If A is a K-vector space, we denote $\mathcal{O}(X^{\mathrm{la}}, A)$, $\mathcal{O}(X^\delta, A)$ etc. the corresponding functions with values in K.

If $X.$ is a smooth simplicial R-scheme, $\mathcal{O}(X.)$ etc. are cosimplicial rings.

0.2. **The classifying space.** We collect some standard material on the classifying space $B.H$ thereby fixing our notations.

Let H/R be a reductive algebraic group over a ring R or a formal group. As usual, let $E.H$ be the simplicial space with

$$E_nH = H \times \cdots \times H \;\; n+1\text{-times}$$

with the usual face and degeneracy maps. The group H acts on $E.H$ on the right via

$$(h_0, \dots, h_n)h := (h_0h, \dots, h_nh).$$

The quotient by this action is the classifying space $B.H$ and has the explicit description

$$B_nH = H \times \cdots \times H \;\; n\text{-times}$$

with degeneracies $\sigma^i(h_1, \dots, h_n) = (h_1, \dots, h_i, 1, h_{i+1}, \dots, h_n)$ and faces

$$\delta^0(h_1, \dots, h_n) = (h_2, \dots, h_n)$$
$$\delta^i(h_1, \dots, h_n) = (h_1, \dots, h_ih_{i+1}, \dots, h_n) \text{ for } i = 1, \dots, n-1$$
$$\delta^n(h_1, \dots, h_n) = (h_1, \dots, h_{n-1}).$$

The map $E.H \to B.H$ sends $(h_0, \dots, h_n) \mapsto (h_0 h_1^{-1}, \dots, h_{n-1} h_n^{-1})$.

0.3. **Group cohomology.** Let H be an abstract group. Let F^* be a complex of $\mathbb{Q}_p$-vector spaces with trivial operation of H. Let $H^i(H, F^*)$ be group hypercohomology of H with coefficients in F, i.e., in the notation introduced above, cohomology of the complex associated to the cosimplicial complex $\mathcal{O}(B.H^\delta, F^*)$.

For a K-Lie group H and a complex F^* of K-vector spaces, let $H^i_{\mathrm{cont}}(H, F^*)$ be *continuous group cohomology*, i.e., cohomology of $\mathcal{O}(B.H^{\mathrm{cont}}, F^*)$ and $H^i_{\mathrm{la}}(H, F^*)$ the *locally analytic group cohomology*, i.e., the cohomology of $\mathcal{O}(B.H^{\mathrm{la}}, F^*)$. The inclusions of function spaces induce natural transformations

$$H^i_{\mathrm{la}}(H, F^*) \to H^i_{\mathrm{cont}}(H, F^*) \to H^i(H, F^*).$$

If H is a smooth algebraic group over R, we abbreviate $H^i_{\mathrm{cont}}(H, F^*) = H^i_{\mathrm{cont}}(H^{\mathrm{cont}}, F^*)$ etc.

0.4. **Lie algebras and their cohomology.**

Definition 0.4.1. Let H be a smooth algebraic group over R. We denote

$$\mathrm{Lie}H = \mathrm{Der}_K(\mathcal{O}(H^{\mathrm{alg}})_e, K)$$

its algebraic tangent space at e. This is the *K-Lie algebra* of H.

Example 0.4.2. For $G = \mathrm{GL}_N$, we have naturally $\mathrm{Lie}G \cong \mathfrak{g}$.

By definition, there is a natural pairing

$$\mathcal{O}(H^{\mathrm{alg}})_e \times \mathrm{Lie}H \to K$$

It extends to locally analytic functions and induces

$$\mathcal{O}(H^{\mathrm{la}}) \to \mathrm{Lie}H^\vee$$
$$f \mapsto df(e)$$

Definition 0.4.3. Let $\mathfrak{h}$ be a Lie algebra over K. Then *Lie algebra cohomology* $H^i(\mathfrak{h}, K)$ is defined as cohomology of the complex $\bigwedge^* \mathfrak{h}_K^\vee$ with differential induced by the dual of the Lie bracket $\mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$.

Example 0.4.4. For $\mathfrak{g} = \mathfrak{gl}_N$, $H^*(\mathfrak{g}, K)$ is an exterior algebra on primitive elements $p_n \in H^{2n-1}(\mathfrak{g}, K)$ for $n = 1, \dots, N$.

We need a precise normalisation of these elements and follow [Bu] Example 5.37.

Definition 0.4.5. Let $\mathfrak{g} = \mathfrak{gl}_N$. For $n \le N$ let $p_n \in H^{2n-1}(\mathfrak{g}, K)$ be the map

$$p_n : \bigwedge^{2n-1} \mathfrak{g} \to K$$

which attaches to $(x_1, \dots, x_{2n-1}) \in \mathfrak{g}^{\oplus 2n-1}$ the value

$$\frac{((n-1)!)^2}{(2n-1)!} \sum_{\sigma \in \mathfrak{S}_{2n-1}} \mathrm{sgn}(\sigma)\, \mathrm{Tr}(x_{\sigma(1)} \circ \dots \circ x_{\sigma(2n-1)})$$

where $\mathfrak{S}_{2n-1}$ is the symmetric group, Tr is the trace map from $\mathfrak{gl}_N$ to K and $\circ$ is matrix multiplication.

Remark 0.4.6. The exact form of this primitive element p_n is not necessary for our main result. What we need is the element $p_n \in H^{2n-1}(\mathfrak{g}, K)$ which is the image of the Chern class $c_n \in H^{2n}_{\mathrm{dR}}(B.\mathrm{GL}_N/K)$ under the suspension map $s_G : H^{2n}_{\mathrm{dR}}(B.\mathrm{GL}_N/K) \to H^{2n-1}_{\mathrm{dR}}(\mathrm{GL}_N/K) \cong H^{2n-1}(\mathfrak{g}, K)$ (see Section 3.2 for more details on the suspension map).

1. Construction of the p-adic Borel regulator and statement of the main results

1.1. **Review of the classical Borel regulator.** We briefly review the construction of the classical Borel regulator

$$b_\infty : K_{2n-1}(\mathbb{C}) \to \mathbb{C}$$

from [Bor1]. Recall that the K-group $K_{2n-1}(\mathbb{C})$ maps via the Hurewicz map to the group homology $H_{2n-1}(\mathrm{GL}(\mathbb{C}), \mathbb{C})$ of the infinite linear group $\mathrm{GL}(\mathbb{C}) := \varinjlim_N \mathrm{GL}_N(\mathbb{C})$. Thus one needs to define a system of compatible maps for all big enough N (also called b_∞)

$$b_\infty : H_{2n-1}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C}) \to \mathbb{C},$$

or, by duality for group cohomology, elements

$$b_\infty \in H^{2n-1}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C}),$$

compatible with enlarging N. In fact, using stabilisation, these elements live in $H_{2n-1}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C})$ for fixed n and N big enough. Borel constructs this b_∞ with the *van Est isomorphism* for $\mathrm{GL}_N(\mathbb{C})$ considered as real Lie group

$$H^{2n-1}(\mathfrak{gl}_N, \mathfrak{u}_N, \mathbb{C}) \cong H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C})$$

between relative Lie algebra cohomology for $\mathfrak{gl}_N$ with respect to the Lie algebra $\mathfrak{u}_N$ of the unitary group U_N and continuous group cohomology. To compute $H^{2n-1}(\mathfrak{gl}_N, \mathfrak{u}_N, \mathbb{C})$ observe that the compact form of $\mathrm{GL}_N(\mathbb{C})$ is $U_N \times U_N$ so that we have isomorphisms

$$H^{2n-1}(\mathfrak{gl}_N, \mathfrak{u}_N, \mathbb{C}) \cong H^{2n-1}(\mathfrak{u}_N \oplus \mathfrak{u}_N, \mathfrak{u}_N, \mathbb{C}) \cong H^{2n-1}(\mathfrak{u}_N, \mathbb{C}) \cong H^{2n-1}(\mathfrak{gl}_N, \mathbb{C}).$$

Recall from Definition 0.4.5 that we have defined a primitive element $p_n \in H^{2n-1}(\mathfrak{gl}_N, \mathbb{C})$.

Definition 1.1.1. The *Borel regulator* b_∞ is the image of p_n under the composition

$$H^{2n-1}(\mathfrak{gl}_N, \mathbb{C}) \cong H^{2n-1}(\mathfrak{gl}_N, \mathfrak{u}_N, \mathbb{C}) \cong H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C}) \to H^{2n-1}(\mathrm{GL}_N(\mathbb{C}), \mathbb{C}).$$

Remark 1.1.2. Beilinson shows in [Be] (see also [Ra], [Bu]) that under the identification of Deligne cohomology $H^1_{\mathcal{D}}(\mathrm{Spec}\mathbb{C}, \mathbb{R}(n)) \cong \mathbb{C}$ his regulator r_∞ coincides with Borel's regulator b_∞ up to a rational number. It was later shown by Burgos [Bu] that in fact $2r_\infty = b_\infty$.

1.2. **A p-adic analogue of Borel's regulator.** Recall that $K/\mathbb{Q}_p$ is a finite extension with valuation ring R. Our aim is to define a p-adic analogue of Borel's regulator. We will construct a map

$$b_p : K_{2n-1}(R) \to K.$$

As before, we have the Hurewicz map $K_{2n-1}(R) \to H_{2n-1}(\mathrm{GL}(R), K)$ and it suffices to define a compatible system of maps

$$b_p : H_{2n-1}(\mathrm{GL}_N(R), K) \to K \quad \text{for } N \geq n$$

or, equivalently, elements

$$b_p \in H^{2n-1}(\mathrm{GL}_N(R), K).$$

To define b_p we replace the van Est isomorphism in Borel's construction with a slight generalisation of the *Lazard isomorphism*: consider $\mathrm{GL}_N(R)$ as a K-analytic group and let $\mathfrak{gl}_N$ be its K-Lie algebra. We will prove:

Theorem 1.2.1 ([L], see also Section 5). *For all $i \geq 0$ one has an isomorphism*

$$H^i_{\mathrm{la}}(\mathrm{GL}_N(R), K) \cong H^i(\mathfrak{gl}_N, K)$$

between the locally analytic group cohomology and the Lie algebra cohomology.

Remark 1.2.2. In the case $R = \mathbb{Z}_p$, this is the combination of two isomorphisms shown by Lazard

$$H^i_{\mathrm{la}}(\mathrm{GL}_N(\mathbb{Z}_p), \mathbb{Q}_p) \cong H^i_{\mathrm{cont}}(\mathrm{GL}_N(\mathbb{Z}_p), \mathbb{Q}_p)$$

and

$$H^i_{\mathrm{cont}}(1 + pM_N(\mathbb{Z}_p), \mathbb{Q}_p) \cong H^i(\mathfrak{gl}_N, \mathbb{Q}_p),$$

(see [L] chapter V) together with a remark of Casselman-Wigner [CW] §3 to pass from the saturated subgroup $1 + pM_N(\mathbb{Z}_p)$ to $\mathrm{GL}_N(\mathbb{Z}_p)$. (In the case $p = 2$ the saturated subgroup is $1 + 4M_N(\mathbb{Z}_2)$.) For the non-trivial argument in the general case see Section 5. In [HKN] Theorem 4.3.1, we extend the result to all open subgroups of $\mathbb{G}(R)$ where $\mathbb{G}/R$ is a formal group scheme.

Now we can define the p-adic analogue of the Borel regulator using again the primitive element $p_n \in H^i(\mathfrak{gl}_N, K)$ from Definition 0.4.5.

Definition 1.2.3. The *p-adic Borel regulator*

$$b_p : K_{2n-1}(R) \to K$$

for $1 \leq n$, is defined by the element $b_p \in H^{2n-1}(\mathrm{GL}_N(R), K)$ (for N big enough), which is the image of p_n under the composition

$$H^i(\mathfrak{gl}_N, K) \cong H^i_{\mathrm{la}}(\mathrm{GL}_N(R), K) \to H^{2n-1}(\mathrm{GL}_N(R), K).$$

Remark 1.2.4. (1) Soulé [So2] was the first to study a p-adic regulator for $K_{2n-1}(K)$ with values in $\mathbb{Z}_p^{[K:\mathbb{Q}_p]}$. His regulator is defined via Iwasawa theory.

(2) Karoubi (see e.g. [Kar2]) has also defined a p-adic regulator with values in topological cyclic homology. His construction is by homotopy theoretical methods. His regulator was further studied in the thesis of Hamida [Ha]. Tamme [Ta] has been able to show that the Karoubi regulator is in fact equal to $\frac{(-1)^{n-1}}{(n-1)!(2n-2)!}b_p$.

1.3. **The main result.** Before we can formulate our main result, we need to recall the Soulé regulator and the Bloch-Kato exponential map.

Soulé [So1] has defined *regulators* for $n > 0$

$$r_p : K_{2n-1}(K) \to H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n)),$$

which are just the Chern classes

$$c_n \in H^{2n-1}(\mathrm{GL}_N(K), H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n)))$$

induced by the universal Chern classes (via restriction to $B.GL_N^\delta$)

$$c_n \in H^{2n}_{\mathrm{et}}(B.GL_N, \mathbb{Q}_p(n)).$$

By abuse of notation, we also let r_p be the composition

$$r_p : K_{2n-1}(R) \to K_{2n-1}(K) \to H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n)).$$

This r_p is given by the restriction of the c_n above to $H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n)))$.

Remark 1.3.1. Note that for $n > 1$ the morphism of $\mathbb{Q}_p$-vector spaces

$$K_{2n-1}(R) \otimes \mathbb{Q}_p \to K_{2n-1}(K) \otimes \mathbb{Q}_p$$

is an isomorphism. This follows from the localisation sequence for K-theory and from Quillen's result that K_i of a finite field is torsion for $i \geq 1$.

Recall ([BK] Definition 3.10) that the *Bloch-Kato exponential*

$$\exp_{\mathrm{BK}} : K \to H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n))$$

is the connecting morphism of the short exact sequence of continuous $\mathrm{Gal}(\bar{K}/K)$-modules ([BK] Proposition 1.17)

$$0 \to \mathbb{Q}_p \to B^{f=1}_{\mathrm{crys}} \oplus B^+_{\mathrm{DR}} \to B_{\mathrm{DR}} \to 0$$

tensored with $\mathbb{Q}_p(n)$. It is an isomorphism for $n > 1$.

