FIBRE SEQUENCES AND LOCALIZATION OF SIMPLICIAL SHEAVES

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Abstract. In this paper, we provide criteria for fibre sequences of simplicial sheaves to be preserved by nullifications. This generalizes a result of Berrick and Dror Farjoun and allows a better understanding of unstable $\mathbb{A}^1$-homotopy theory.

Contents

1. Introduction

In this paper, we discuss aspects of Bousfield localization for simplicial sheaves. One of the main phenomena of interest is the behaviour of fibre sequences under a Bousfield localization. In general, fibre sequences are not preserved by a Bousfield localization, and it is an interesting question to find suitable criteria under which they are preserved. An extensive discussion of issues related to this can be found in [DF96]. A general criterion for nullifications has been obtained by Berrick and Dror Farjoun in [BF03]. The main goal of this paper is to provide a generalization of this result to the setting of simplicial sheaves.

One of the technical tools used in [BF03] is the fibrewise localization in the context of simplicial sets. For a discussion of fibrewise localization of simplicial sets resp. topological spaces, see [DF96] or [Hir03]. A construction of fibrewise localization for model categories satisfying certain axioms was provided in [CS06]. We explain why the result of Chataur and Scherer holds for model structures on categories of simplicial sheaves. With this technical tool available, the proofs from [BF03] can be carried over without much problems. The main theorem is then the following, cf. Theorem 5.2, generalizing [BF03, Theorem 4.1].

1991 *Mathematics Subject Classification.* 18F20, 55P60, 14F42.

Key words and phrases. Bousfield localization, simplicial sheaves, $\mathbb{A}^1$-homotopy theory.

Theorem 1. *Let T be a site, and let $f : X \to Y$ be a* null-homotopic *morphism of simplicial sheaves in $\Delta^{op}\mathcal{S}hv(T)$. Furthermore, let $F \to E \xrightarrow{p} B$ be a fibre sequence.*

Let $L_f F \to \overline{E} \to B$ be a fibrewise localization of p, and let $L_f F \to E' \to A_{L_f} B$ be the pullback of this fibrewise localization of p to $A_{L_f} B = \mathrm{hofib}(B \to L_f B)$. Then the following are equivalent:

(i) The fibre sequence p is preserved by L_f.
(ii) $E' \simeq L_f F \times A_{L_f} B$ and therefore there is an f-local weak equivalence $E' \simeq L_f F$.
(iii) The following composition of morphisms is null-homotopic:

$$A_{L_f} B \to B \xrightarrow{p} B^f F \to B^f L_f F.$$

Note that (iii) only applies in situations where there is a classifying space available. By the results from [Wen09], this is the case e.g. if the fibre sequence is locally trivial in some refinement of the topology of T.

As an interesting application, we arrive at conditions when morphisms induce fibre sequences in $\mathbb{A}^1$-homotopy theory. In the case where the morphisms are locally trivial in the Nisnevich topology, the homotopy theory criteria reduce to a simple condition on the sheaf of homotopy self-equivalences of the fibre.

Theorem 2. *Let F be a simplicial sheaf on Sm_k. If $\pi_0 \,\mathrm{hAut}_\bullet\, L_{\mathbb{A}^1} F$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups, then any morphism $p : E \to B$ which is locally trivial in the Nisnevich topology with fibre F induces an $\mathbb{A}^1$-local fibre sequence $F \to E \to B$.*

Structure of the Paper: The paper is structured as follows: in Section 2, we repeat preliminaries on Bousfield localization for simplicial sheaves. Then Section 3 recalls the construction of fibrewise localization by Chataur and Scherer. In Section 4, we give a characterization of nullification functors. In Section 5, we prove the criterion for fibre sequences to be preserved by a nullification. Some remarks on properness of the local model structure are provided in Section 6. Finally, the locality of classifying spaces in $\mathbb{A}^1$-homotopy theory is discussed in Section 7.

Acknowledgements: The results presented here are taken from my PhD thesis [Wen07] which was supervised by Annette Huber-Klawitter. I would like to use the opportunity to thank her for her encouragement and interest in my work. I would also like to thank Denis-Charles Cisinski for interesting remarks on the relation between the results presented here and properness of the local model structure.

2. Localization Functors for Simplicial Sheaves

This section collects elementary facts concerning localizations and nullifications of simplicial sheaves. These facts are well-known for simplicial sets resp. topological spaces and proofs mostly carry over directly. Most of the basic properties are formal, once the theorem on existence, universality and continuity of localization functors is established. This theorem is proven in [GJ98].

There are two approaches to localizations in the literature for simplicial sets: on the one hand, the whole book [DF96] does not require a single word on model categories, and defines localizations as certain homotopy-universal coaugmented functors. On the other hand, starting from a proper, simplicial, cofibrantly generated model structure on the category $\mathcal{C}$, it is possible to define a new model structure by keeping the cofibrations, and defining a new class of *local weak equivalences*. The relation between the two approaches is easily described: The proof of existence of localization functors in [DF96] is the same as the small object argument proving that Quillen's axioms hold for the local model structure. On the other hand, assuming known the existence of this model structure, the localization functor is basically the fibrant replacement in the local model structure.

2.1. **Model Categories of Simplicial Sheaves:** We will be working in categories of simplicial sheaves. The underlying site is usually denoted by T, the category of sheaves on it by $\mathcal{S}hv(T)$, and the category of simplicial sheaves by $\Delta^{op}\mathcal{S}hv(T)$. On this category, there are several model structures all yielding the same homotopy theory. We will use the injective model structure, cf. [Jar96, Theorems 18 and 27].

Theorem 2.1. *Let $\mathcal{E}$ be a topos. Then the category $\Delta^{op}\mathcal{E}$ of simplicial objects in $\mathcal{E}$ has a model structure, where the*

(i) cofibrations are monomorphisms,
(ii) weak equivalences are detected on a fixed Boolean localization,
(iii) fibrations are determined by the right lifting property.

Moreover, the above definition of weak equivalences does not depend on the Boolean localization.

The following proposition recalls the basic properties of this model structure.

Proposition 2.2. *Let T be any Grothendieck site. Then the injective model structure of Jardine on the category of (pre-)sheaves of simplicial sets on T is a proper simplicial and cellular model structure.*

Next, we repeat several basic statements on the behaviour of homotopy limits and colimits in categories of simplicial sheaves. Results and preliminaries can be found in [Wen09, Section 3].

Corollary 2.3 (Mather's Cube Theorem)**.** *Let $\mathcal{E}$ be any Grothendieck topos. Consider the following diagram of simplicial objects in $\mathcal{E}$:*

$$\begin{tikzcd}
& X_1 \arrow[rr] \arrow[dl] \arrow[dd] & & X_2 \arrow[dl] \arrow[dd] \\
X_3 \arrow[rr] \arrow[dd] & & X_4 \arrow[dd] & \\
& Y_1 \arrow[rr] \arrow[dl] & & Y_2 \arrow[dl] \\
Y_3 \arrow[rr] & & Y_4 &
\end{tikzcd}$$

Assume that the bottom face, i.e. the one consisting of the spaces Y_i, is a homotopy pushout, and that all the vertical faces are homotopy pullbacks. Then the top face is a homotopy pushout.

Moreover, taking the homotopy fibre commutes with homotopy pushouts: For a commutative diagram

$$\begin{tikzcd} E_2 \arrow[d, "p_2"'] & E_0 \arrow[l] \arrow[r] \arrow[d, "p_0"] & E_1 \arrow[d, "p_1"] \\ B_2 & B_0 \arrow[l] \arrow[r] & B_1 \end{tikzcd}$$

in which the squares are homotopy pullbacks, we have weak equivalences

$$\operatorname{hofib} p_i \xrightarrow{\cong} \operatorname{hofib}(p : E_1 \cup^h_{E_0} E_2 \to B_1 \cup^h_{B_0} B_2).$$

Proposition 2.4 (Puppe's Theorem). *Let $\mathcal{E}$ be a Grothendieck topos, and let $\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{E}$ be a diagram of simplicial objects over a fixed base simplicial object Y, i.e. the following diagram commutes for every $\alpha : i \to j$ in $\mathcal{I}$:*

$$\begin{tikzcd} \mathcal{X}(i) \arrow[rr, "\mathcal{X}(\alpha)"] \arrow[dr] & & \mathcal{X}(j) \arrow[dl] \\ & Y & \end{tikzcd}$$

There is an associated diagram of homotopy fibres

$$\mathcal{F} : \mathcal{I} \to \Delta^{op}\mathcal{E} : i \mapsto \operatorname{hofib}(\mathcal{X}(i) \to Y)$$

Denoting $X = \operatorname{hocolim}_{\mathcal{I}} \mathcal{X}$ and $F = \operatorname{hocolim}_{\mathcal{I}} \mathcal{F}$, we have a weak equivalence $\operatorname{hofib}(X \to Y) \simeq F$.

2.2. **Internal Mapping Spaces:** We recall basic facts on mapping spaces. In a general model category, one can only consider homotopy classes of maps $\hom_{\mathrm{Ho}\,\mathcal{C}}(X, Y)$. If the model category is simplicial, one can define mapping spaces $\operatorname{Hom}(X, Y)$ which are simplicial sets. Categories of sheaves are cartesian closed, and this implies that we indeed have internal mapping spaces, i.e. for any two simplicial sheaves X and Y, there is a simplicial sheaf $\mathbf{Hom}(X, Y)$.

In a category of simplicial sheaves, these mapping spaces can be defined as follows: Let T be a site, and let X be a fibrant simplicial sheaf. By Proposition 2.2, the simplicial sheaves on T form a simplicial model category, hence for any two simplicial sheaves X, Y there is a simplicial set, the function complex $\operatorname{Hom}(X, Y)$, whose n-simplices are given by

$$\operatorname{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X \times \Delta^n, Y).$$

We have a contravariant functor

$$T^{\mathrm{op}} \to \Delta^{op}\mathcal{S}et : (U \in T) \mapsto \operatorname{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X \times U, Y).$$

This functor is representable by a simplicial sheaf which we again denote by $\mathbf{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X, Y)$. This is discussed in [MV99].

As the internal mapping spaces play a fundamental role in the description of localizations, we recall some details and the difference between pointed and unpointed mapping spaces.

There are versions of internal mapping spaces for both unpointed and pointed categories, and their relation is given by the following is a fibration sequence:

$$\mathbf{Hom}_*(X,Y) \to \mathbf{Hom}(X,Y) \to Y.$$

For simplicial sheaves X and Y on a site T with enough points, this can be easily proven using the above fibre sequence for simplicial sets [DF96, Definition 1.A.1], the fact that the internal mapping spaces at the points are given by the simplicial set mapping spaces, and then putting everything together using [Wen09, Proposition 3.17]. This implies that if we use the pointed mapping spaces in the definition of localization, only the connected component of the chosen base point will be localized, the others remain unchanged. This is one reason for connectivity restrictions on fibrewise localizations using pointed functors in Section 3.

2.3. **Bousfield Localization:** We repeat the standard definitions of local objects and local weak equivalences. These definitions can be found in [DF96, Hir03] for the case of simplicial sets, and in [MV99] for the case of simplicial sheaves.

Let $\mathcal{C}$ be a model category, and let $f : A \to B$ be a morphism of cofibrant objects.

Definition 2.5 (Local Objects, Weak Equivalences)**.** *An object $X \in \mathcal{C}$ is called* f-local *if X is fibrant and the following morphism is a bijection for each $Y \in \mathrm{Ho}\,\mathcal{C}$:*

$$\hom_{\mathrm{Ho}\,\mathcal{C}}(Y \times B, X) \to \hom_{\mathrm{Ho}\,\mathcal{C}}(Y \times A, X).$$

A morphism $g : X \to Y \in \mathcal{C}$ is called an f-local weak equivalence *if for any f-local object Z, the following morphism is a bijection:*

$$\hom_{\mathrm{Ho}\,\mathcal{C}}(Y, Z) \to \hom_{\mathrm{Ho}\,\mathcal{C}}(X, Z).$$

Remark 2.6. *(i) The above is the definition of local given in* [MV99]*. It is easy to check that it coincides with the definition in* [GJ98]*, where one requires a weak equivalence of simplicial sets:*

$$\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z).$$

This in turn is equivalent to requiring weak equivalences on internal homs:

$$\mathbf{Hom}(Y, Z) \to \mathbf{Hom}(X, Z).$$

(ii) Note that there is a difference between pointed and unpointed. The definition above is for a general model category, using unpointed mapping spaces. In a pointed model category, one uses the pointed mapping spaces. For connected spaces both notions coincide.

Of course, one can consider more general localizations, i.e. localizations with respect to a *set* of maps as in [MV99, Section 2.2], or homology localization as in [GJ98, Section 3]. If f is null-homotopic such a localization is also called *nullification*, and we also use L_W to denote the corresponding localization functor. The most important applications we have in mind are the $\mathbb{A}^1$-nullification functors $L_{\mathbb{A}^1}$ on $\Delta^{op}\mathcal{S}hv(\mathrm{Sm}_S)$.

2.4. **Localization Functors:** This paragraph repeats the theorem on existence and universality of localization functors for simplicial sheaves. Most of the elementary facts in this section are easy consequences of this theorem, which is proved in [MV99, Theorem 2.2.5] and in similar form in [GJ98, Theorem 4.4].

We start recalling the definition of localization functor in a general model category.