Our main result is the following:

Theorem 1.3.2. *The following diagram commutes for all $n > 0$:*

$$\begin{array}{ccc} K_{2n-1}(R) & \xrightarrow{r_p} & H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n)) \\ & \searrow^{b_p} & \uparrow \exp_{\mathrm{BK}} \\ & & K \end{array} \qquad (2)$$

Idea of proof: (1) By construction, the statement of the theorem can be formulated as follows: the p-adic Borel regulator $b_p \in H^{2n-1}(\mathrm{GL}_N(R), K)$ is mapped under the Bloch-Kato exponential $\exp_{\mathrm{BK}} : K \to H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n))$ to the étale Chern class $c_n \in H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n)))$

(2) In Proposition 2.2.9 together with Proposition 2.3.4 we show that the étale Chern class c_n is the image of the syntomic Chern class $H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{syn}}(K,n))$ under the natural map

$$H^1_{\mathrm{syn}}(R,n) \to H^1_{\mathrm{et}}(K,\mathbb{Q}_p(n)).$$

(3) Using the evaluation map from syntomic cohomology (Theorem 2.4.1) we consider the image of this syntomic Chern class in analytic group cohomology.

(4) We apply the Lazard morphism to this element and show that it can be identified with the primitive element in Lie algebra cohomology. For this we have to relate the Lazard morphism to the suspension map (see Theorem 3.1.1)

(5) This finishes the proof because it is well known that the Chern class (in de Rham cohomology) is mapped to the primitive element under the suspension map. In fact, this is one of the possible definitions of Chern classes.

The detailed proof will be given in Section 5. □

Remark 1.3.3. One can see the main theorem as a kind of generalised explicit reciprocity law for the formal group of $\mathbb{G}_m$. More precisely, the main theorem generalises the diagram

$$\begin{array}{ccc} R^* & \xrightarrow{\partial} & H^1_{\mathrm{et}}(K,\mathbb{Q}_p(1)) \\ & \searrow^{\log_p} & \uparrow \exp_{\mathrm{BK}} \\ & & K, \end{array}$$

(where ∂ is the Kummer map) established by Bloch and Kato in their work on the exponential map [BK] 3.10.1.

Corollary 1.3.4. (1) *The Soulé regulator* $c_n \in H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{et}}(K,\mathbb{Q}_p(n)))$ *is the image of an element*

$$c_n^{\mathrm{cont}} \in H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_N(R), H^1_{\mathrm{et}}(K,\mathbb{Q}_p(n)))$$

in continuous group cohomology for N big enough.

(2) *There is an element*

$$(\exp^*_{\mathrm{BK}} c_n)^{\mathrm{la}} \in H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K)$$

in locally analytic group cohomology for N big enough, such that the composition with the Bloch-Kato-exponential map $K \to H^1_{\mathrm{et}}(K,\mathbb{Q}_p(n))$ gives c_n^{cont}.

Proof. After identifying r_p (i.e, the étale Chern class) with b_p this follows from the definition of b_p. In fact, however, we are going to prove this directly as a step in the proof of the Main Theorem, see Corollary 2.5.3. □

Remark 1.3.5. In her thesis [Ha] Hamida also obtains the result that Karoubi's regulator is induced from a continuous (hence also from a locally analytic) group cohomology class.

1.4. **The Lazard isomorphism as a Taylor series expansion.** In our proof of the main theorem we will give a new description of the Lazard isomorphism at least for formal groups associated to smooth algebraic groups.

Let $H/\mathbb{Z}_p$ be a smooth linear algebraic group and $\mathcal{H}$ a p-saturated group of finite rank (see [L] III Definition 2.1.3 and 2.1.6) with valuation ω which is an open subgroup of $H(\mathbb{Z}_p)$. The group $\mathcal{H}$ will always be considered as a $\mathbb{Q}_p$-analytic manifold. Our main example is $H = \mathrm{GL}_N$, $\mathcal{H} = 1 + pM_N(\mathbb{Z}_p)$ with ω the inf-valuation.

We show in Proposition 4.3.1 that the Lie algebra $\mathcal{L}^*$ of $\mathcal{H}$ in the sense of Lazard (cf. Definition 4.1.1) can be identified with the algebraic Lie algebra

$$\mathfrak{h} \cong \mathcal{L}^* \otimes \mathbb{Q}_p.$$

Lazard ([L] chapter V) shows in his paper that one has a chain of isomorphisms

$$H^i_{\mathrm{la}}(\mathcal{H}, \mathbb{Q}_p) \cong H^i_{\mathrm{cont}}(\mathcal{H}, \mathbb{Q}_p) \cong H^i(\mathcal{L}^*, \mathbb{Q}_p)$$

where the last isomorphism is induced by an isomorphism between the saturations of the continuous group algebra of $\mathcal{H}$ and the universal enveloping algebra of $\mathcal{L}^*$. We give a much simpler description of this isomorphism:

Definition 1.4.1. Define a map of complexes

$$\Phi : \mathcal{O}(B_n\mathcal{H}^{\mathrm{la}}) \cong \mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\hat{\otimes} n} \to \bigwedge^n \mathfrak{h}^\vee$$
$$f_1 \otimes \ldots \otimes f_n \mapsto df_1(e) \wedge \cdots \wedge df_n(e).$$

Theorem 1.4.2 (see Proposition 4.6.1)**.** *Let us identify $\mathfrak{h} \cong \mathcal{L}^* \otimes \mathbb{Q}_p$ so that $H^i(\mathcal{L}^*, \mathbb{Q}_p) \cong H^i(\mathfrak{h}, \mathbb{Q}_p)$. The Lazard isomorphism coincides with the map which is induced by Φ on cohomology:*

$$\Phi : H^i_{\mathrm{la}}(\mathcal{H}, \mathbb{Q}_p) \cong H^i(\mathfrak{h}, \mathbb{Q}_p).$$

Remark 1.4.3. The map used here is well-known in connection with cyclic homology and the Hochschild-Kostant-Rosenberg theorem, see [Lod] 1.3.14. It would be interesting to study this relation further.

2. Syntomic cohomology

As before let R be a discrete valuation ring finite over $\mathbb{Z}_p$ with field of fractions K and residue field $k \cong \mathbb{F}_q$. Let K_0 be the maximal absolutely unramified subfield of K and $R_0 \subset R$ its ring of integers. There are no conditions on ramification.

Let X be a smooth scheme over R. We are going to review the definition of (a certain version of) syntomic cohomology and prove properties of the syntomic Chern classes on $K_i(R)$. The construction follows Besser's $\tilde{H}^i_{\mathrm{syn}}(X, n)$ [Be] Definition 9.3. We simplify the construction through systematic use of †-spaces.

We then construct a natural map from syntomic cohomology to locally analytic de Rham cohomology. This is used to show that syntomic Chern classes (and hence étale Chern classes) factor through locally analytic group cohomology.

2.1. **Weakly formal schemes and †-spaces.** We review the properties of Große-Klönne's theory of †-spaces that we need. See [GK1], [GK2] for the complete treatment. Loosely speaking, a †-space is a rigid analytic space with structure sheaf of *overconvergent functions*. If $X^\dagger$ is a †-space, we denote $\mathcal{O}(X^\dagger)$ the ring of overconvergent functions on X.

Example 2.1.1. Let $B^\dagger$ be the "closed unit disk" over K, i.e.,

$$B(\mathbb{C}_p) = \{x \in \mathbb{C}_p \mid |x|_p \leq 1\}.$$

Then

$$\mathcal{O}(B^\dagger) = \left\{ f(t) = \sum_{n=0}^{\infty} a_n t^n \mid a_n \in K, f \text{ convergent on } |x| \leq 1+\epsilon \text{ for some } \epsilon \right\}$$

On the other hand, let $\Delta^\dagger$ be the "open unit disk" over K, i.e.,

$$\Delta^\dagger(\mathbb{C}_p) = \{x \in \mathbb{C}_p \mid |x| < 1\}.$$

Then

$$\mathcal{O}(\Delta^\dagger) = \left\{ f(t) = \sum_{n=0}^{\infty} a_n t^n \mid a_n \in K, f \text{ convergent on } |x| < 1 \right\}$$

Weakly formal schemes play the role in the theory of †-spaces which formal schemes play in the theory of rigid analytic spaces.

Let $\mathfrak{X}$ be a weakly formal R-scheme ([M] or [GK1] Kapitel 3). We denote $\mathfrak{X}^\dagger$ its *generic fibre* (loc. cit. Korollar 3.4) as †-space. Let $\mathfrak{X}_k$ be the special fibre of $\mathfrak{X}$. This is a k-scheme locally of finite type. There is a natural *specialisation map*

$$\mathrm{sp} : \mathfrak{X}^\dagger \to \mathfrak{X}_k$$

on the underlying topological spaces.

Example 2.1.2. There is a natural functor

$$\widehat{(\cdot)} : R\text{-schemes of finite type} \to \text{ weakly formal } R\text{-schemes}$$

the weak completion of the special fibre. It preserves special fibres. For an R scheme of finite type, we put

$$X^\dagger = (\widehat{X})^\dagger.$$

Note that this is *not* the †-space attached to the variety X_K if X is not proper. We have $X^\dagger(K) = X(R)$.

Definition 2.1.3. A *†-space with reduction* is a triple $\mathfrak{X} = (\mathfrak{X}^\dagger, \mathfrak{X}_k, \mathrm{sp})$ of a †-space $\mathfrak{X}^\dagger$, a k-scheme locally of finite type $\mathfrak{X}_k$ and a continuous map $\mathrm{sp} : \mathfrak{X}^\dagger \to \mathfrak{X}_k$ of the underlying topological spaces. Morphisms are defined in the obvious way. A morphism $\mathfrak{X} \to \mathfrak{Y}$ of †-spaces is called closed immersion if it is a closed immersion on both components. It is called smooth if it is smooth on both components.

As discussed before, any weakly formal scheme $\mathfrak{X}$ gives rise to a †-space with reduction. If $\mathfrak{P}$ is another weakly formal R-scheme, Y_k a k-scheme locally of finite type and $Y_k \to \mathfrak{P}_k$ a closed immersion, then the *tubular neighbourhood* $]Y_k[_{\mathfrak{P}}= \mathrm{sp}_{\mathfrak{P}}^{-1}(Y_k)$ of Y_k in $\mathfrak{P}_K$ is also a natural †-space with reduction.

Remark 2.1.4. The tubular neighbourhood $]Y_k[_{\mathfrak{P}}$ with its reduction should be induced by some weakly formal scheme, namely the weak completion of Y_k in $\mathfrak{P}$. However, the theory of †-spaces has not yet been developed up to this point.

Example 2.1.5. For $Y = \mathrm{Spec} R$, the †-space $Y^\dagger$ consists of one point. For $P = \mathbb{A}^1$, we have $P^\dagger = \{x \in \mathbb{C}_p \mid |x| \leq 1\} = B^\dagger$ (see Example 2.1.1). For an R-valued point $a : Y \to P$, we have $]Y_k[_P(\mathbb{C}_p) = \{x \in \mathbb{C}_p \mid |x - a| < 1\}$.

Smooth †-spaces have a well-behaved theory of differential forms ([GK2] 4.1). For a †-space $Y^\dagger$, let $\Omega^*_{Y^\dagger}$ be the complex of sheaves of (overconvergent) differential forms on $Y^\dagger$. For a closed immersion of †-spaces $Z^\dagger \to Y^\dagger$ with ideal of definition I, let

$$\mathrm{Fil}^n_{Z^\dagger}\Omega^*_{Y^\dagger} = I^n \to I^{n-1}\Omega^1_{Y^\dagger} \to I^{n-2}\Omega^2_{Y^\dagger} \to \dots$$

be the *Hodge filtration*. For $Z^\dagger = Y^\dagger$ this yields the stupid filtration $\Omega^{\geq n}_{Y^\dagger}$. The complexes $\mathrm{Fil}^n_{Z^\dagger}\Omega^*_{Y^\dagger}$ are functorial with respect to such pairs.

If $\mathfrak{Y} \to \mathfrak{P}$ is a closed immersion of weakly formal schemes, then $\mathfrak{Y}^\dagger \to]\mathfrak{Y}_k[_{\mathfrak{P}}$ is a closed immersion of †-spaces.

Proposition 2.1.6 (Poincaré Lemma)**.** *Let $i : \mathfrak{X} \to \mathfrak{P}$ and $i' : \mathfrak{X} \to \mathfrak{P}'$ be closed immersions of smooth weakly formal schemes. Let $u : \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism compatible with the inclusion of $\mathfrak{X}$, i.e., $u \circ i' = i$. Then*

$$\mathrm{Fil}^n_{\mathfrak{X}^\dagger}\Omega^*_{]\mathfrak{X}_k[_{\mathfrak{P}}} \to u_*\mathrm{Fil}^n_{\mathfrak{X}^\dagger}\Omega^*_{]\mathfrak{X}_k[_{\mathfrak{P}'}} = Ru_*\mathrm{Fil}^n_{\mathfrak{X}^\dagger}\Omega^*_{]\mathfrak{X}_k[_{\mathfrak{P}'}}$$

are quasi-isomorphisms of complexes of sheaves.

Proof. The cohomological assertion depends on a weak fibration formula as in rigid cohomology, [Ber] 1.3.2.

Claim: Locally $]\mathfrak{X}_k[_{\mathfrak{P}'} \cong]\mathfrak{X}_k[_{\mathfrak{P}} \times \Delta^d$ and $(i')^\dagger : \mathfrak{X}^\dagger \to]\mathfrak{X}_k[_{\mathfrak{P}} \times \Delta^d$ is of the form $(i^\dagger, 0)$.

It suffices to consider the case $\mathfrak{X}$, $\mathfrak{P}$, $\mathfrak{P}'$ affine. Then $\mathfrak{X}^\dagger$, $\mathfrak{P}^\dagger$ and $(\mathfrak{P}')^\dagger$ are affinoid. By making them small enough we can assume that the conormal bundle of $\mathfrak{X}$ in $\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{P}} \mathfrak{P}'$ is free of rank d. Let $t_1, \dots, t_d \in \mathcal{O}(\mathfrak{P}')$ be a regular sequence defining $\mathfrak{X}$ in $\mathfrak{X}'$. Then $dt_1, \dots, dt_d$ are a basis of $\Omega^1_{\mathfrak{P}'/\mathfrak{P}}$ in a neighbourhood of $\mathfrak{X}$. These sections define a morphism

$$\mathfrak{P}' \to \mathfrak{P}'' = \mathfrak{P} \times (\hat{\mathbb{A}}^1)^d$$

étale in a neighbourhood of $\mathfrak{X}$. The composition $\mathfrak{X} \to \mathfrak{P}'' \to (\hat{\mathbb{A}}^1)^d$ is given by the zero-section.

By [Ber] Proposition 1.3.1

$$]\mathfrak{X}_k[_{\mathfrak{P}'} \to]\mathfrak{X}_k[_{\mathfrak{P}''}$$

is an isomorphism for the corresponding rigid analytic varieties. By [GK2] Theorem 1.12 (a) this implies that the morphism of dagger-spaces is an isomorphism. On the other hand

$$]\mathfrak{X}_k[_{\mathfrak{P}''} \cong]\mathfrak{X}_k[_{\mathfrak{P}} \times (\mathbb{A}^{1\dagger})^d =]\mathfrak{X}_k[_{\mathfrak{P}} \times \Delta^d$$

where Δ is the open unit disc. (There is an obvious map to the right hand side which is an isomorphism of rigid analytic spaces and hence also of dagger-spaces). This proves the claim.

We now turn to the statement on differential forms. By [GK2] Satz 4.12

$$\Omega^*(]\mathfrak{X}_k[_{\mathfrak{P}} \times \Delta^d) = \Omega^*(]\mathfrak{X}_k[_{\mathfrak{P}}) \otimes \Omega^*(\Delta)^{\otimes d}$$

The filtration is compatible with this decomposition. This reduces the proof to the case $\mathfrak{P}' = \Delta$ and $\mathfrak{X} = \mathfrak{P}$ the zero-section.

As Δ is Stein, $Ru_* = u_*$. Let t be the parameter of Δ. The filtration has two steps:

$$\mathrm{Fil}^0\Omega^*(\Delta) = [\mathcal{O}(\Delta) \to \Omega^1(\Delta)] \qquad \mathrm{Fil}^1\Omega^*(\Delta) = [t\mathcal{O}(\Delta) \to \Omega^1(\Delta)]$$

The differential is an isomorphism on Fil^1, i.e. the complex is acyclic. The kernel on Fil^0 consists of constants functions, i.e. the cohomology of a single point. □

Proposition 2.1.7 (Rigid Cohomology). *Let $\mathfrak{X} \to \mathfrak{P}$ be a closed immersion of smooth weakly formal schemes. Then $H^i(]\mathfrak{X}_k[_{\mathfrak{P}}, \Omega^*)$ is naturally isomorphic to rigid cohomology of $\mathfrak{X}_k$ in the sense of Berthelot. If $\mathfrak{X}_k$ is affine, it agrees with Monsky-Washnitzer cohomology.*

Proof. This is [GK1] Proposition 8.1 (b) or [GK2] Theorem 5.1. In the affine case, rigid cohomology is known to agree with Monsky-Washnitzer cohomology. □

2.2. **Syntomic cohomology.** We define syntomic cohomology on affine †-spaces with reduction. The case of most interest is the one of the weak completion of an affine R-scheme. The restriction to the affine case is not essential. It simplifies the construction slightly because all †-spaces which occur are acyclic for cohomology of coherent sheaves.

Definition 2.2.1. A *syntomic data* for an affine †-space with reduction $\mathfrak{X}$ is a collection of

- a smooth affine weakly formal R_0-scheme $\mathfrak{P}_0$ together with a σ-linear lift Φ of absolute Frobenius on the special fibre $\mathfrak{P}_{0k}$;
- a closed immersion $\mathfrak{X}_k \to \mathfrak{P}_{0k}$;
- a smooth affine weakly formal R-scheme $\mathfrak{P}$;
- a closed immersion $\mathfrak{X} \to \mathfrak{P}$ of †-spaces with reduction and a morphism of weakly formal schemes $\mathfrak{P}_0 \to \mathfrak{P}$ such that the two maps $\mathfrak{X}_k \to \mathfrak{P}_k$ agree.

A morphism of syntomic data for $\mathfrak{X}$ is a pair of smooth morphisms $u_0 : \mathfrak{P}'_0 \to \mathfrak{P}_0$, $u : \mathfrak{P}' \to \mathfrak{P}$ such that the obvious diagrams commute.

Let $n \in \mathbb{Z}$. The *syntomic complex* $R\Gamma_{\mathrm{syn}}(\mathfrak{X}, n)_{\mathfrak{P}_0,\mathfrak{P}}$ attached to this data is defined as

$$\mathrm{Cone}\left(\mathrm{Fil}^n_{\mathfrak{X}^\dagger}\Omega^*(]\mathfrak{X}_k[_\mathfrak{P}) \oplus \Omega^*(]\mathfrak{X}_k[_{\mathfrak{P}_0}) \to \Omega^*(]\mathfrak{X}_k[_\mathfrak{P}) \oplus \Omega^*(]\mathfrak{X}_k[_{\mathfrak{P}_0})\right)[-1]$$

where the map is given by $(a, b) \mapsto (a - b, (1 - \Phi^*/p^n)b)$. Its cohomology is called *syntomic cohomology* $H^i_{\mathrm{syn}}(\mathfrak{X}, n)$ of X. If X is a smooth affine scheme, $H^i_{\mathrm{syn}}(X, n)$ is defined as syntomic cohomology of its weakly formal completion.

Remark 2.2.2. By the Poincaré Lemma 2.1.6, a morphism of syntomic data induces an isomorphism on syntomic cohomology, i.e., a quasi-isomorphism of syntomic complexes. Note, however, that the system of syntomic data is not filtering: a pair of morphism of syntomic data $\alpha, \beta : (\mathfrak{P}_0, \Phi, \mathfrak{P}) \to (\mathfrak{P}'_0, \Phi', \mathfrak{P}')$ is not equalised on a third data. To obtain a complex independent of choices (and hence a functorial theory), one has to proceed as Besser in [Be] Definition 4.11 - Definition 4.13. We do not go into the details.

Remark 2.2.3. The restriction to the affine case is not necessary. In the general case one has to replace the global sections $\Omega^*(]\mathfrak{X}_k[_\mathfrak{P})$ by global sections of a functorial injective resolution of $\Omega^*_{]\mathfrak{X}_k[_\mathfrak{P}}$. Indeed, in the affine case the complexes $\Omega^*_{]\mathfrak{X}_k[_\mathfrak{P}}$ are the ones computing Monsky-Washnitzer cohomology of $\mathfrak{X}_k$. The use of dagger-spaces could be thus be avoided for the needs of the present paper.

Example 2.2.4. Let $\mathfrak{X} = \mathrm{Sp}\widehat{R}$ the weakly formal completion of Spec$R$. Then $\mathfrak{P}_0 = \mathrm{Sp}\widehat{R}_0$ with $\Phi = \sigma$ and $\mathfrak{X} = \mathfrak{P}$ is a syntomic data. We have $]\mathfrak{X}_k[_\mathfrak{X} = \mathfrak{X}^\dagger$ (a single point) and hence the ideal of definition I vanishes. For $n > 0$, the complex $\Omega^{\geq n}(\mathfrak{X}^\dagger)$ vanishes. Moreover, $\Omega^0(\mathfrak{X}^\dagger) = K$ (constant functions on a single point). Hence the syntomic complex $R\Gamma_{\mathrm{syn}}(\mathrm{Spec}R, n)_{\mathfrak{P}_0,\mathfrak{X}}$ is simply

$$\mathrm{Cone}\left(K_0 \xrightarrow{(1,1-\sigma/p^n)} K \oplus K_0\right)[-1]$$

The $\mathbb{Q}_p$-linear map $(1 - \sigma/p^n)$ is bijective, hence

$$\eta^{-1} : K[-1] \to R\Gamma_{\mathrm{syn}}(\mathrm{Spec}R, n)_{\mathfrak{P}_0,\mathfrak{X}}$$

is a quasi-isomorphism. Hence for $n > 0$

$$H^i_{\mathrm{syn}}(\mathrm{Spec}R, n) = \begin{cases} K & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

This identification will be used very often in the sequel.

Definition 2.2.5. Let $n > 0$. We denote

$$\eta : H^1(R, n) \to K$$

the isomorphism of Example 2.2.4.

Remark 2.2.6. Let X be a smooth affine R-scheme, $a \in X(R)$, $c \in H^1_{\text{syn}}(X,n)$. Then $\eta(a^*c) \in K$. This means that c induces a (set-theoretic) map

$$c : X(R) \to K$$

We are going to show in the next section that this map is in fact locally analytic on $X(R)$ (but not necessarily rigid analytic in general).

Proposition 2.2.7. *For smooth R-schemes X, syntomic cohomology as defined above agrees with $\tilde{H}^i_{\text{syn}}(X,n)$ defined by Besser* [Be] *Definition 9.3. For $R = R_0$, it agrees with syntomic cohomology defined by Gros* [G].

Proof. Besser uses direct limits over rigid analytic functions in strict neighbourhoods of $]\mathfrak{X}_k[_{\mathfrak{P}}$ rather than †-spaces. By Proposition 2.1.7 this amounts to the same. Apart from this point, the definitions agree. The second statement is [Be] Proposition 9.4. □

Remark 2.2.8. The theory immediately extends to simplicial schemes over R, in particular to $B\mathrm{GL}_N$.

Proposition 2.2.9 ([Be] 9.10, [Ni2])**.** *Let X be a smooth affine R-scheme. Then there is a natural morphism*

$$H^i_{\text{syn}}(X,n) \to H^i_{\text{et}}(X_k, \mathbb{Q}_p(n))$$

of cohomology theories compatible with Chern classes.

Proof. Besser constructs such a transformation for his version of syntomic cohomology. (The main step in the proof is due to Niziol, see [Ni2].) In his proof the map factors by construction through his $\tilde{H}^i_{\text{syn}}(X,n)$ which is our $H^i_{\text{syn}}(X,n)$ (Proposition 2.2.7). □

2.3. **Evaluation maps and Chern classes.** Let X be a smooth affine R-scheme.

Definition 2.3.1. Let

$$X^\delta = \coprod_{a \in X(R).} \mathrm{Spec} R$$

Recall that $\mathcal{O}(X^\delta, F)$ denotes the F-valued set-theoretic functions on $X(R)$.

Lemma 2.3.2. *Let $n > 0$. The natural morphism of schemes $X^\delta \to X$ induces a natural map*

$$\mathrm{ev} : R\Gamma_{\text{syn}}(X,n) \to \mathcal{O}(X^\delta, H^1_{\text{syn}}(\mathrm{Spec}R, n))[-1] \xrightarrow{\eta} \mathcal{O}(X^\delta)[-1]$$

with η as in Definition 2.2.5.

Proof. We use the functoriality of syntomic complexes

$$R\Gamma_{\text{syn}}(X,n) \to R\Gamma_{\text{syn}}(X^\delta, n)$$

This gives the formula of the Lemma by Example 2.2.4. □

Applying this map to the simplicial scheme $B.G$ for our smooth affine group-scheme $G = \mathrm{GL}_N$, we get by definition of group cohomology a natural map

$$\mathrm{ev} : H^{2n}_{\mathrm{syn}}(B.G, n) \to H^{2n-1}(G(R), K)\ .$$

Gros (for $R = R_0$) and Besser (general case) have established the existence of Chern classes in syntomic cohomology. The key ingredient is a universal Chern class for $i \leq N$

$$c_i \in H^{2i}_{\mathrm{syn}}(B.\mathrm{GL}_N, i)$$

It is uniquely characterised by the fact that its image in $\mathrm{Fil}^i H^{2i}(B\mathrm{GL}^{\dagger}_{N,K}, \Omega^*)$ is the usual Chern class in de Rham cohomology ([Be] Proposition 7.4 and the discussion following it).

Definition 2.3.3 ([Be] Theorem 7.5)**.** The syntomic Chern class

$$c_n \in H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{syn}}(R, n))$$

is given by applying the evaluation map of Lemma 2.3.2 for $B.\mathrm{GL}_N$ to the universal Chern class.

We denote $R\Gamma_{\mathrm{et}}(K, \mathbb{Q}_p(n))$ the complex computing continuous étale cohomology of K with coefficients in $\mathbb{Q}_p(n)$. This agrees with continuous cohomology of the group $\mathrm{Gal}(\bar{K}/K)$.

Proposition 2.3.4. *As before let $G = \mathrm{GL}_N$ considered as smooth group scheme over R. For $n > 1$, there is a natural commutative diagram*

$$\begin{array}{ccc}
H^{2n}_{\mathrm{syn}}(B.G, n) & \longrightarrow & H^{2n}_{\mathrm{et}}(B.G_K, \mathbb{Q}_p(n)) \\
\downarrow & & \downarrow \\
H^{2n-1}(G(R), H^1_{\mathrm{syn}}(R, n)) & \longrightarrow & H^{2n-1}(G(R), H^1_{\mathrm{et}}(K, \mathbb{Q}_p(n))) \\
& \searrow^{\eta} & \uparrow \exp_{\mathrm{BK}} \\
& & H^{2n-1}(G(R), K)
\end{array}$$

For $n = 1$ the diagram reads

$$\begin{array}{ccccc}
H^2_{\mathrm{syn}}(B.G, 1) & \longrightarrow & & \longrightarrow & H^2_{\mathrm{et}}(B.G_K, \mathbb{Q}_p(1)) \\
\downarrow & & & & \downarrow \\
H^1(G(R), H^1_{\mathrm{syn}}(R, 1)) & \longrightarrow & H^1(G(R), H^1_{\mathrm{et}}(K, \mathbb{Q}_p(1))) & \longrightarrow & H^1(G(R), R\Gamma_{\mathrm{et}}(K, \mathbb{Q}_p(1))) \\
& \searrow^{\eta} & \uparrow \exp_{\mathrm{BK}} & & \\
& & H^1(G(R), K) & &
\end{array}$$

For $n \geq 1$ and N big enough, the universal syntomic Chern class is mapped to the universal étale Chern class.

Proof. The vertical maps are the ones from Lemma 2.3.2 and their étale analogue respectively. Note that $R\Gamma_{\mathrm{et}}(K, \mathbb{Q}_p(n))$ is concentrated in degrees $1, 2$ for $n \neq 0$ (even in degree 1 for $n \neq 0, 1$). The natural transformation of Proposition 2.2.9 gives the horizontal maps. This yields the upper commutative square. By [Be] proof of Proposition 9.9 it is compatible with Chern classes.

In [Be] Proposition 9.11 the relation to the exponential is made explicit. This gives the lower triangle. □

Remark 2.3.5. This proposition reduces the proof of our Main Theorem 1.3.2 to a statement on universal Chern classes in syntomic cohomology.

2.4. **Analyticity of evaluation maps.** We are going to show that the syntomic Chern classes c_n is an element of the continuous group cohomology of $\mathrm{GL}_N(R)$. We want to prove:

Theorem 2.4.1. *Let X be a smooth affine R-scheme, $n \geq 1$. Then the evaluation map of Lemma 2.3.2 factors naturally via a morphism in the derived category of $\mathbb{Q}_p$-vector spaces*

$$R\Gamma_{\mathrm{syn}}(X, n) \xrightarrow{\eta} \Omega^{<n}(X^{\mathrm{la}})[-1] \xrightarrow{\pi} \mathcal{O}(X^{\mathrm{la}})[-1]$$

Moreover, η is represented by a natural sequence of morphisms of complexes and formal inverses of quasi-isomorphisms of complexes. For $X = \mathrm{Spec}R$, the map agrees with the one defined previously (see Definition 2.2.5).