Definition 2.7. *A functor $F : \mathcal{C} \to \mathcal{C}$ is called* coaugmented *if there is a natural transformation $j : \mathrm{id}_{\mathcal{C}} \to F$. A coaugmented functor F is called* idempotent *if the two natural maps $j_{FX}, Fj_X : FX \rightrightarrows FFX$ are weak equivalences and homotopic to each other. The coaugmentation map j_X is homotopy universal with respect to maps into local spaces if any map $X \to T$ into a local space T factors uniquely (up to homotopy) through $j_X : X \to FX$. The functor F is called* simplicial *if it is compatible with the simplicial structure, i.e. if there exist functorial morphisms $\sigma : (FX) \otimes K \to F(X \otimes K)$ for any object $X \in \mathcal{C}$ and any simplicial set K. These morphisms have to satisfy some rather obvious conditions described in* [DF96, Definition 1.C.8]*. The functor F is called* continuous *if it induces a morphism on inner function spaces*

$$\mathbf{Hom}(X, Y) \to \mathbf{Hom}(FX, FY),$$

which is compatible with composition.

We recall the existence of localizations for simplicial sheaf categories from [GJ98, Theorem 4.4], which is the proper generalization of [DF96, Theorem A.3]. The existence of the f-local model structure is proven in [GJ98, Theorem 4.8]. Note that the existence of localizations for simplicial sheaves is a global result, in the sense that it does not simply follow from the existence of localizations of simplicial sets by looking at the points of the topos.

Theorem 2.8. *Let $f : A \to B$ be any cofibration in $\Delta^{op}Shv(T)$ and suppose α is an infinite cardinal which is an upper bound for the cardinalities of both B and the set of morphisms of T. Then there exists a functor L_f, called the f-*localization functor*, which is coaugmented and homotopically idempotent. Any two such functors are naturally weakly equivalent to each other. The map $X \to L_f X$ is a homotopically universal map to f-local spaces. Moreover, L_f can be chosen to be simplicial and continuous.*

There is a simplicial model structure on $\Delta^{op}\mathcal{S}hv(T)$ where the cofibrations are monomorphisms, weak equivalences are f-local weak equivalences and fibrations are defined via the right lifting property.

Remark 2.9 (Properness)**.** *In* [Hir03, Chapter 3]*, Bousfield localizations of general model categories are investigated. As shown in* [Hir03, Proposition 3.4.4 and Theorem 4.1.1]*, left Bousfield localizations preserve left properness, i.e. the left Bousfield localization of a left proper model category is again left proper.*

The f-local model structure for a morphism $f : A \to B$ is not in general right proper. It is known [Jar00, Theorem A.5]*, that the f-local model structure is proper if f is of the form $* \to I$. A special case of this is the properness of the homotopy theory of a site with interval, which is proved in*

[MV99, Theorem 2.2.7 and Section 2.3]. *Note however that in the formulation of the theorem left and right properness have to be interchanged.*

2.5. **Elementary facts concerning f-local spaces and L_f:** There is a collection of elementary facts concerning localization functors in [DF96]. These also hold for localization functors on categories of simplicial sheaves. We will neither state nor prove all of them, we only discuss those that are directly relevant to fibre sequences.

Proposition 2.10. *Any homotopy limit of f-local simplicial sheaves is again f-local. In particular, we have the following conclusions:*

(i) The homotopy fibre of a morphism of f-local simplicial sheaves is f-local.
(ii) The product of any family of f-local simplicial sheaves is f-local. The morphism
$$L_f(X \times Y) \to L_f X \times L_f Y$$
is a homotopy equivalence.
(iii) For f-local Y and cofibrant X, the simplicial sheaves $\mathbf{Hom}(X,Y)$ and $\mathbf{Hom}_*(X,Y)$ are f-local. Therefore, the loop spaces $\Omega^n Y$ are f-local for f-local Y.

Proof. Let T be a Grothendieck site, and let $\mathcal{I}$ be a small category. Furthermore, let $\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{S}hv(T)$ be a diagram of f-local simplicial sheaves. We want to show that

$$\mathbf{Hom}(f, \operatorname{holim}_{\mathcal{I}} \mathcal{X}) : \mathbf{Hom}(B, \operatorname{holim}_{\mathcal{I}} \mathcal{X}) \to \mathbf{Hom}(A, \operatorname{holim}_{\mathcal{I}} \mathcal{X})$$

is a weak equivalence. By [MV99, Lemma 2.1.19], this map is the same as the map $\operatorname{holim}_{\mathcal{I}} \mathbf{Hom}(f, \mathcal{X})$. By assumption, the map

$$\mathbf{Hom}(f, \mathcal{X}(i)) : \mathbf{Hom}(B, \mathcal{X}(i)) \to \mathbf{Hom}(A, \mathcal{X}(i))$$

is a weak equivalence for any $i \in \mathcal{I}$. By homotopy invariance, cf. [Hir03, Theorem 18.5.3], this implies that $\operatorname{holim}_{\mathcal{I}} \mathbf{Hom}(f, \mathcal{X})$ is a weak equivalence as well.

(i) and (ii) are then clear by applying the above to the relevant diagrams, and the homotopy equivalence $L_f(X \times Y) \to L_f X \times L_f Y$ follows from universality and adjointness, cf. [DF96, Section 1.G].

For (iii), we have by adjointness and locality of Y the weak equivalences:

$$\begin{aligned}\mathbf{Hom}(A, \mathbf{Hom}(X,Y)) &\simeq \mathbf{Hom}(X, \mathbf{Hom}(A,Y)) \\ &\simeq \mathbf{Hom}(X, \mathbf{Hom}(B,Y)) \\ &\simeq \mathbf{Hom}(B, \mathbf{Hom}(X,Y)).\end{aligned}$$

The statement on loop spaces is a consequence of this. □

2.6. **Interaction of Localization and Homotopy Colimits:**

Proposition 2.11 ([DF96], Proposition 1.D.2)**.** *Let T be a site, let $\mathcal{I}$ be a small category and let $f : A \to B$ be a morphism of cofibrant spaces. Furthermore, let $g : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{I}$-diagram of cofibrant objects of simplicial* *sheaves in $\Delta^{op}\mathcal{S}hv(T)$. Assume that for any $i \in \mathcal{I}$ the morphism $g(i) : \mathcal{X}(i) \to \mathcal{Y}(i)$ is an L_f-equivalence. Then*

$$\operatorname*{hocolim}_{\mathcal{I}} g : \operatorname*{hocolim}_{\mathcal{I}} \mathcal{X} \to \operatorname*{hocolim}_{\mathcal{I}} \mathcal{Y}$$

is also an L_f-equivalence for both pointed and unpointed homotopy colimits.

Proof. Using [MV99, Proposition 2.2.9], it suffices to show for any f-local object Z that

$$\mathbf{Hom}(\operatorname*{hocolim}_{\mathcal{I}} g, Z) : \mathbf{Hom}(\operatorname*{hocolim}_{\mathcal{I}} \mathcal{X}, Z) \to \mathbf{Hom}(\operatorname*{hocolim}_{\mathcal{I}} \mathcal{Y}, Z)$$

is a weak equivalence. Using [MV99, Lemma 2.1.19], it suffices to show that the morphism $\operatorname{holim}_{\mathcal{I}} \mathbf{Hom}(g, Z)$ is a weak equivalence. By assumptions the objects $\mathbf{Hom}(\mathcal{X}(i), Z)$ resp. $\mathbf{Hom}(\mathcal{Y}(i), Z)$ are fibrant and $\mathbf{Hom}(g(i), Z)$ is a weak equivalence for any $i \in \mathcal{I}$. Therefore, using homotopy invariance of holim, we conclude that the morphism $\operatorname{holim}_{\mathcal{I}} \mathbf{Hom}(g, Z)$ is a weak equivalence. □

Applying the above to the diagram of augmentation maps $g : \mathcal{X} \to L_f\mathcal{X}$, we obtain that

$$L_f(\operatorname*{hocolim}_{\mathcal{I}} g) : L_f \operatorname*{hocolim}_{\mathcal{I}} \mathcal{X} \to L_f \operatorname*{hocolim}_{\mathcal{I}} L_f\mathcal{X}$$

is a weak equivalence, generalizing [DF96, Theorem D.3]. In particular we get $L_f(X \vee Y) \simeq L_f(L_fX \vee L_fY)$. Note that a homotopy colimit of local spaces need not be local again. Note also that the interaction of homotopy limits and localizations is not as easy as the homotopy *colimit* case. Although we have seen in Proposition 2.10, that the homotopy fibre of a morphism of local spaces is local, it is not the case that the homotopy fibre of the localization of a morphism g is the localization of the homotopy fibre of g. The rest of this paper is devoted to studying under which circumstances a homotopy pullback diagram in $\Delta^{op}\mathcal{S}hv(T)$ is a homotopy pullback diagram in a localization of $\Delta^{op}\mathcal{S}hv(T)$.

3. Fibrewise Localization for Simplicial Sheaves

In this section, we recall the existence of fibrewise versions of simplicial coaugmented functors in categories of simplicial sheaves. For a treatment of fibrewise localization for simplicial sets resp. topological spaces, see [DF96, Section 1.F] and [Hir03, Chapter 6].

3.1. **Fibrewise Localization:** The main motivation for studying fibrewise localization is a gain of control on the behaviour of a fibration under a localization functor. For a locally trivial morphism $f : E \to B$ of topological spaces with fibre F, one can explain quite easily how to construct the fibrewise localization: Take a trivialization of f, i.e. a covering U_i of X over which $f|_{U_i} : E \times_B U_i \cong U_i \times F \to U_i$. Then apply the simplicial coaugmented functor: On the level of the trivialization one simply replaces the space F by the space LF. On the level of transition morphisms, one applies the functor L to the transition map. For this to work we need the functor L to be continuous. This produces an explicit recipe to construct an LF-bundle over B.

In this section we recall the construction of fibrewise localizations for general model categories of simplicial sheaves. We first define precisely what we mean by fibrewise application of functors.

Definition 3.1. *Let L be a simplicial coaugmented functor on $\Delta^{op}Shv(T)$. We say that L can be* applied fibrewise *resp. that* there is a fibrewise version *of L, if every fibre sequence of simplicial sheaves $F \to E \to B$ can be mapped via a homotopy commutative diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] \\ LF \arrow[r] & \overline{E} \arrow[r] & B \end{tikzcd}$$

to a fibre sequence with fibre LF over B.

Let L_f be a localization functor. Then we say that L_f admits a fibrewise version*, if there exists a homotopy commutative diagram as above, where the morphism $E \to \overline{E}$ is an f-local weak equivalence.*

Remark 3.2. *There is a difference between the above notion of fibrewise localization and the one used in* [DF96]*. Whereas Dror Farjoun requires a concrete model for the fibrewise localization, the morphism $\overline{E} \to B$ being a fibration with fibre LF, we only require the existence of a fibre sequence. This is weaker, in that LF need not actually be the fibre of a fibrant replacement of $\overline{E} \to B$. For the homotopy groups this is not essential.*

We want to note that pointed and unpointed simplicial sets behave rather differently with respect to fibrewise localization. For unpointed simplicial sets, one can construct fibrewise localizations in various different ways, whereas for pointed simplicial sets, one always has to make special connectivity assumptions on the base resp. the fibre because usually there is no continuous choice of base point in a nontrivial fibre sequence $F \to E \xrightarrow{p} B$.

This difference between the unpointed and the pointed setting is also illustrated by [Hir03, Proposition 6.1.4]. See also the discussion in [DF96, Remark 1.A.7].

In [DF96], several constructions of fibrewise localization are given. All of them can be translated to categories of simplicial sheaves on a Grothendieck site T. The key input in all of these methods is again the homotopy distributivity from [Wen09]. Here, we focus on the method used by Chataur and Scherer [CS06]. For the other methods, see [Wen07].

3.2. **Fibrewise Nullification after Chataur and Scherer:** One of the versions of fibrewise localization [DF96, Proposition 1.F.7] has been extended to general model categories by Chataur and Scherer [CS06]. The advantage of this construction is that it works even for functors which are not simplicial resp. continuous. On the other hand, this approach works for *pointed spaces* only under the assumption that the fibre is connected.

There are some axioms that are discussed and used in [CS06]. We remark that the cube axiom used in that paper is nothing else but Mather's cube theorem Corollary 2.3, and the ladder axiom is a form of homotopy distributivity applied to sequential colimit diagrams saying that sequential homotopy colimits commute with homotopy pullbacks. From our discussion in [Wen09, Section 3] we know that both axioms are satisfied in any model category of simplicial sheaves.

Now we can state the fibrewise localization result of Chataur and Scherer, cf. [CS06, Theorem 4.3]

Theorem 3.3. *Let $\mathcal{M}$ be a model category which is pointed, left proper, cellular and in which the cube axiom and the ladder axiom holds. Let $L_f : \mathcal{M} \to \mathcal{M}$ be a localization functor which preserves products, and let $p : F \to E \to B$ be a fibre sequence in $\mathcal{M}$. Then there exists a fibrewise f-localization of p.*

We first note that localization functors of simplicial sheaves commute with finite products as remarked in the proof of [MV99, Lemma 2.2.32], cf. Proposition 2.10.

A result similar to the above can be formulated for fibre sequences over simply-connected base spaces, replacing the product condition on L_f by the join axiom, cf. [CS06].

For the convenience of the reader, we provide a of the Chataur-Scherer result, specialized to the simplicial sheaf setting. Note that although we are working in an unpointed setting, we will speak of fibre sequences. What we mean is that F is a homotopy fibre of $p : E \to B$ over a fixed base point $b \in B$, i.e. the obvious diagram is a homotopy pullback. This makes sense even in an unpointed setting.