Definition 2.4.2. Let Y be a smooth affine R-scheme, Let $\mathring{Y}$ the †-space with reduction with special and generic fibre

$$\mathring{Y}_k = \coprod_{a \in Y(k)} \mathrm{Spec} k$$

$$\mathring{Y}^\dagger = \coprod_{a \in Y(k)}]a[_Y$$

together with the natural specialisation map.

Remark 2.4.3. Note that by definition $(\mathring{Y})^\dagger(K) = Y^\dagger(K) = Y(R)$. The locally analytic manifolds on these spaces agree.

Lemma 2.4.4. *Let $(\mathfrak{P}_0, \Phi, \mathfrak{P})$ a syntomic data for Y. Then $(\mathfrak{P}'_0, \Phi', \mathfrak{P}')$ with $\mathfrak{P}' = \coprod_{a \in Y(k)} \mathfrak{P}_0$, $\mathfrak{P}' = \coprod_{a \in Y(k)} \mathfrak{P}$ with Φ' operating as σ on $Y(k)$ and via Φ on the $\mathfrak{P}_0$ is a syntomic data for $\mathring{Y}$. Let $n > 0$. Then there is a natural isomorphism*

$$R\Gamma_{\mathrm{syn}}(\mathring{Y}, n)_{(\mathfrak{P}'_0, \mathfrak{P}')} \to \Omega^{<n}(\mathring{Y}^\dagger)[-1]$$

in the derived category of $\mathbb{Q}_p$-vector spaces. Moreover, the morphism is represented by a natural sequence of quasi-isomorphisms of complexes going either direction.

Proof. All properties of a syntomic data follow from functoriality.

By the Poincaré Lemma 2.1.6, the natural inclusion

$$\bigoplus_{a\in Y(k)} K_0 \to \Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'_0})$$

is a quasi-isomorphism. Hence

$$\mathrm{Cone}\left(\mathrm{Fil}^n_{\mathring{Y}^\dagger}\Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'}) \oplus \bigoplus_{a\in Y(k)} K_0 \to \Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'}) \oplus \bigoplus_{a\in Y(k)} K_0\right)[-1]$$

is quasi-isomorphic to $R\Gamma_{\mathrm{syn}}(\mathring{Y},n)$. The map Φ^* operates as σ on K_0 and permutes the elements of $Y(k)$. This map preserves the norm on $\bigoplus_{a\in Y(k)} K_0$. Hence the map $1-\Phi^*/p^n$ is an isomorphism for $n>0$. Hence the inclusion of

$$\mathrm{Cone}\left(\mathrm{Fil}^n_{\mathring{Y}^\dagger}\Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'}) \to \Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'})\right)[-1]$$

is a quasi-isomorphism. Now we apply the Poincaré Lemma again: $\mathrm{Fil}^n_{\mathring{Y}^\dagger}\Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'})$ is quasi-isomorphic to the subcomplex

$$\mathrm{Fil}^n_{\mathring{Y}^\dagger}\Omega^*(\mathring{Y}^\dagger) = \Omega^{\geq n}(\mathring{Y}^\dagger)$$

and $\Omega^*(]\mathring{Y}_k[_{\mathfrak{P}'})$ to $\Omega^*(\mathring{Y}^\dagger)$. Finally the cone

$$\mathrm{Cone}\left(\Omega^{\geq n}(\mathring{Y}^\dagger) \to \Omega^*(\mathring{Y}^\dagger)\right)[-1]$$

is quasi-isomorphic to the quotient. □

Proof of Theorem 2.4.1. We define η as the composition of the natural map

$$R\Gamma_{\mathrm{syn}}(Y,n) \to R\Gamma_{\mathrm{syn}}(\mathring{Y},n)$$

with the quasi-isomorphism of Lemma 2.4.4

$$R\Gamma_{\mathrm{syn}}(\mathring{Y},n) \to \Omega^{<n}(\mathring{Y}^\dagger)[-1]$$

Note finally that $\mathring{Y}^\dagger(K) = Y(R)$ as K-manifolds and that overconvergent differentials are locally analytic.

Now assume $Y = \mathrm{Spec}R$. Then $\mathring{Y} = Y$, $\mathcal{O}(\mathring{Y}^\dagger) = K$. The the chain of isomorphisms in Lemma 2.4.4 agrees with the one in Example 2.2.4.

As η is natural, this implies that the evaluation map of Lemma 2.3.2 factors through η. □

We apply the arguments to $B.G$. For later use we record a couple of commutative diagrams:

Proposition 2.4.5. *As before let $G = \mathrm{GL}_N$ as smooth affine algebraic group over R. Let $\mathcal{G}^\dagger = \mathring{G} = \coprod_{a\in G(k)}]a[_G$ as $\dagger$-space and $\mathcal{G} = \mathcal{G}^\dagger(K) = G(R)$ as* *locally analytic K-manifold. Then the following diagram commutes:*

$$\begin{array}{ccc}
H^{2n}(\Omega^{\geq n}(B.\mathcal{G}^\dagger)) & \longrightarrow & H^{2n}(\Omega^{\geq n}(B.\mathcal{G}^\dagger)) \\
\uparrow d & & \uparrow \partial \\
H^{2n}_{\mathrm{syn}}(B.G,n) & \xrightarrow{\eta} & H^{2n-1}(\Omega^{<n}(B.\mathcal{G}^\dagger)) \\
\downarrow \mathrm{ev} & & \downarrow \pi \\
H^{2n-1}(\mathcal{G},K) & \longleftarrow & H^{2n-1}_{\mathrm{la}}(\mathcal{G},K)
\end{array}$$

where d is induced from the natural map $R\Gamma_{\mathrm{syn}}(B.G,n) \to \Omega^{\geq n}(B.G^\dagger)$ (see Definition 2.2.1), ∂ is induced from the connecting map of the short exact sequence of complexes

$$0 \to \Omega^{\geq n}(B.\mathcal{G}^\dagger) \to \Omega^*(B.\mathcal{G}^\dagger) \to \Omega^{<n}(B.\mathcal{G}^\dagger) \to 0$$

ev *is the evaluation map 2.3.2, η is the map of Theorem 2.4.1 and π is induced by the projection $\Omega^{<n}(B.\mathcal{G}^\dagger) \to \mathcal{O}(B.\mathcal{G}^\dagger) \to \mathcal{O}(B.\mathcal{G}^{\mathrm{la}})$.*

For $1 \leq n$ and N big enough, we define the Chern class $c_n^{\mathrm{an}} \in H^{2n}(\Omega^{\geq n}B.\mathcal{G}^\dagger)$ as image of the Chern class in algebraic de Rham cohomology. Then the universal syntomic Chern class is mapped to c_n^{an} in the top right corner.

Proof. We apply Lemma 2.4.4 to $B.G$. This defines the map η in the middle. Recall that there are natural map $\mathcal{O}(B.\mathcal{G}^\dagger) \to \mathcal{O}(B.\mathcal{G}^{\mathrm{la}}) \to \mathcal{O}(B.\mathcal{G}^\delta)$ and that the latter two define locally analytic and discrete group cohomology. The commutativity of the lower square then follows from Theorem 2.4.1.

By definition of syntomic Chern classes they are mapped to the standard Chern classes in algebraic and hence also overconvergent de Rham cohomology, see [Be] Proposition 7.4 and the discussion following it. (Note that Besser uses a more refined version of syntomic cohomology than we do. His version of d takes values in *algebraic* de Rham cohomology.)

It remains to check commutativity of the upper square. By construction of η, we have (for Y a smooth affine R-scheme):

$$\begin{array}{ccccc}
\Omega^{\geq n}(Y^\dagger) & \longrightarrow & \Omega^{\geq n}(\mathring{Y}^\dagger) & & \\
\uparrow d & & \uparrow d & \nwarrow \partial & \\
R\Gamma_{\mathrm{syn}}(Y,n) & \longrightarrow & R\Gamma_{\mathrm{syn}}(\mathring{Y},n) & \longrightarrow & \mathrm{Cone}\left(\Omega^{\geq n}(\mathring{Y}^\dagger) \to \Omega^*(\mathring{Y}^\dagger)\right)[-1]
\end{array}$$

□

Remark 2.4.6. The above diagrams work without changes for all smooth algebraic group schemes over R.

2.5. **Analyticity of Chern classes.**

Theorem 2.5.1. *Let $n > 0$ and $N > n$. There exists an element*

$$(\eta c_n)^{\mathrm{la}} \in H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K)$$

which has the same image in $H^{2n-1}(\mathrm{GL}_N(R), K)$ *as the syntomic Chern classes*

$$c_n \in H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{syn}}(R, n))$$

(cf. Definition 2.3.3) under the map $\eta : H^1_{\mathrm{syn}}(R, n) \to K$*. In particular,* c_n *is the image of an element*

$$c_n^{\mathrm{cont}} \in H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_N(R), H^1_{\mathrm{syn}}(R, n))$$

Proof. The syntomic Chern class $c_n \in H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{syn}}(R, n))$ is by Definition 2.3.3 the image of the universal Chern class. By the lower part of the diagram in Proposition 2.4.5 it is the image of a locally analytic class. In particular it is the image of a continuous class. □

Example 2.5.2. Let $N = 1$, $n = 1$. By [G] Proposition 4.1, the first Chern class

$$c_1 : R^* \otimes \mathbb{Q}_p \to H^1_{\mathrm{syn}}(R, 1) = K$$

is given by the p-adic logarithm $\log_p$. This function is locally analytic but not rigid analytic. However, it is (overconvergent) rigid analytic on the residue discs. Our proof shows the same behaviour also for higher Chern classes.

Corollary 2.5.3. *Let* $n > 0$*. The étale Chern class*

$$c_n \in H^{2n-1}(\mathrm{GL}_N(R), H^1_{\mathrm{et}}(K, n))$$

is the image of an element

$$c_n^{\mathrm{cont}} \in H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_N(R), H^1_{\mathrm{et}}(K, n))$$

in continuous group cohomology.

Proof. Combine Theorem 2.5.1 with Proposition 2.3.4. □

Theorem 2.5.1 allows to reduce the proof of our Main Theorem 1.3.2 to continuous group cohomology.

3. The suspension map and locally analytic group cohomology

As before let $G = \mathrm{GL}_N$ as algebraic group over R, let $\mathcal{G}^\dagger = \coprod_{a \in G(k)}]a[_G$ the union of residue discs viewed as dagger-space, and $\mathcal{G} = \mathcal{G}^\dagger(K) = \mathrm{GL}_N(R)$ as K-Lie group. Let $\mathfrak{g} = \mathfrak{gl}_N$ be the K-Lie algebra of G (see Definition 0.4.1).

In the last section, we constructed an element

$$(\eta c_n)^{\mathrm{la}} \in H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) = H^{2n-1}_{\mathrm{la}}(\mathcal{G}, K)$$

(see Theorem 2.5.1). In this section we are going to define (see Definition 3.4.5) a natural map

$$\Psi : H^{2n-1}_{\mathrm{la}}(\mathcal{G}, K) \to H^{2n-1}(\mathfrak{g}, K) \ .$$

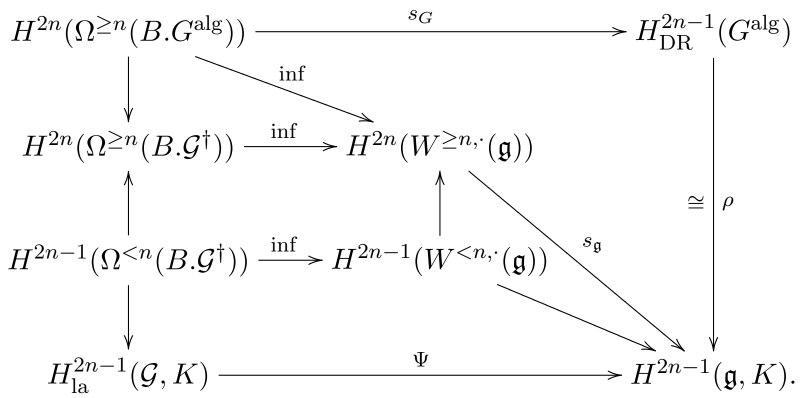
Eventually, we want to show that $\Psi((\eta c_n)^{\mathrm{la}}) = p_n$, the primitive element in Lie algebra cohomology.

The aim of this section is to embed Ψ into a huge commutative diagram relating it to the suspension for $B.G$. As the image of the universal Chern class in algebraic de Rham cohomology under this suspension map is precisely p_n, this will allow to deduce the claim (see Section 5).

We follow closely the ideas of Beilinson [Be] as outlined by Rapoport [Ra] and Burgos [Bu].

3.1. **A commutative diagram.** We state the result of this chapter.

Theorem 3.1.1. *There is a natural commutative diagram*



The suspension map s_G will be introduced in Section 3.2, ρ in Lemma 3.3.1 and the algebraic Lazard isomorphism Ψ in Definition 3.4.5. The Weil algebra $W^{*,\cdot}(\mathfrak{g})$ and the map $s_{\mathfrak{g}}$ are defined in Section 3.5. The various maps inf will be introduced in Section 3.6. Finally the proof of the Theorem will be given in Section 3.8.

Remark 3.1.2. The same arguments yield the above diagram in the case of a reductive group over K.