Proposition 3.4. *Let T be a Grothendieck site, and let $L_f : \Delta^{op}\mathcal{S}hv(T) \to \Delta^{op}\mathcal{S}hv(T)$ be a localization functor. For any fibration $p : E \to B$ and any base point $b \in B$ there exists a fibrewise f-localization of the homotopy pullback $p^{-1}(b) \to E \to B$.*

Proof. Let $F \to E \xrightarrow{p} B$ be a fibre sequence, assuming that p is a fibration of fibrant simplicial sheaves on T, and that $F = p^{-1}(b)$ for a chosen base point $b \in B$.

(i) We construct a new fibre sequence. This argument is the same as [CS06, Proposition 4.1]. We assume without loss of generality that the augmentation map $F \to L_f F$ is a cofibration. Then we construct the following diagram:

$$\begin{tikzcd}
F \arrow[r, "\simeq_f"'] \arrow[d] & L_f F \arrow[r] \arrow[d] & F_1 \arrow[d] \\
E \arrow[r, "\simeq_f"'] \arrow[d, "p"'] & E \cup_F L_f F \arrow[r, "\simeq"'] \arrow[d, "q"] & E_1 \arrow[d, "p_1"] \\
B \arrow[r, "="'] & B \arrow[r, "="'] & B.
\end{tikzcd}$$

The left column is the original fibre sequence. The space in the centre is constructed as the pushout of $F \to E$ along the augmentation $F \to L_f F$. We have assumed that this is a cofibration, and therefore the centre space is also a homotopy pushout. The map q comes from the universal property of the pushout. Then we factor $q : E \cup_F L_f F \to B$ as a trivial cofibration $E \cup_F L_f F \to E_1$ and a fibration $p_1 : E_1 \to B$. Finally, F_1 is the fibre of p_1

over $b \in B$. An argument similar to the one applied in [Wen09, Proposition 4.22] implies that F_1 is the homotopy pushout of the homotopy fibres of $L_f F \leftarrow F \rightarrow E$. Therefore the composition $F \rightarrow L_f F \rightarrow F_1$ is an f-local weak equivalence.

(ii) The problem is that the rectification of $L_f F \rightarrow E \cup_F L_f F \rightarrow B$ to the fibre sequence $F_1 \rightarrow E_1 \rightarrow B$ destroys the property of the fibre being f-local. Therefore one has to use something similar to a small object argument, constructing ever better approximations to the total space of the fibrewise localization.

Transfinitely iterating this construction until a well-defined ordinal κ, see (iii), produces telescopes $E \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots$ resp. $F \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots$. Applying Puppe's theorem 2.4 to the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E_1 \arrow[r] \arrow[d, "p_1"'] & E_2 \arrow[r] \arrow[d, "p_2"'] & \cdots \\ B \arrow[r, "="'] & B \arrow[r, "="'] & B \arrow[r, "="'] & \cdots \end{tikzcd}$$

we find that the homotopy fibre of $(\operatorname{hocolim}_\kappa E_i) \rightarrow B$ is $\operatorname{hocolim}_\kappa F_i$. We therefore have a morphism of fibre sequences

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, "="] \\ \operatorname{hocolim}_\kappa F_i \arrow[r] & \operatorname{hocolim}_\kappa E_i \arrow[r] & B. \end{tikzcd}$$

Since $F \rightarrow F_i$ and $E \rightarrow E_i$ are local weak equivalences by (i), the vertical maps in the above are local weak equivalences. Moreover, any $F_i \rightarrow F_{i+1}$ factors through $L_f F$ and therefore we have a local weak equivalence $L_f F \simeq \operatorname{hocolim}_\kappa F_i$.

(iii) The size issues differ from the argument in [CS06]. There a countable number of steps is used, and the set of detectors is used to make sure countably many steps are enough. In an arbitrary topos, this does not work, since a set of detectors in the sense of [CS06, Definition 1.1] is the same as a set of compact generators in the sense of [Jar09]. However, we can choose a cardinal in a similar way as [GJ98]: We let $f : A \rightarrow B$ be a cofibration of simplicial sheaves and assume L_f is the f-localization functor constructed in [GJ98, Theorem 4.4]. Then let α be a cardinal which is an upper bound for the cardinality of the sets of morphisms of the site T and all sets of sections of the simplicial sheaf B. The cardinal up to which the construction in (ii) has to be iterated is any cardinal κ with $\kappa > 2^\alpha$. □

Remark 3.5. *Note that the above construction is functorial since we have only used universal constructions such as the localization augmentation $F \rightarrow L_f F$ and pushouts along this morphism.*

3.3. **Fibrewise Localization via Classifying Spaces:** We return to the remark made earlier that in the locally trivial case, it is quite easy to construct the fibrewise version explicitly. If the fibre sequence $F \rightarrow E \rightarrow B$ is locally trivial, it is classified by a morphism $B \rightarrow B\,\mathrm{hAut}_\bullet F$. Composing with the morphism of classifying space induced from the coaugmentation, we obtain a morphism $B \to B\,\mathrm{hAut}_\bullet F \to B\,\mathrm{hAut}_\bullet LF$. Pulling back the universal LF-fibre sequence along this morphism produces an LF-fibre sequence $LF \to \overline{E} \to B$ over B, which is the fibrewise localization of the fibre sequence we started with. This implies that in the above situation any F-fibre sequence of simplicial sheaves $F \to E \to B$ can be mapped via a homotopy commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] \\ LF \arrow[r] & \overline{E} \arrow[r] & B \end{tikzcd}$$

to a fibre sequence over B, i.e. a fibrewise localization exists. Note that we can get such a morphism between the classifying spaces by applying Proposition 3.4 to the universal fibre sequence $F \to E^f F \to B^f F$. Then we obtain a new fibre sequence $LF \to \overline{E^f F} \to B^f F$ which then is classified by a morphism of the classifying spaces $B^f F \to B^f LF$.

3.4. **Some Consequences:** The existence of a fibrewise localization functor implies the following two results from [DF96, Section 1.H]. It also yields the characterization of nullification functors given later on.

Corollary 3.6. *If $F \to E \to B$ is a fibre sequence with $L_f F \simeq *$ and B path-connected, then $L_f(p) : L_f E \to L_f B$ is a homotopy equivalence, and $* \to L_f E \to L_f B$ is a fibre sequence.*

Note that if we apply the pointed localization, we have to assume ΩB path-connected. For the unpointed version, the unconditional statement obtains, since the unpointed localization functor localizes the induced fibre sequences over each connected component of B. Of course this statement can also be proven without the use of fibrewise localization, cf. [MV99, Example 3.2.3]. Note however that in the above corollary there is no restriction like being locally trivial in the Nisnevich topology, the only restriction is that $F \to E \to B$ is a fibre sequence.