3.2. **The suspension map s_G.** Consider the simplicial schemes $E.G$ and $B.G$ over R. Let $G.$ be the constant simplicial scheme and $\Delta : G. \to E.G$ be the diagonal inclusion. Then we have a fibre diagram

$$\begin{array}{ccc} G. & \xrightarrow{\Delta} & E.G \\ & & \downarrow \\ & & B.G \end{array}$$

As $E.G$ is contractible the suspension for this G-torsor gives us a morphism for $n > 0$

$$s_G : H^{2n}_{\mathrm{DR}}(B.G^{\mathrm{alg}}) \to H^{2n-1}_{\mathrm{DR}}(G^{\mathrm{alg}})$$

(compare [Bu] Example 4.16 and recall that $H^{2n}_{\mathrm{DR}}(B.G^{\mathrm{alg}})$ and $H^{2n-1}_{\mathrm{DR}}(G^{\mathrm{alg}})$ are the de Rham cohomology of the generic fibre). We will use another description of the suspension map in terms of the Eilenberg-Moore spectral sequence

$$E_1^{p,q} = H^q_{\mathrm{DR}}(B_pG^{\mathrm{alg}}) \Longrightarrow H^{p+q}_{\mathrm{DR}}(B.G^{\mathrm{alg}}).$$

As $E_1^{0,q} = 0$ for $q > 0$ we get an edge morphism

$$s_G : H^{2n}_{\mathrm{DR}}(B.G^{\mathrm{alg}}) \to E_1^{1,2n-1} = H^{2n-1}_{\mathrm{DR}}(G^{\mathrm{alg}}), \tag{3}$$

which is none other than the suspension s_G. In particular, the suspension is compatible with the "Hodge filtration" (defined by $\Omega^{\geq n}(B.G^{\mathrm{alg}})$ and $\Omega^{\geq n}(G^{\mathrm{alg}})$) on both sides and we get a map

$$s_G : H^{2n}(\Omega^{\geq n}(B.G^{\mathrm{alg}})) \to H^{2n-1}(\Omega^{\geq n}(G^{\mathrm{alg}})). \tag{4}$$

3.3. **Lie algebra and de Rham cohomology.** Let $\mathfrak{g}^\vee := \mathrm{Hom}_K(\mathfrak{g}, K)$ be the dual of $\mathfrak{g}$. Let $\mathcal{C}^{\cdot}(\mathfrak{g})$ be the standard complex of Lie algebra cohomology with coefficients in K and

$$H^i(\mathfrak{g}) := H^i(\mathfrak{g}, K).$$

Let $\Omega^{\cdot}(G^{\mathrm{alg}})$ be the de Rham complex of the generic fibre $G \times_R K$ of G. Identifying $\mathfrak{g}$ with the left invariant vector fields on G, one has an embedding

$$\mathfrak{g}^\vee \subset \Omega^1(G^{\mathrm{alg}}) \tag{5}$$

of the dual of $\mathfrak{g}$ into the 1-forms on $G \times_R K$, which induces a map of complexes

$$\mathcal{C}^{\cdot}(\mathfrak{g}) \subset \Omega^{\cdot}(G^{\mathrm{alg}}).$$

It has a splitting by evaluation at the identity e.

Lemma 3.3.1 ([Ho] Lemma 4.1)**.** *The above inclusion induces an isomorphism*

$$\rho : H^{\cdot}(\mathfrak{g}) \cong H^{\cdot}_{\mathrm{DR}}(G^{\mathrm{alg}}).$$

3.4. **The infinitesimal version of $B.G$ and Lie algebra cohomology.** Recall that $\mathcal{O}(B.G^{\mathrm{alg}})$ is the cosimplicial ring of K-valued algebraic functions on $B.G$. If we extend Spec in the obvious way to cosimplicial rings, we have

$$\mathrm{Spec}\mathcal{O}(B.G^{\mathrm{alg}}) = B.G \times_R K.$$

We need an infinitesimal version of $B.G$. For this let $\mathfrak{m}_e^{\mathrm{alg}} \subset \mathcal{O}(B_1 G^{\mathrm{alg}}) = \mathcal{O}(G^{\mathrm{alg}})$ be the kernel of the augmentation map defined by the unit element $e \in G(R)$. Let $J^{\cdot} \subset \mathcal{O}(B.G^{\mathrm{alg}})$ be the cosimplicial ideal generated by $(\mathfrak{m}_e^{\mathrm{alg}})^2$.

Definition 3.4.1. Let $\mathcal{O}(B^1_{\cdot} G^{\mathrm{alg}}) := \mathcal{O}(B.G^{\mathrm{alg}})/J^{\cdot}$. The *infinitesimal version of $B.G$* is the simplicial scheme

$$B^1_{\cdot} G := \mathrm{Spec}\mathcal{O}(B^1_{\cdot} G^{\mathrm{alg}}).$$

The closed immersion $B^1_{\cdot} G \subset B.G$ is defined by the canonical map

$$\mathcal{O}(B.G^{\mathrm{alg}}) \to \mathcal{O}(B^1_{\cdot} G^{\mathrm{alg}}).$$

It is central for our arguments that this has a counterpart for (overconvergent) rigid analytic functions.

Lemma 3.4.2. *The natural map of cosimplicial rings*

$$\inf : \mathcal{O}(B.G^{\mathrm{alg}}) \to \mathcal{O}(B^1_{.}G^{\mathrm{alg}})$$

factors naturally via

$$\inf : \mathcal{O}(B.\mathcal{G}^{\dagger}) \to \mathcal{O}(B^1_{.}G^{\mathrm{alg}}) \tag{6}$$

Proof. First consider more generally a smooth R-scheme X and $x \in X(R)$. Let J be an ideal of $\mathcal{O}_{X^{\mathrm{alg}},x}$ and put $J^{\dagger} = J\mathcal{O}_{X^{\dagger},x}$. For $J = \mathfrak{m}_x$ the maximal ideal, $\mathfrak{m}_x^{\dagger}$ is indeed the maximal ideal of $\mathcal{O}_{X^{\dagger},x}$. Hence for J containing a power of $\mathfrak{m}_x$,

$$\mathcal{O}_{X^{\dagger},x}/J^{\dagger} \cong \mathcal{O}_{X^{\mathrm{alg}},x}/J$$

(Note that stalks of the structure sheaf of $X^{\dagger}$ as dagger-space and the corresponding rigid analytic variety agree.) All components of the simplicial scheme $B^1_{.}G$ are of this form. This yields the claim. □

To formulate the next two propositions, we need the concept of normalisation for cosimplicial rings.

Definition 3.4.3. For any cosimplicial object $A^{\cdot}$ in an abelian category let $\mathcal{C}A^{\cdot}$ be the complex with $\mathcal{C}A^n = A^n$ and differential $d = \sum_{i=0}^{n+1}(-1)^i\delta_i$. The *normalisation* is the subcomplex

$$\mathcal{N}A^{\cdot} = \bigcap_{i=0}^{n-1} \ker \sigma^i.$$

Proposition 3.4.4. *There is a natural isomorphism of complexes*

$$\mathcal{N}\mathcal{O}(B^1_{.}G^{\mathrm{alg}}) \cong \mathcal{C}^{\cdot}(\mathfrak{g}).$$

Proof. This is [Ra] Lemma 3.1 or [Bu] Proposition 8.9. The first reference works over $\mathbb{R}$ and the second over $\mathbb{C}$. But by inspection both proofs work without any changes over an arbitrary field of characteristic 0. □

Definition 3.4.5. Let $\mathbb{G} = G^{\mathrm{la}}, G^{\mathrm{alg}}$ or $\mathcal{G}^{\dagger}$. Define a map of complexes

$$\Psi : \mathcal{O}(B.\mathbb{G}) \to \mathcal{C}^{\cdot}(\mathfrak{g})$$

as the composition $\pi^* : \mathcal{O}(B_n\mathbb{G}) \to \mathcal{O}(E_n\mathbb{G}) \cong \mathcal{O}(\mathbb{G})^{\hat{\otimes}(n+1)}$ with the map

$$\begin{aligned} \mathcal{O}(\mathbb{G})^{\hat{\otimes}(n+1)} &\to \bigwedge^n \mathfrak{g}^{\vee} \\ f_0 \otimes \ldots \otimes f_n &\mapsto f_0(e)df_1(e) \wedge \cdots \wedge df_n(e) \end{aligned}$$

Here $\pi : E_n\mathbb{G} \to B_n\mathbb{G}$ is the map $(g_0, \ldots, g_n) \mapsto (g_0g_1^{-1}, \ldots, g_{n-1}g_n^{-1})$ The induced map on cohomology is called *algebraic Lazard morphism.*

Lemma 3.4.6. *The morphism* Ψ *agrees with the natural morphism of complexes*

$$\mathcal{N}\mathcal{O}(B.\mathcal{G}^{\dagger}) \to \mathcal{N}\mathcal{O}(B^1_{.}G^{\mathrm{alg}}) \cong \mathcal{C}^{\cdot}(\mathfrak{g})$$

induced by inf*, where the isomorphism is the one from Proposition 3.4.4.*

Proof. This follows from Burgos ([Bu] Theorem 8.4. and Proposition 8.9). Burgos first defines a map

$$\begin{aligned} \mathcal{O}(E_n G^{\mathrm{alg}}) &\to \Omega^n_{G/K} \\ f_0 \otimes \cdots \otimes f_n &\mapsto f_0 df_1 \wedge \cdots \wedge df_n, \end{aligned}$$

which factors over $\mathcal{O}(E^1_\cdot G^{\mathrm{alg}})$ and induces an isomorphism $\mathcal{N}\mathcal{O}(E^1_\cdot G^{\mathrm{alg}}) \cong \Omega^\cdot_{G/K}$, which is compatible with the G-action. As $\mathcal{C}^n(\mathfrak{g}) \subset \Omega^n_{G/K}$ are the G-invariant differential forms the above map factors naturally as

$$\begin{array}{ccc} \mathcal{O}(E_n G^{\mathrm{alg}}) & \longrightarrow & \Omega^n_{G/K} \\ {\scriptstyle \pi^*}\uparrow & & \uparrow \\ \mathcal{O}(B_n G^{\mathrm{alg}}) & \xrightarrow{\Psi} & \mathcal{C}^n(\mathfrak{g}) \end{array}$$

Evaluation at e is a splitting of the right vertical map. As $\Psi : \mathcal{O}(B_n G^{\mathrm{alg}}) \to \mathcal{C}^n(\mathfrak{g})$ factors through $\mathcal{O}(B_n \mathcal{G}^\dagger)$ and $\mathcal{N}\mathcal{O}(E^1_\cdot G^{\mathrm{alg}})^G = \mathcal{N}\mathcal{O}(B^1_\cdot G^{\mathrm{alg}})$ by [Bu] Proposition 8.9, we get the desired result. □

3.5. **The Weil algebra and the infinitesimal suspension map $s_{\mathfrak{g}}$.** In this section we will define the bigraded Weil algebra $W^{*,\cdot}(\mathfrak{g})$ and the suspension on the level of the Lie algebra $\mathfrak{g}$.

Definition 3.5.1. The *Weil algebra* is the bigraded algebra with

$$W^{p,q}(\mathfrak{g}) := \mathrm{Sym}^p \mathfrak{g}^\vee \otimes \Lambda^{q-p} \mathfrak{g}^\vee$$

and $W^{p,q}(\mathfrak{g})$ has total degree $p+q$ (this means that $\mathrm{Sym}^p \mathfrak{g}^\vee$ has degree $2p$). Write $W^n(\mathfrak{g}) := \bigoplus_{p+q=n} W^{p,q}(\mathfrak{g})$.

The Weil algebra has a differential $\delta : W^n(\mathfrak{g}) \to W^{n+1}(\mathfrak{g})$, which is uniquely determined on $\mathrm{Sym}^1 \mathfrak{g}^\vee$ and $\Lambda^1 \mathfrak{g}^\vee$ as follows: let $X_1, \ldots, X_k$ be a basis of $\mathfrak{g}$ and $X_1^\vee, \ldots, X_k^\vee$ be the dual basis and $h : \mathrm{Sym}^1 \mathfrak{g}^\vee \to \Lambda^1 \mathfrak{g}^\vee$ the identity map, then

$$\begin{aligned} \delta : \Lambda^1 \mathfrak{g}^\vee &\to \mathrm{Sym}^1 \mathfrak{g}^\vee \oplus \Lambda^2 \mathfrak{g}^\vee \\ X^\vee &\mapsto h(X^\vee) + dX^\vee, \end{aligned} \tag{7}$$

where $d : \mathcal{C}^\cdot(\mathfrak{g}) \to \mathcal{C}^{\cdot+1}(\mathfrak{g})$ is the differential in the Lie algebra complex and

$$\begin{aligned} \delta : \mathrm{Sym}^1 \mathfrak{g}^\vee &\to \mathrm{Sym}^1 \mathfrak{g}^\vee \otimes \Lambda^1 \mathfrak{g}^\vee \\ X^\vee &\mapsto \sum_{i=1}^k \theta(X_i) X^\vee \otimes h(X_i^\vee). \end{aligned} \tag{8}$$

Here $\theta(X_i)X^\vee(Y) := X^\vee([Y, X_i])$. It is clear from this definition that

$$W^{\geq n,\cdot}(\mathfrak{g}) := \bigoplus_{p \geq n} W^{p,\cdot}(\mathfrak{g})$$

is a subcomplex of $W^{*,\cdot}(\mathfrak{g})$. On the other hand,

$$W^{<n,\cdot}(\mathfrak{g}) := \bigoplus_{p<n} W^{p,\cdot}(\mathfrak{g})$$

is a quotient of $W^{*,\cdot}(\mathfrak{g})$ and canonically isomorphic to $W^{*,\cdot}(\mathfrak{g})/W^{\geq n,\cdot}(\mathfrak{g})$. For $n = 1$ we have

$$W^{<1,\cdot}(\mathfrak{g}) = \mathcal{C}^{\cdot}(\mathfrak{g}).$$

The following lemma is classical (see Cartan [Ca] or [Ra] Lemma 2.10. for a proof).

Lemma 3.5.2. *One has:*

a) $H^0(W^{*,\cdot}) = K$
b) $H^n(W^{*,\cdot}) = 0$ *for* $n > 0$
c) $H^{2n}(W^{\geq n,\cdot}) = (\mathrm{Sym}^n \mathfrak{g}^\vee)^{\mathfrak{g}}$

Consider the exact sequence

$$0 \to W^{\geq 1,\cdot}(\mathfrak{g}) \to W^{*,\cdot}(\mathfrak{g}) \to \mathcal{C}^{\cdot}(\mathfrak{g}) \to 0.$$

This induces a connecting homomorphism

$$\partial : H^{2n-1}(\mathfrak{g}) \to H^{2n}(W^{\geq 1,\cdot}(\mathfrak{g})),$$

which is an isomorphism for $n > 0$ by Lemma 3.5.2.

Definition 3.5.3. The *suspension* morphism for $\mathfrak{g}$ is the composition

$$s_{\mathfrak{g}} : H^{2n}(W^{\geq n,\cdot}(\mathfrak{g})) \to H^{2n}(W^{\geq 1,\cdot}(\mathfrak{g})) \xrightarrow{\partial^{-1}} H^{2n-1}(\mathfrak{g}).$$

The suspension $s_{\mathfrak{g}}$ has a different description:

Lemma 3.5.4. *The following diagram commutes:*

$$\begin{array}{ccc} H^{2n}(W^{\geq n,\cdot}(\mathfrak{g})) & & \\ \uparrow \partial & \searrow s_{\mathfrak{g}} & \\ H^{2n-1}(W^{<n,\cdot}(\mathfrak{g})) & \longrightarrow & H^{2n-1}(\mathfrak{g}). \end{array}$$

Here the horizontal map is induced from the canonical projection $W^{<n,\cdot}(\mathfrak{g}) \to \mathcal{C}^{\cdot}(\mathfrak{g})$.

Proof. This is just the compatibility of the two coboundary maps for the two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^{\geq 1,\cdot}(\mathfrak{g}) & \longrightarrow & W^{*,\cdot}(\mathfrak{g}) & \longrightarrow & \mathcal{C}^{\cdot}(\mathfrak{g}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow = & & \uparrow & & \\ 0 & \longrightarrow & W^{\geq n,\cdot}(\mathfrak{g}) & \longrightarrow & W^{*,\cdot}(\mathfrak{g}) & \longrightarrow & W^{<n,\cdot}(\mathfrak{g}) & \longrightarrow & 0, \end{array}$$

where the right vertical map is the canonical projection. □

3.6. **The cosimplicial de Rham complex and the Weil algebra.** In this section we formulate the extension of Proposition 3.4.4 to the de Rham complex and the Weil algebra.

Proposition 3.6.1 ([Bu] Proposition 8.10.)**.** *There is a natural bigraded isomorphism*

$$\mathcal{N}\Omega^*(B^1_\cdot G^{\mathrm{alg}}) \cong W^{*,\cdot}(\mathfrak{g}).$$

The bigrading gives:

Corollary 3.6.2. *There are natural isomorphisms*

$$\mathcal{N}\Omega^{\geq n}(B^1_\cdot G^{\mathrm{alg}}) \cong W^{\geq n,\cdot}(\mathfrak{g}),$$
$$\mathcal{N}\Omega^{< n}(B^1_\cdot G^{\mathrm{alg}}) \cong W^{< n,\cdot}(\mathfrak{g}).$$

Definition 3.6.3. We also denote by inf the map of complexes induced by $\inf : \mathcal{O}(B.G^{\mathrm{alg}}) \to \mathcal{O}(B^1_\cdot G^{\mathrm{alg}})$

$$\mathcal{N}\Omega^*(B.G^{\mathrm{alg}}) \to \mathcal{N}\Omega^*(B^1_\cdot G^{\mathrm{alg}}) \cong W^{*,\cdot}(\mathfrak{g})$$

(resp. for $\geq n$ and $< n$) and the isomorphisms of Corollary 3.6.2.

Lemma 3.6.4. *The map* inf *factors through $\mathcal{N}\Omega^*(B.\mathcal{G}^\dagger)$ and the diagram*

$$\begin{array}{ccc} H^{2n}(\Omega^{\geq n}(B.\mathcal{G}^\dagger)) & \xrightarrow{\inf} & H^{2n}(W^{\geq n,\cdot}(\mathfrak{g})) \\ \partial\uparrow & & \uparrow\partial \\ H^{2n-1}(\Omega^{< n}(B.\mathcal{G}^\dagger)) & \xrightarrow{\inf} & H^{2n-1}(W^{< n,\cdot}(\mathfrak{g})) \end{array}$$

commutes, where the vertical maps are the boundary maps for the obvious exact sequences.

Proof. As $\inf : \mathcal{O}(B.G^{\mathrm{alg}}) \to \mathcal{O}(B^1_\cdot G^{\mathrm{alg}})$ factors through $\mathcal{O}(B.\mathcal{G}^\dagger)$ the first statement is clear. The morphism of short exact sequences

$$\begin{array}{ccccc} \mathcal{N}\Omega^{\geq n}(B.\mathcal{G}^\dagger) & \longrightarrow & \mathcal{N}\Omega^*(B.\mathcal{G}^\dagger) & \longrightarrow & \mathcal{N}\Omega^{< n}(B.\mathcal{G}^\dagger) \\ \inf\downarrow & & \inf\downarrow & & \inf\downarrow \\ W^{\geq n,\cdot}(\mathfrak{g}) & \longrightarrow & W^{*,\cdot}(\mathfrak{g}) & \longrightarrow & W^{< n,\cdot}(\mathfrak{g}). \end{array}$$

induces natural boundary maps. □

3.7. **Comparison of the suspension maps.** Now we can state the relation between the suspension s_G for $B.G$ and its infinitesimal version $s_\mathfrak{g}$ on the Weil algebra. The proof is taken from Burgos [Bu].

Consider s_G as the composition

$$s_G : H^{2n}(\Omega^{\geq n}(B.G^{\mathrm{alg}})) \to H^{2n-1}(\Omega^{\geq n}(G^{\mathrm{alg}})) \to H^{2n-1}_{\mathrm{DR}}(G^{\mathrm{alg}})$$

as in Section 3.2.

Proposition 3.7.1. *There is a commutative diagram*

$$\begin{array}{ccc} H^{2n}(\Omega^{\geq n}(B.G^{\mathrm{alg}})) & \xrightarrow{s_G} & H^{2n-1}_{\mathrm{DR}}(G^{\mathrm{alg}}) \\ {\scriptstyle \mathrm{inf}}\downarrow & & \uparrow{\scriptstyle \rho} \\ H^{2n}(W^{\geq n,\cdot}(\mathfrak{g})) & \xrightarrow{s_{\mathfrak{g}}} & H^{2n-1}(\mathfrak{g}) \end{array}$$

where ρ is the isomorphism of Lemma 3.3.1.

Proof. Consider the map of complexes

$$\mathcal{N}\Omega^{\geq n}(B.G^{\mathrm{alg}}) \to \mathcal{N}\Omega^{\geq n}(B^1_{\cdot}G^{\mathrm{alg}}) \cong W^{\geq n,\cdot}(\mathfrak{g})$$

defined in Definition 3.6.3. According to [Bu] Theorem 8.12 inf induces a map

$$\omega^{-1}_{E.G} : H^{2n}(\Omega^{\geq n}(B.G^{\mathrm{alg}})) \to H^{2n}(W^{\geq n,\cdot}(\mathfrak{g})),$$

which is an algebraic description of the inverse of the Chern-Weil homomorphism. Proposition 5.33 in [Bu] says that s_G can be factored

$$H^{2n}(\Omega^{\geq n}(B.G^{\mathrm{alg}})) \xrightarrow{\omega^{-1}_{E.G}} H^{2n}(W^{\geq n,\cdot}(\mathfrak{g})) \xrightarrow{s_{\mathfrak{g}}} H^{2n-1}(\mathfrak{g}).$$

The proof in loc. cit. is only over $\mathbb{C}$, but works without any changes over an arbitrary field of characteristic zero. This gives the desired commutativity. □

3.8. **Proof of Theorem 3.1.1.** We need one more lemma.

Lemma 3.8.1. *The diagram*

$$\begin{array}{ccc} H^{2n-1}(\Omega^{<n}(B.\mathcal{G}^{\dagger})) & \xrightarrow{\mathrm{inf}} & H^{2n-1}(W^{<n,\cdot}(\mathfrak{g})) \\ \downarrow & & \downarrow \\ H^{2n-1}_{\mathrm{la}}(\mathcal{G}, K) & \xrightarrow{\Psi} & H^{2n-1}(\mathfrak{g}) \end{array}$$

commutes, where Ψ is the map defined in Definition 3.4.5.

Proof. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{N}\Omega^{<n}(B.\mathcal{G}^{\dagger}) & \xrightarrow{\mathrm{inf}} & W^{<n,\cdot}(\mathfrak{g}) \\ \downarrow & & \downarrow \\ \mathcal{N}\Omega^{<1}(B.\mathcal{G}^{\dagger}) & \xrightarrow{\mathrm{inf}} & W^{<1,\cdot}(\mathfrak{g}) \\ \downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{O}(B.\mathcal{G}^{\dagger}) & \xrightarrow{\mathrm{inf}} & \mathcal{C}^{\cdot}(\mathfrak{g}). \end{array}$$

By Lemma 3.4.6 inf agrees with Ψ. Finally Ψ factors naturally through $\mathcal{O}(B.\mathcal{G}^{\mathrm{la}})$. □

Proof of Theorem 3.1.1. We only have to combine the commutative diagrams that we have established: The statements on inf were shown in Lemma 3.6.4. The diagram for s_G is Lemma 3.7.1. The small triangle was considered in Lemma 3.5.4. Finally the diagram for Ψ is the above Lemma 3.8.1. □

4. The identification of Φ with the Lazard Isomorphism

In this section we work with a smooth algebraic group $H/\mathbb{Z}_p$ and a p-saturated group of finite rank $\mathcal{H}$ (see [L] III Definition 2.1.3 and 2.1.6) with valuation ω which is an open subgroup of $H(\mathbb{Z}_p)$. The group $\mathcal{H}$ will always be considered as a $\mathbb{Q}_p$-analytic manifold. Our main example is $H = \mathrm{GL}_N$, $\mathcal{H} = 1 + pM_N(\mathbb{Z}_p)$ with ω the inf-valuation.

4.1. **The Lazard Lie algebra.** Let $\mathrm{Al}\mathcal{H}$ be the completed group ring of $\mathcal{H}$ over $\mathbb{Z}_p$ (as defined in [L] II 2.2.1.). Note that by [L] III. 3.3.2.1 $\mathcal{H}$ is an analytic Taylor manifold, in particular, that $\mathcal{H}$ can be identified with $\mathbb{Z}_p^r$ as an analytic manifold. Fix

$$\phi : \mathcal{H} \to \mathbb{Z}_p^r \text{ with } \phi(e) = 0$$

such an identification.

If we are only interested in the structure of $\mathrm{Al}\mathcal{H}$ as a $\mathbb{Z}_p$-module, we can identify $\mathcal{H}$ with $\mathbb{Z}_p^r$ via ϕ and obtain as in [L] III 3.3.5 a topological basis $(z^\alpha)_{\alpha\in J}$, where $J := \mathbb{N}^r$ (see [L] III 2.3.8 and III 2.3.11.3). Here $z_i := x_i - 1$ for an ordered basis $x_1, \dots, x_r$ of $\mathcal{H}$ and $z^\alpha := \prod_{i=1}^r z_i^{\alpha_i}$. The valuation of z^α is

$$w(z^\alpha) := \sum_{i=1}^r \alpha_i \omega(x_i).$$

This means that every element $x \in \mathrm{Al}\mathcal{H}$ can be written in the form

$$x = \sum_{\alpha\in J} \lambda_\alpha z^\alpha$$

with $\lambda_\alpha \in \mathbb{Z}_p$ and the valuation is defined by

$$w(\sum_{\alpha\in J} \lambda_\alpha z^\alpha) := \inf_{\alpha\in J}\{v(\lambda_\alpha) + w(z^\alpha)\}$$

(see [L] I 2.1.17). The map $\mathcal{H} \to \mathrm{Al}\mathcal{H}$ is explicitly given by

$$\prod_{i=1}^r x_i^{\lambda_i} = \sum_{\alpha\in J} \binom{\lambda}{\alpha} z^\alpha.$$

The saturation of $\mathrm{Al}\mathcal{H}$ contains by definition the elements μz^α, $\mu \in \mathbb{Q}_p$ with $w(\mu z^\alpha) \geq 0$, i.e., with $v(\mu) + \sum_{i=1}^r \alpha_i\omega(x_i) \geq 0$. As $\alpha_i\omega(x_i) > \frac{\alpha_i}{p-1}$ (see [L] III 2.2.7.1) and $v(\alpha_i!) \leq \frac{\alpha_i}{p-1}$ we see that

$$e_\alpha := \frac{z^\alpha}{\alpha!} \in \mathrm{Sat}\mathrm{Al}\mathcal{H}.$$

The saturated group algebra $\mathrm{SatAl}\mathcal{H}$ (see [L] I 2.2.11) has also a structure of a valued, diagonal $\mathbb{Z}_p$-algebra, i.e., one has a valued $\mathbb{Z}_p$-algebra morphism

$$\Delta : \mathrm{SatAl}\mathcal{H} \to \mathrm{SatAl}\mathcal{H} \otimes_{\mathbb{Z}_p} \mathrm{SatAl}\mathcal{H}$$

of supplemented algebras. This is defined using the diagonal map $\mathcal{H} \to \mathcal{H} \times \mathcal{H}$ and [L] I 3.2.8, II 2.2.8.

Definition 4.1.1 ([L] IV 1.3.1)**.** The *Lazard Lie algebra* $\mathcal{L}^*$ of $\mathcal{H}$ is defined to be

$$\mathcal{L}^* := \mathcal{L}^*\mathrm{SatAl}\mathcal{H} = \{x \in \mathrm{SatAl}\mathcal{H} | \Delta(x) = x \otimes 1 + 1 \otimes x \text{ and } w(x) > \frac{1}{p-1}\}.$$

We would like to make this more explicit. Let us first define special elements in $\mathrm{SatAl}\mathcal{H}$:

Definition 4.1.2. Let $\partial_i \in \mathrm{SatAl}\mathcal{H}$ be the element

$$\partial_i := \sum_{\alpha_i > 0} \frac{(-1)^{\alpha_i - 1} z_i^{\alpha_i}}{\alpha_i} = \log(z_i + 1) = \log(x_i).$$

Lemma 4.1.3. *The elements ∂_i are primitive in* $\mathrm{SatAl}\mathcal{H}$*, i.e.,*

$$\Delta(\partial_i) = \partial_i \otimes 1 + 1 \otimes \partial_i.$$

Moreover, they give a basis of $\mathcal{L}^*$.

Proof. This is just lemma IV 3.3.6 in [L] as $z_i + 1 = x_i$. □

Corollary 4.1.4. *There is a morphism*

$$\mathcal{U}(\mathcal{L}^*) \to \mathrm{SatAl}\mathcal{H}.$$

Proof. Clear from the universal property of $\mathcal{U}(\mathcal{L}^*)$. □

4.2. **Distributions of locally analytic functions.** We show that the space of distributions of locally analytic functions $\mathcal{D}_{\mathrm{cont}}(\mathcal{H})$ is a subspace of $\mathrm{SatAl}\mathcal{H}$. It turns out that the elements $\partial_i \in \mathrm{SatAl}\mathcal{H}$ can also be viewed as distributions of locally analytic functions.

Recall that a continuous function defined by the Mahler series

$$f(\lambda) = \sum_{\alpha \in J} c_\alpha \binom{\lambda}{\alpha} \text{ with } c_\alpha \in \mathbb{Z}_p$$

is locally analytic if $\liminf_{|\alpha| \to \infty} \frac{v(c_\alpha)}{|\alpha|} > 0$ (see [L] III 1.3.9.2).