Corollary 3.7. *Let L_W be the W-nullification functor with respect to any space W. Let X be a pointed path-connected space. Then we have*

$$L_W A_{L_W} X \simeq *,$$

where $A_{L_W} X$ is the homotopy fibre of the nullification $X \to L_W X$.

The proofs carry over verbatim from [DF96, Section 1.H], as we have established all the necessary facts about the nullification functors for simplicial sheaves.

4. Characterizing Nullification Functors

In this section, we discuss properties which distinguish nullification functors from general localization functors. A characterization result for nullification functors of simplicial sets was given in [BF03, Theorem 2.1]. We will show in this section, that the result also holds for simplicial sheaves. The main technical tool are the fibrewise localizations from the previous section.

The assumption that $f : X \to Y$ is null-homotopic, i.e. the localization functor L_f is equivalent to a nullification $L_{*\to W}$, is essential in the characterization of local fibrations. Theorem 4.7 provides some evidence for this.

We first discuss the two notions of behaviour of fibre sequences under localizations. Recall from Section 2 that there were two approaches to localization: One via an axiomatic definition of localization functors, and another one via the definition of an f-local model structure. These two approaches have a different terminology in dealing with local fibre sequences. Looking at the localization functors approach, the natural question to ask is if a fibre sequence is preserved by a localization functor L_f. On the other hand, the natural question for the model category approach is rather if a given sequence is a fibre sequence in the local model structure, using [Hov98, Definition 6.2.6]. Although it seems almost clear that these two terminologies basically mean the same thing, we still give a proof of the fact. This fact also generalizes to homotopy pullbacks in case the local model structure is proper.

4.1. **Localization of Fibre Sequences:** In this paragraph, we show that to determine if $F \to E \to B$ is a fibre sequence in a localized model structure, it suffices to check if $LE \to LF \to LB$ is also a fibre sequence. A similar statement holds for homotopy pullbacks and their localizations. Note, that the fibre sequence statement does not depend on properness of the local model structure, since the localization of the fibre sequence automatically yields morphisms between fibrant objects. However, the result for homotopy pullbacks is not unconditional, as we need properness of both $\mathcal{C}$ and its Bousfield localization $L_f\mathcal{C}$ to even talk about homotopy pullbacks. Therefore the homotopy pullback part seems to depend on the fact that L_f is a nullification.

We begin with a definition of what it means for a fibre sequence resp. a homotopy pullback to be preserved by a localization functor.

Definition 4.1. *Following* [BF03] *we say that a fibre sequence $F \to E \to B$ is preserved by a localization L_f if applying L_f to p yields a morphism of fibre sequences:*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ LF & \longrightarrow & LE & \longrightarrow & LB \end{array}$$

Similarly, a homotopy pullback is preserved by the localization L_f if applying L_f to the homotopy pullback diagram yields a (homotopy) commutative diagram of homotopy pullbacks:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\
& L_f X \arrow[rr] \arrow[dd] & & L_f Y \arrow[dd] \\
Z \arrow[rr] \arrow[dr] & & W \arrow[dr] & \\
& L_f Z \arrow[rr] & & L_f W
\end{tikzcd}$$

The next proposition shows that being an f-local fibre sequence is the same as being preserved by a localization.

Proposition 4.2. *Let $p : F \to E \to B$ be a fibre sequence in a proper model category $\mathcal{C}$. Let $f : A \to B$ be a morphism with associated localization functor L_f. Then p is preserved by L_f if and only if p is a fibre sequence in the f-local model structure.*

Proof. Assume that p is preserved by L_f. Then we have a commutative diagram of fibre sequences:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d] & B \arrow[d] \\
LF \arrow[r] & LE \arrow[r] & LB
\end{tikzcd}$$

The morphism f is equivariant for the holonomy action of ΩB, because the functor L_f is continuous, and preserves products, cf. Theorem 2.8 and Proposition 2.10. The fibre sequence $LF \to LE \to LB$ is a fibre sequence in the f-local model structure since it is a non-local fibre sequence and both LE and LB are local, cf. [Hir03, Proposition 3.3.16]. Therefore $F \to E \to B$ is also a fibre sequence in the local model structure.

Conversely, let p be an f-local fibre sequence. Then there is an f-local fibre sequence $X \to Y \to Z$ where $q : Y \to Z$ is an f-local fibration of f-local fibrant objects. Therefore q is homotopy equivalent to $L_f p$ and we have $\operatorname{hofib} L_f p \simeq X \simeq L_f \operatorname{hofib} p$. The latter weak equivalence follows since X is f-locally equivalent to F. So p is preserved by L_f. $\square$

The following proposition shows that a homotopy pullback is preserved by localization exactly when it is a homotopy pullback in the local model structure. This generalizes the previous statement on fibre sequences under the condition that both the model category $\mathcal{C}$ and its Bousfield localization $L_f\mathcal{C}$ are proper.

Proposition 4.3. *Let the following square be a homotopy pullback in a proper model category $\mathcal{C}$:*

$$\begin{tikzcd}
A \arrow[r, "\phi_0"] \arrow[d, "g"'] & X \arrow[d, "h"] \\
B \arrow[r, "\phi_1"'] & Y.
\end{tikzcd}$$

Assume the Bousfield localization $L_f\mathcal{C}$ is also proper. Then the above diagram is a homotopy pullback in the f-local model structure if and only if its localization is a homotopy pullback in $\mathcal{C}$.

Proof. We will occasionally refer to the pullback diagram as $\phi : g \to h$.

(i) Assume the square above is an f-local homotopy pullback. We can assume without loss of generality that h is an f-local fibration. To see this, consider the following diagram obtained by factoring h into the f-local trivial cofibration a and the f-local fibration c:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "b"] \arrow[dd, "g"'] & & X \arrow[dr, "{a,\simeq_f}"] \arrow[dd] & \\
& B \times_Y Z \arrow[rr, "h"] \arrow[dl] & & Z \arrow[dl, "c"] \\
B \arrow[rr] & & Y &
\end{tikzcd}$$

By assumption the morphism b is an f-local weak equivalence. As we also assumed functorial factorizations, this diagram continues to commute after f-localization, and the f-local weak equivalences descend to weak equivalences between the localized objects [Hir03, Theorem 3.2.18]. So the upper square is a homotopy pullback. It remains to show that the lower square is, then by the homotopy pullback lemma the localization $L_f g \to L_f h$ is a homotopy pullback. This argument uses properness of $L_f\mathcal{C}$ since the homotopy pullback lemma [GJ99, Lemma II.8.22] only holds in a proper model category.