Recall that $\mathcal{O}(\mathcal{H}^{\mathrm{la}}) \cong \mathcal{O}((\mathbb{Z}_p^r)^{\mathrm{la}}) = \cup_{h>0} LA^h(\mathbb{Z}_p^r, \mathbb{Q}_p)$, where $LA^h(\mathbb{Z}_p^r, \mathbb{Q}_p)$ are the locally analytic functions of order h on $\mathbb{Z}_p^r$ with values in $\mathbb{Q}_p$ (see [L] III 1.3.7). Each $LA^h(\mathbb{Z}_p^r, \mathbb{Q}_p)$ is a p-adic Banach space with norm v_{LA^h} (see [Co] 1.4.2 for the definition) and $\mathcal{O}(\mathcal{H}^{\mathrm{la}})$ gets the inverse limit topology. We define

$$\mathcal{D}_{\mathrm{cont}}(\mathcal{H}) := \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}(\mathcal{H}^{\mathrm{la}}), \mathbb{Q}_p).$$

Amice shows in the case $r = 1$, which extends immediately to $r \geq 1$, the following proposition:

Proposition 4.2.1 (Amice, cf. [Co] Théorème 2.3). *The ring of distributions* $\mathcal{D}_{\mathrm{cont}}(\mathcal{H})$ *is identified via the* Amice or Fourier transformation

$$\mathcal{A} : \mathcal{D}_{\mathrm{cont}}(\mathcal{H}) \to \mathbb{Q}_p[[T_1, \ldots, T_r]]$$
$$\mu \mapsto \sum_{\alpha} \mu(\binom{\lambda}{\alpha})T^{\alpha}$$

with the ring of power series $G(T) = \sum_{\alpha} \lambda_{\alpha} T^{\alpha}$, *with* $\lambda_{\alpha} \in \mathbb{Q}_p$, *which converge for* $v(T) > 0$ *(i.e.,* $G(T)$ *defines an analytic function on the open* r*-dimensional unit ball).*

Corollary 4.2.2. *There is an injection*

$$\mathcal{D}_{\mathrm{cont}}(\mathcal{H}) \to \mathrm{SatAl}\mathcal{H} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
$$\mu \mapsto \sum_{\alpha} \mu\left(\binom{\lambda}{\alpha}\right)z^{\alpha}.$$

An element $D := \sum_{\alpha} \rho_{\alpha} z^{\alpha} \in \mathrm{SatAl}\mathcal{H}$ *is an element in* $\mathcal{D}_{\mathrm{cont}}(\mathcal{H})$ *if and only if the power series* $G_D(T) := \sum_{\alpha} \rho_{\alpha} T^{\alpha}$ *converges for* $v(T) > 0$. *Explicitly, for* $f(\lambda) = \sum_{\alpha \in J} c_{\alpha} \binom{\lambda}{\alpha} \in \mathcal{O}(\mathcal{H}^{\mathrm{la}})$ *one has*

$$Df = \sum_{\alpha} c_{\alpha} \rho_{\alpha}.$$

Proof. Using Proposition 4.2.1 we see that $\sum_{\alpha} \mu(\binom{\lambda}{\alpha})z^{\alpha}$ converges in $\mathrm{SatAl}\mathcal{H}$ as $w(z_i) = \omega(x_i) > \frac{1}{p-1}$ and that the map is injective. The rest of the corollary is just a restatement of Proposition 4.2.1. □

As $\partial_i = \sum_{\alpha_i > 0} \frac{(-1)^{\alpha_i - 1}}{\alpha_i} z_i^{\alpha_i}$ we get:

Corollary 4.2.3. *The elements* ∂_i *are contained in* $\mathcal{D}_{\mathrm{cont}}(\mathcal{H})$ *and the inclusion* $\mathcal{L}^* \subset \mathcal{D}_{\mathrm{cont}}(\mathcal{H})$ *defines a ring homomorphism*

$$\mathcal{U}(\mathcal{L}^*) \to \mathcal{D}_{\mathrm{cont}}(\mathcal{H}).$$

4.3. **Identification of the algebraic Lie algebra $\mathfrak{h}$ with the Lazard Lie algebra $\mathcal{L}^*$.** Recall that the algebraic $\mathbb{Q}_p$-Lie algebra of a smooth linear algebraic group scheme $H/\mathbb{Z}_p$ is defined in Definition 0.4.1 as the derivations

$$\mathfrak{h} := \mathrm{Der}_{\mathbb{Z}_p}(\mathcal{O}(H^{\mathrm{alg}})_e, \mathbb{Q}_p).$$

Inside $H(\mathbb{Z}_p)$ we have the open, p-saturated subgroup $\mathcal{H}$ considered as $\mathbb{Q}_p$-analytic manifold. Using the inclusion $\mathcal{O}(H^{\mathrm{alg}})_e \subset \mathcal{O}(\mathcal{H}^{\mathrm{la}})_e \subset \widehat{\mathcal{O}(H^{\mathrm{alg}})}_e$ (completion w.r.t. the maximal ideal), we can compare this with the Lazard Lie algebra $\mathcal{L}^*$.

Proposition 4.3.1. *The inclusion* $\mathcal{O}(H^{\mathrm{alg}})_e \subset \mathcal{O}(\mathcal{H}^{\mathrm{la}})_e$ *induces an isomorphism*

$$\mathfrak{h} \cong \mathcal{L}^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Moreover, let $f(\lambda)=\sum_{\alpha\in J}c_\alpha\binom{\lambda}{\alpha}$ *be a locally analytic function and* $\partial_i\in\mathcal{L}^*$*, then*

$$\partial_i f=\frac{\partial f}{\partial\lambda_i}(e).$$

Proof. We compute

$$\frac{\partial f}{\partial\lambda_i}=\sum_{\alpha\in J}c_\alpha\frac{\partial}{\partial\lambda_i}\binom{\lambda}{\alpha}$$

and

$$\frac{\partial}{\partial\lambda_i}\binom{\lambda}{\alpha}|_{\lambda=0}=\begin{cases}\frac{(-1)^{\alpha_i-1}}{\alpha_i} & \text{if } \alpha=(0,\cdots,\alpha_i,\cdots,0)\\ 0 \text{ else}\end{cases}.$$

Thus,

$$\frac{\partial f}{\partial\lambda_i}|_{\lambda=0}=\sum_{\alpha_i>0}\frac{(-1)^{\alpha_i-1}c_{\alpha_i}}{\alpha_i}=\partial_i f,$$

which gives the desired result. To prove that $\mathfrak{h}\cong\mathcal{L}^*\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, first observe that the maximal ideal of $\mathcal{O}(\mathcal{H}^{\mathrm{la}})_e$ is $\mathfrak{m}_e\mathcal{O}(G^{\mathrm{alg}})_e$ and that the ∂_i by the above calculation form a basis of this ideal. As $\mathcal{L}^*$ is generated by the ∂_i we get that $\mathcal{L}^*\otimes_R K\cong\mathfrak{m}_e\mathcal{O}(G^{\mathrm{alg}})_e/\mathfrak{m}_e^2\mathcal{O}(G^{\mathrm{alg}})_e$. As $\mathfrak{h}=\mathfrak{m}_e/\mathfrak{m}_e^2$ this proves that $\mathfrak{h}\cong\mathcal{L}^*\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. □

4.4. **Standard complexes for group and Lie algebra cohomology.** For any augmented $\mathbb{Z}_p$-algebra A with augmentation $\epsilon: A\to\mathbb{Z}_p$ we consider two *standard complexes* $(T.A,d)$ and $(\widetilde{T}.A,\widetilde{d})$ with

$$T_nA=\widetilde{T}_nA=A^{\otimes n+1},$$

where for simplicity

$$A^{\otimes n+1}=\begin{cases}A\otimes_{\mathbb{Z}_p}\ldots\otimes_{\mathbb{Z}_p}A & n+1\text{-times; if } A \text{ has no topology}\\ A\widehat{\otimes}_{\mathbb{Z}_p}\ldots\widehat{\otimes}_{\mathbb{Z}_p}A & n+1\text{-times; if } A \text{ complete.}\end{cases}$$

and differentials

$$\begin{aligned}d(a_0\otimes\ldots\otimes a_n)&:=\sum_{i=0}^{n}(-1)^i\epsilon(a_i)a_0\otimes\ldots\otimes\widehat{a}_i\otimes\ldots\otimes a_n\\ \widetilde{d}(a_0\otimes\ldots\otimes a_n)&:=\sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n\\ &\quad+(-1)^n a_0\otimes\ldots\otimes a_{n-1}\epsilon(a_n)\end{aligned}$$

To motivate these constructions, we consider two realizations $E.\mathcal{H}$ and $\widetilde{E}.\mathcal{H}$ of the universal bundle over the classifying space $B.\mathcal{H}$. Recall that $E_n\mathcal{H}=\mathcal{H}^{n+1}$ with face maps $\delta_i(h_0,\ldots,h_n)=(h_0,\ldots,\hat{h}_i,\ldots,h_n)$ and $E_n\mathcal{H}\to B_n\mathcal{H}$

given by $(h_0,\dots,h_n)\mapsto (h_0h_1^{-1},\dots,h_{n-1}h_n^{-1})$ (see 0.2). Let $\widetilde{E}_n\mathcal{H}=\mathcal{H}^{n+1}$ with face maps

$$\widetilde{\delta}_i(h_0,\dots,h_n)=\begin{cases}(h_0,\dots,h_{i-1},h_ih_{i+1},\dots,h_n) & \text{if } i=0,\dots,n-1\\ (h_0,\dots,h_{n-1}) & \text{if } i=n\end{cases}$$

The classifying space $B.\mathcal{H}$ is then the quotient of these spaces by the $\mathcal{H}$-right action : diagonally on $E.\mathcal{H}$ and as

$$(h_0,\dots,h_n)h=(h^{-1}h_0,h_1,\dots,h_n)$$

on $\widetilde{E}.\mathcal{H}$. The map $(h_0,\dots,h_n)\mapsto (h_0^{-1},h_0h_1^{-1},\dots,h_{n-1}h_n^{-1})$ is an equivariant isomorphism from $E.\mathcal{H}$ to $\widetilde{E}.\mathcal{H}$ that is compatible with the map to $B.\mathcal{H}$.

Associated to these contractible simplicial spaces we have complexes $\mathcal{D}_{\text{cont}}(E.\mathcal{H})$, $\text{Al}(E.\mathcal{H})$, $\text{SatAl}(E.\mathcal{H})$ and similarly for $\widetilde{E}.\mathcal{H}$ with differential the alternating sums of the δ_i's respectively the $\widetilde{\delta}_i$'s.

Lemma 4.4.1. *One has*

$$\mathcal{D}_{\text{cont}}(E.\mathcal{H})\cong T.\mathcal{D}_{\text{cont}}(\mathcal{H})$$

and

$$\mathcal{D}_{\text{cont}}(\widetilde{E}.\mathcal{H})\cong \widetilde{T}.\mathcal{D}_{\text{cont}}(\mathcal{H})$$

and similar results for $\text{Al}\mathcal{H}$ *and* $\text{SatAl}\mathcal{H}$*.*

Proof. Clear from the definition. □

From the isomorphism of simplicial spaces $E.\mathcal{H}\cong\widetilde{E}.\mathcal{H}$ one sees that there is an isomorphism of complexes $\mathcal{D}_{\text{cont}}(E.\mathcal{H})\cong\mathcal{D}_{\text{cont}}(\widetilde{E}.\mathcal{H})$. In a similar way one sees:

Lemma 4.4.2. *There is an isomorphism of complexes*

$$T.\mathcal{UL}^*\cong\widetilde{T}.\mathcal{UL}^*$$

and both complexes are projective resolutions of $\mathbb{Q}_p$ *as trivial* $\mathcal{UL}^*$*-module.*

4.5. **Review of the Lazard isomorphism.** Recall that $\mathcal{H}$ is a p-saturated group of finite rank with valuation ω and $\mathcal{L}^*$ is its Lazard Lie algebra. The main theorem of chapter V in [L] can be formulated as follows:

Theorem 4.5.1 ([L] V 2.4.9)**.** *There is an isomorphism*

$$H^i_{\text{cont}}(\mathcal{H},\mathbb{Q}_p)\cong H^i(\mathcal{L}^*,\mathbb{Q}_p).$$

Let us review how Lazard constructs this isomorphism. First Lazard [L] V 1.2.9 shows that $\text{Hom}_{\text{Al}\mathcal{H}}(\widetilde{T}.\text{Al}\mathcal{H},\mathbb{Q}_p)$ computes continuous group cohomology. Then the isomorphism is obtained from the following three quasi-isomorphisms:

$$\text{Hom}_{\text{Al}\mathcal{H}}(\widetilde{T}.\text{SatAl}\mathcal{H},\mathbb{Q}_p)\xrightarrow{qis}\text{Hom}_{\text{Al}\mathcal{H}}(\widetilde{T}.\text{Al}\mathcal{H},\mathbb{Q}_p)\tag{9}$$

induced by $\mathrm{Al}\mathcal{H} \subset \mathrm{SatAl}\mathcal{H}$,

$$\mathrm{Hom}_{\mathrm{SatAl}\mathcal{H}}(\widetilde{T}.\mathrm{SatAl}\mathcal{H}, \mathbb{Q}_p) \xrightarrow{qis} \mathrm{Hom}_{\mathcal{UL}^*}(\widetilde{T}.\mathcal{UL}^*, \mathbb{Q}_p), \tag{10}$$

induced by $\mathcal{UL}^* \to \mathrm{SatAl}\mathcal{H}$ and finally

$$\mathrm{Hom}_{\mathcal{UL}^*}(\widetilde{T}.\mathcal{UL}^*, \mathbb{Q}_p) \xrightarrow{qis} \mathrm{Hom}_{\mathcal{UL}^*}(\mathcal{UL}^* \otimes \bigwedge^{\cdot} \mathcal{L}^*, \mathbb{Q}_p) \tag{11}$$

induced by the anti-symmetrisation map as_n

$$as_n : \bigwedge^n \mathcal{L}^* \to \mathcal{U}(\mathcal{L}^*)^{\otimes n}$$

given by

$$as_n(X_1 \wedge \ldots \wedge X_n) = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) X_{\sigma^{-1}(1)} \otimes \ldots \otimes X_{\sigma^{-1}(n)}.$$

The fact that the latter is a quasi-isomorphism follows from [CE] XIII Theorem 7.1.

4.6. **Explicit description of the Lazard isomorphism.** We describe the Lazard isomorphism as a kind of Taylor series expansion.