(ii) Let h be an f-local fibration, and consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & L_f X \arrow[d] \\
B \arrow[r] & Y \arrow[r] & L_f Y.
\end{tikzcd}$$

The outer rectangle is an f-local homotopy pullback by the homotopy pullback lemma [GJ99, Lemma II.8.22], since the left-hand square is an f-local homotopy pullback by assumption, and the right-hand square is a homotopy pullback by [Hir03, Proposition 3.4.8(1)]. By the characterization of f-local homotopy pullbacks, the morphism $A \to L_f X \times_{L_f Y} L_f B$ is an f-local weak equivalence.

Since we assumed h to be an f-local fibration which therefore is preserved by L_f, cf. Proposition 4.2, we can assume that $L_f h$ is a fibration, and therefore an f-local fibration [Hir03, Proposition 3.3.16(1)]. By properness the following pullback is also an f-local homotopy pullback:

$$\begin{tikzcd}
L_f B \times_{L_f Y} L_f X \arrow[r] \arrow[d] & L_f X \arrow[d, "L_f h"] \\
L_f B \arrow[r, "L_f \phi_1"'] & L_f Y
\end{tikzcd}$$

By 2-out-of-3, all morphisms in the composition $A \to L_f A \to L_f B \times_{L_f Y} L_f X$ are f-local weak equivalences, where the latter map is the universal one from the definition of pullbacks. By assumption, $L_f h$ is an f-local fibration, so is $L_f B \times_{L_f Y} L_f X \to L_f B$, and in particular $L_f B \times_{L_f Y} L_f X$ is f-local. Using again [Hir03, Theorem 3.2.18], $L_f A \to L_f B \times_{L_f Y} L_f X$ is a weak equivalence, which readily implies (using properness) that $L_f g \to L_f h$ is a homotopy pullback.

(iii) Assume now $L_f g \to L_f h$ is a homotopy pullback. Factor h into an f-local trivial cofibration a and an f-local fibration c. We need to prove that the pullback map $b : A \to B \times_Y X$ is an f-local weak equivalence. Consider the following diagram:

$$\begin{tikzcd}
L_f A \arrow[rr] \arrow[dr, "L_f b"] \arrow[ddd, "L_f g"'] & & L_f X \arrow[dr, "L_f a"] \arrow[ddd, "L_f h"] & \\
& L_f(B \times_Y Z) \arrow[rr] \arrow[d, "f"] & & L_f Z \arrow[d, "d"] \\
& L_f B \times_{L_f Y} \tilde{X} \arrow[rr] \arrow[dl] & & \tilde{X} \arrow[dl, "e"] \\
L_f B \arrow[rr, "L_f \phi_1"'] & & L_f Y &
\end{tikzcd}$$

Since we can not guarantee that $L_f c$ is again an f-local fibration, we have to factor it as f-local trivial cofibration d followed by the f-local fibration e. We have by construction that the square $\phi_1^* c \to c$ is an f-local homotopy pullback, and using (ii), the square $L_f(\phi_1^* c) \to L_f c$ is a homotopy pullback. Therefore $f : L_f(B \times_Y Z) \to L_f B \times_{L_f Y} \tilde{X}$ is a weak equivalence. By assumption, $L_f g \to L_f h$ is a homotopy pullback, so the weak equivalence $d \circ L_f a$ induces a weak equivalence $f \circ L_f b$, which by 2-out-of-3 implies that $L_f b$ is a weak equivalence. Both $L_f A$ and $L_f(B \times_Y Z)$ are f-local, so employing [Hir03, Theorem 3.2.18] again, we find that $b : A \to B \times_Y Z$ is an f-local weak equivalence, which proves the claim.

□

Remark 4.4. *Note that the properness hypothesis in the above result is really necessary: If any homotopy pullback in the proper model category $\mathcal{C}$ with one morphism an f-local fibration is preserved by f-localization, then the Bousfield localization $L_f\mathcal{C}$ is already proper. The following is a short argument for this.*

Let $\mathcal{C}$ be a proper model category, and let the following homotopy pullback in $\mathcal{C}$ be given:

$$\begin{tikzcd}
A \arrow[r, "\phi_0"] \arrow[d, "g"'] & X \arrow[d, "h"] \\
B \arrow[r, "\phi_1"'] & Y.
\end{tikzcd}$$

Let ϕ_1 be an f-local weak equivalence, let h be an f-local fibration. We have assumed that the homotopy pullback is preserved by f-localization. The morphism $L_f\phi_1$ is a weak equivalence, and by properness of $\mathcal{C}$, the morphism $L_f\phi_0$ is also a weak equivalence. Therefore $\phi_0 : A \to X$ is an f-local weak equivalence, and we have proved properness of the f-local model structure.

The argument that a homotopy pullback with one weak equivalence has another parallel weak equivalence is as follows: Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "d"'] & X \arrow[d, "c"] \\ B \times_Y \tilde{X} \arrow[r, "k"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ B \arrow[r, "i"'] & Y \end{tikzcd}$$

We assume the outer rectangle is a homotopy pullback. Then we consider a factorization of $X \to Y$ as a trivial cofibration c and a fibration p. We assume i is a weak equivalence and we want to show j is a weak equivalence. Then k is a weak equivalence by properness. Since the rectangle is a homotopy pullback, the morphism $d : A \to B \times_Y \tilde{X}$ is a weak equivalence. By 2-out-of-3 and since c is a weak equivalence we obtain that j is a weak equivalence.

4.2. **Characterizing Nullifications:** This paragraph is concerned with special properties that distinguish nullification functors, i.e. localizations with respect to *null-homotopic* morphisms $W \to *$, from localizations with respect to general morphisms $f : A \to B$. This is a generalization of results that appeared in [BF03] to the case of simplicial sheaves. The first is a simple lemma appearing as [BF03, Lemma 1.7].

Lemma 4.5. *Let T be a Grothendieck site, and let $F \to E \xrightarrow{p} B$ be a fibre sequence in $\Delta^{op}\mathcal{S}hv(T)$. Let L_f be an arbitrary localization functor on $\Delta^{op}\mathcal{S}hv(T)$. Consider a fibrewise localization $LF \to \overline{E} \xrightarrow{\overline{p}} B$. Then L_f preserves p if and only if L_f preserves $\overline{p}$.*

Proof. The definition of fibrewise localization, cf. Definition 3.1, requires that the morphism $a : E \to \overline{E}$ is an f-local weak equivalence, hence $L_f a : L_f E \to L_f\overline{E}$ is a weak equivalence and therefore $\operatorname{hofib} p \simeq \operatorname{hofib} \overline{p}$. □

We need yet another lemma on ladder diagrams of simplicial sheaves, which is derived from the corresponding fact on simplicial sets:

Lemma 4.6. *Let T be a site, and let a commutative ladder diagram of fibre sequences be given:*

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "f"'] & E_1 \arrow[r] \arrow[d, "e"'] & B_1 \arrow[d, "b"] \\ F_2 \arrow[r] & E_2 \arrow[r] & B_2 \end{tikzcd}$$

Assume that f and b are weak equivalences. Then e is also a weak equivalence.