In the last section the Lazard isomorphism was shown to be induced from the map of complexes

$$\mathcal{UL}^* \otimes \bigwedge^{\cdot} \mathcal{L}^* \to \widetilde{T}.\mathcal{UL}^* \to \widetilde{T}.\mathrm{SatAl}\mathcal{H}.$$

In Corollary 4.2.2 we saw that the map $\mathcal{UL}^* \to \mathrm{SatAl}\mathcal{H} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ factors through $\mathcal{D}_{\mathrm{cont}}(\mathcal{H})$. We get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{SatAl}\mathcal{H}}(\widetilde{T}.\mathrm{SatAl}\mathcal{H}, \mathbb{Q}_p) & \longrightarrow & \mathrm{Hom}_{\mathcal{UL}^*}(\mathcal{UL}^* \otimes \bigwedge^{\cdot} \mathcal{L}^*, \mathbb{Q}_p) \\ \downarrow & \nearrow & \\ \mathrm{Hom}_{\mathcal{D}_{\mathrm{cont}}(\mathcal{H})}(\widetilde{T}.\mathcal{D}_{\mathrm{cont}}(\mathcal{H}), \mathbb{Q}_p) & & \end{array}$$

Using the identification

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{cont}}(\mathcal{H})}(\widetilde{T}_n \mathcal{D}_{\mathrm{cont}}(\mathcal{H}), \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathcal{D}_{\mathrm{cont}}(\mathcal{H})^{\otimes n}, \mathbb{Q}_p) \cong \mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\otimes n}$$

and

$$\mathrm{Hom}_{\mathcal{UL}^*}(\mathcal{UL}^* \otimes \bigwedge^{\cdot} \mathcal{L}^*, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\bigwedge^{\cdot} \mathcal{L}^*, \mathbb{Q}_p)$$

the diagram gives:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{SatAl}\mathcal{H}}(\widetilde{T}.\mathrm{SatAl}\mathcal{H}, \mathbb{Q}_p) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(\bigwedge^{\cdot} \mathcal{L}^*, \mathbb{Q}_p) \\ \downarrow & \nearrow & \\ \mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\otimes \cdot} & & \end{array}$$

We want to make the map $\mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\otimes n} \to \mathrm{Hom}_{\mathbb{Z}_p}(\bigwedge^n \mathcal{L}^*, \mathbb{Q}_p)$ more explicit.

Recall that we have defined in Section 0.4 for each $f \in \mathcal{O}(\mathcal{H}^{\mathrm{la}})$ a linear form

$$df(e) \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{L}^*, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{h}, \mathbb{Q}_p) = \mathfrak{h}^\vee,$$

where the isomorphism comes from Proposition 4.3.1. One has

$$df(e)(\partial_j) = \frac{\partial f}{\partial \lambda_j}(e).$$

This is the differential of f evaluated at $e \in \mathcal{H}$.

Proposition 4.6.1. *The map*

$$\mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\otimes n} \to \mathrm{Hom}_{\mathbb{Z}_p}(\bigwedge^n \mathcal{L}^*, \mathbb{Q}_p) \cong \bigwedge^n \mathfrak{h}^\vee$$

is given by

$$f_1 \otimes \ldots \otimes f_n \mapsto df_1(e) \wedge \ldots \wedge df_n(e).$$

Thus the Lazard isomorphism agrees with Φ as defined in Definition 1.4.1.

Proof. By definition, $f_1 \otimes \ldots \otimes f_n$ maps to the linear form, which maps $\partial_1 \wedge \ldots \wedge \partial_n \in \bigwedge^n \mathfrak{h}$ to

$$\sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \partial_{\sigma^{-1}(1)} f_1 \ldots \partial_{\sigma^{-1}(n)} f_n.$$

But this is exactly the linear form $df_1(e) \wedge \ldots \wedge df_n(e)$ with $df_i(e)$ as defined above. □

Remark 4.6.2. It is not hard to see that the map

$$\mathcal{O}(\mathcal{H}^{\mathrm{la}})^{\otimes n} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{U}(\mathcal{L}^*)^{\otimes n}, \mathbb{Q}_p)$$

is given by the whole Taylor series and not just the first coefficient. For our purposes the above result suffices.

4.7. **Comparison of Φ with Ψ.** We return to the situation in Section 4.3, where we had a smooth algebraic group scheme $H/\mathbb{Z}_p$ and $\mathcal{H}$ was a p-saturated open subgroup of $H(\mathbb{Z}_p)$. Using Proposition 4.3.1 we identify $\mathcal{L}^* \otimes \mathbb{Q}_p \cong \mathfrak{h}$.

In this section we relate the map of complexes

$$\begin{aligned} \Phi : \mathcal{O}(B_n \mathcal{H}^{\mathrm{la}}) &\to \mathcal{C}^n(\mathfrak{h}) \\ f_1 \otimes \ldots \otimes f_n &\mapsto df_1(e) \wedge \ldots \wedge df_n(e) \end{aligned}$$

to the map of complexes

$$\begin{aligned} \Psi : \mathcal{O}(B_n \mathcal{H}^{\mathrm{la}}) \subset \mathcal{O}(E_n \mathcal{H}^{\mathrm{la}}) &\to \mathcal{C}^n(\mathfrak{h}) \\ f_0 \otimes \ldots \otimes f_n &\mapsto f_0(e) df_1(e) \wedge \ldots \wedge df_n(e) \end{aligned}$$

defined in Definition 3.4.5 (for $\mathcal{H} = 1 + pM_N(\mathbb{Z}_p)$). We have the following theorem:

Theorem 4.7.1. *The map Φ and the map Ψ are homotopic maps of complexes. In particular, they induce the same map on cohomology*

$$\Phi = \Psi : H^i_{\mathrm{la}}(\mathcal{H}, \mathbb{Q}_p) \cong H^i(\mathfrak{h}, \mathbb{Q}_p).$$

Proof. Write $\mathcal{C}^n(\mathfrak{h}) = \mathrm{Hom}_{\mathcal{U}\mathfrak{h}}(\mathcal{U}\mathfrak{h} \otimes \bigwedge^n \mathfrak{h}, \mathbb{Q}_p)$, then the map $\mathcal{O}(E_n\mathcal{H}^{\mathrm{la}}) \to \mathcal{C}^n(\mathfrak{h})$, which defines Ψ is induced by a map of complexes

$$\mathcal{U}\mathfrak{h} \otimes \bigwedge^{\cdot} \mathfrak{h} \to T.\mathcal{U}\mathfrak{h} \to T.\mathcal{D}_{\mathrm{cont}}(\mathcal{H}).$$

On the other hand, $\Phi : \mathcal{O}(B_n\mathcal{H}^{\mathrm{la}}) \to \mathcal{C}^n(\mathfrak{h})$ is induced in degree n by a map

$$\bigwedge^n \mathfrak{h} \to \mathcal{U}\mathfrak{h}^{\otimes n} \to \mathcal{D}_{\mathrm{cont}}(\mathcal{H})^{\otimes n}.$$

We extend the first map $\mathcal{U}\mathfrak{h}$-linearly to a map of complexes

$$\mathcal{U}\mathfrak{h} \otimes \bigwedge^{\cdot} \mathfrak{h} \to \widetilde{T}.\mathcal{U}\mathfrak{h}.$$

Using the isomorphism $\widetilde{T}.\mathcal{U}\mathfrak{h} \cong T.\mathcal{U}\mathfrak{h}$ from Lemma 4.4.2, we get two maps from the projective resolution $\mathcal{U}\mathfrak{h} \otimes \bigwedge^{\cdot} \mathfrak{h}$ of $\mathbb{Q}_p$ to the resolution $T.\mathcal{U}\mathfrak{h}$, which must be homotopic by general facts for projective resolutions. Composing this with the commutative diagram

$$\begin{array}{ccc} T.\mathcal{U}\mathfrak{h} & \longrightarrow & T.\mathcal{D}_{\mathrm{cont}}(\mathcal{H}) \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{T}.\mathcal{U}\mathfrak{h} & \longrightarrow & \widetilde{T}.\mathcal{D}_{\mathrm{cont}}(\mathcal{H}) \end{array}$$

gives still homotopic maps, so that Φ and Ψ coincide on cohomology. □

5. Proof of the Main Theorem

In this section we are going to put together the proofs of the results announced in Chapter 1.

Let throughout $G = \mathrm{GL}_N$ as algebraic group over R, $\mathcal{G} = \mathrm{GL}_N(R)$ as K-Lie group and $\mathcal{G}^\dagger = \coprod_{a \in G(k)}]a[_G$ as underlying dagger-space. Recall that $\mathcal{G}^\dagger(K) = \mathcal{G}$. Let $1 < n \leq N$.

Proof of Theorem 1.2.1 and of Theorem 1.3.2. We have to check injectivity and surjectivity of the Lazard map and that primitive elements p_n in Lie algebra cohomology are related to the étale Chern classes.

Step 1: Injectivity of the Lazard map

Let $\mathcal{G}_n = 1 + \pi^n M_N(R)$. They are open and closed normal subgroups of $\mathcal{G}$ of finite index. They are also a neighbourhood basis of $e \in \mathcal{G}$. Hence $\mathcal{O}(G^{\mathrm{la}})_e$, the ring of germs of analytic functions, is given by $\lim\limits_{n\to\infty} \mathcal{O}(\mathcal{G}_n^{\mathrm{la}})$. For $n' > n$ the natural restriction maps

$$H^i_{\mathrm{la}}(\mathcal{G}_n, K) \to H^i_{\mathrm{la}}(\mathcal{G}_{n'}, K)$$

are injective as we working with rational coefficients. Passing to the limit the map

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i(\mathcal{O}(B.G^{\mathrm{la}})_e)$$

is also injective. We define

$$\Phi_\infty : H^i(\mathcal{O}(B.G^{\mathrm{la}})_e) \to H^i(\mathfrak{g}, K)$$

by

$$f_1 \otimes \ldots \otimes f_n \mapsto df_1(e) \wedge \ldots \wedge df_n(e).$$

Note that this is the same formula as in Definition 1.4.1.

Now let $R = \mathbb{Z}_p$. Then Φ is an isomorphism on the saturated subgroup $\mathcal{G}_1$ by Theorem 4.5.1 together with Proposition 4.6.1. (For $p = 2$ the saturated subgroup is $\mathcal{G}_2$. The argument remains the same.) Hence Φ_∞ is an also isomorphism.

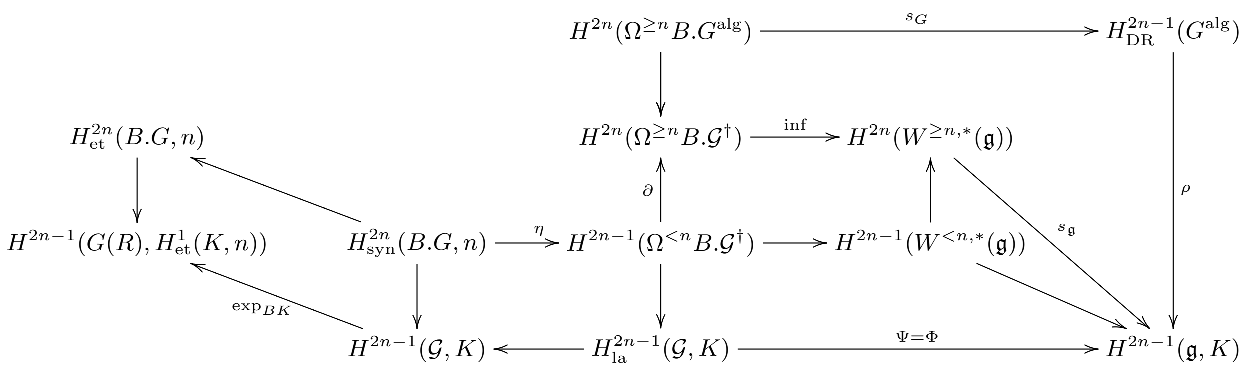
Moreover,

$$\mathrm{Lie}(\mathrm{GL}_N(R)) \cong \mathrm{Lie}(\mathrm{GL}_N(\mathbb{Z}_p)) \otimes K$$
$$H^i(\mathrm{Lie}(\mathrm{GL}_N(R)), K) \cong H^i(\mathrm{Lie}(\mathrm{GL}_N(\mathbb{Z}_p)), \mathbb{Q}_p) \otimes K$$
$$\mathcal{O}(G_R^{\mathrm{la}})_e \cong \mathcal{O}(G_{\mathbb{Z}_p}^{\mathrm{la}})_e \otimes K$$

and the map Φ is compatible with extension of scalars. Hence Φ_∞ is also an isomorphism for general K. This implies that Φ is injective for general K.

Step 2: The commutative diagram

Putting together the commutative diagrams of Proposition 2.3.4, Proposition 2.4.5 and Theorem 3.1.1, we have established the following big commutative diagram:



For $R = \mathbb{Z}_p$ we have $\Psi = \Phi$ by Theorem 4.7.1. By compatibility with base change, we get that $\Psi = \Phi$ is the algebraic Lazard map for any R.

Step 3: Chasing elements

Consider the primitive element $p_n \in H^{2n-1}(\mathfrak{g}, K)$. The Chern class $c_n^{\mathrm{alg}} \in H^{2n}(\Omega^{\geq n} B.G^{\mathrm{alg}})$ is mapped to p_n under the suspension map s_G. Using Proposition 3.7.1, [GHV] VI.6.19. (which computes the image of the Chern class under $s_{\mathfrak{g}}$) and [Bu] Lemma 8.11, one verifies the normalization of p_n used in Definition 0.4.5. By Proposition 2.4.5 the image of c_n^{alg} in $H^{2n}(\Omega^{\geq n} B.\mathcal{G}^\dagger)$ agrees with the image of the syntomic Chern class $c_n^{\mathrm{syn}} \in H^{2n}_{\mathrm{syn}}(B.G, n)$ under $\partial \circ \eta$. By the commutativity of the diagram this implies that the image of c_n^{syn} in $H^{2n-1}(\mathfrak{g}, K)$ is p_n.

Step 4: Surjectivity of the Lazard map Lie algebra cohomology $H^*(\mathfrak{g}, K)$ is generated by the p_n as an algebra and Φ is compatible with cup-product.

We have established in Step 3 that all p_n are in the image of Φ. Hence Φ is surjective. Together with step 1 this means that Φ is invertible.
Step 5: p_n maps to c_n^{et} We have already related p_n to the syntomic Chern class. By Proposition 2.3.4 the universal syntomic Chern class is mapped to the universal étale Chern class. Together this proves the theorem.

□

Remark 5.0.2. In the case $n = 1$ basically the same argument works.

Remark 5.0.3. In the case $R = \mathbb{Z}_p$ surjectivity of Φ can be established more directly, cf. [CW] §3: the operation of Gl_N on $H^*(\mathfrak{g}, K)$ is algebraic. Hence the stabilizer is Zariski-closed subset. It contains the open subset $\mathcal{G}_1$ (for the analytic topology) by Lazard's result 4.5.1. Hence it is all of Gl_N. This implies that

$$H^*(G(\mathbb{Z}_p), K) \cong H^*(\mathrm{Lie}(G), K)^{G(\mathbb{Z}_p)} \cong H^*(\mathrm{Lie}(G), K) .$$

In [HKN] Section 4 the Casselman-Wigner argument is pushed to show that the algebraic Lazard map is surjective (and hence an isomorphism) for all R and open subgroups of $\mathbb{G}(R)$ where $\mathbb{G}$ is a smooth group scheme over $\mathbb{R}$. Loc. cit. Theorem 4.3.1 thus can replace the argument in Step 4 of the above proof.

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