Proof. Fibre sequences are preserved by passing to points, cf. [Wen09, Proposition 3.17]. Also weak equivalences are determined on points by definition. It therefore suffices to prove the above assertion for simplicial sets.

If we consider the above diagram in the category of simplicial sets, then we can assume without loss of generality that $B_1 \to B_2$ is a weak equivalence of *connected* simplicial sets. Then we have a diagram of the corresponding long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(F_1) \arrow[r] \arrow[d, "\pi_n(f)"'] & \pi_n(E_1) \arrow[r] \arrow[d, "\pi_n(e)"'] & \pi_n(B_1) \arrow[r] \arrow[d, "\pi_n(b)"] & \cdots \\ \cdots \arrow[r] & \pi_n(F_2) \arrow[r] & \pi_n(E_2) \arrow[r] & \pi_n(B_2) \arrow[r] & \cdots \end{tikzcd}$$

Using the five lemma together with the fact that $\pi_n(f)$ and $\pi_n(b)$ are isomorphisms for all $n \geq 0$, we obtain that $\pi_n(e)$ is an isomorphism for all $n \geq 0$. Hence e is a weak equivalence. □

The following characterization of nullification functors was proved for the case of simplicial sets resp. topological spaces in [BF03, Thm. 2.1]. Using the elementary facts and the existence of fibrewise localization, we find that it holds for any model category of simplicial sheaves.

Theorem 4.7. *Let T be a Grothendieck site, and let $f : X \to Y$ be a morphism of simplicial sheaves in $\Delta^{op}\mathcal{S}hv(T)$. Denoting by L_f the localization functor, consider the following statements:*

(i) L_f is equivalent to a nullification, i.e. there exists a simplicial sheaf W such that L_f and $L_{*\to W}$ have the same local spaces resp. induce Quillen-equivalent Bousfield localizations.

(ii) If in any fibre sequence $F \to E \xrightarrow{p} B$ both F and B are local, then so is E.

(iii) Every fibre sequence with B local is preserved by L_f.

(iv) For every space Z the space $L_f A_{L_f} Z$ is contractible.

Then (ii), (iii) and (iv) are equivalent, and (i) implies (ii). If we assume that X and Y are π_0-connected, i.e. the sheaves $\pi_0(X)$ and $\pi_0(Y)$ are trivial, then (iv) implies (i).

Proof. Note that L_f and L_g for $g : * \to W$ have the same local spaces if and only if the corresponding local model structures are Quillen equivalent. This can be seen from the following argument: The local model categories have the same underlying categories, namely the category $\Delta^{op}\mathcal{S}hv(T)$. The cofibrations in both model structures are the same, i.e. the monomorphisms of simplicial sheaves. Then we have that $h : X \to Y$ is a f-local weak equivalence if and only if $L_f h : L_f X \to L_f Y$ is a weak equivalence if and only if $L_g h : L_g X \to L_g Y$ is a weak equivalence if and only if $h : X \to Y$ is a g-local weak equivalence. The outer equivalences follow from [Hir03, Theorem 3.2.18], the inner equivalence is in turn equivalent to the assertion that L_f and L_g have the same local spaces. Therefore both local model structures have the same cofibrations and weak equivalences, therefore also the same fibrations. Hence the identity on $\Delta^{op}\mathcal{S}hv(T)$ is both a left and right Quillen functor.

Now we prove the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(i) implies (ii) is elementary fact (e.6) in [DF96, Section 1.A.8]. To get rid of the connectedness assumption, we have to use a slightly extended argument. This is proved as follows: Let $F \to E \to B$ be any fibre sequence. Then the following is also a fibre sequence $\mathbf{Hom}(W, F) \to \mathbf{Hom}(W, E) \to \mathbf{Hom}(W, B)$ for any cofibrant W, [MV99, Lemma 2.1.19]. Moreover, this holds for any choice of base points, so it does not need connectedness. For a pointed and *connected* space X, we have that X is W-local if and only if $\mathbf{Hom}_*(W, X) \simeq *$. If this holds for B and F, then by the long exact homotopy sequence associated to the $\mathbf{Hom}$-sequence, this also holds for E. For the general case, we note that the internal hom-functor is coaugmented $X \to \mathbf{Hom}(W, X)$ by the constant maps. Therefore we obtain a morphism of fibre sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] \\
\mathbf{Hom}(W,F) \arrow[r] & \mathbf{Hom}(W,E) \arrow[r] & \mathbf{Hom}(W,B).
\end{tikzcd}$$

Now X is W-local if and only if $X \to \mathbf{Hom}(W, X)$ is a weak equivalence. Assuming F and B are W-local in the above diagram, the left and right vertical morphisms are weak equivalences. This implies that also the morphism $E \to \mathbf{Hom}(W, E)$ is a weak equivalence by Lemma 4.6. Therefore E is W-local.

For (ii) implies (iii) it suffices to show by Lemma 4.5 that the fibrewise localization $L_f F \to \overline{E} \xrightarrow{\overline{p}} B$ is preserved. But this follows from (ii), since both B and $L_f F$, hence by assumption also $\overline{E}$ are local. Note that this is the place where we use the existence of fibrewise localization for simplicial sheaves!

(iii) implies (iv): By assumption the fibre sequence $A_{L_f} Z \to Z \to L_f Z$ is preserved by L_f, so $L_f A_{L_f} Z$ is the fibre of the homotopy equivalence $L_f Z \to L_f L_f Z$, hence contractible.

(iv) implies (ii): Let $F \to E \to B$ be a fibre sequence with F and B local. Then $E \to B$ factors as $l_E : E \to L_f E$ and $q : L_f E \to B$, by (homotopy) universality of L_f. Consider the following diagram:

$$\begin{tikzcd}
F \arrow[r, "m"] \arrow[d] & \mathrm{hofib}(q) \arrow[r] \arrow[d] & * \arrow[d] \\
E \arrow[r, "l_E"'] & L_f E \arrow[r, "q"'] & B
\end{tikzcd}$$

From [Hir03, Proposition 3.3.16], a fibration between local objects is an f-local fibration. Therefore $\mathrm{hofib}(q)$ is local. This also implies that $\mathrm{hofib}(m)$ is local, since F is local by assumption. Now we apply the (non-local!) homotopy pullback lemma: The outer rectangle is a homotopy pullback, since $F \to E \to B$ is a fibre sequence. The same holds for the right square. So the left square is a homotopy pullback and therefore $A_{L_f}E = \mathrm{hofib}(l_E) \simeq \mathrm{hofib}(m)$, which is local. Hence $A_{L_f}E \simeq L_f A_{L_f}E$ is contractible by assumption. Then l_E is a weak equivalence, so E is already local. The above is independent of the choice of base points, since the homotopy pullback argument and the comparison of the spaces can be done at the points of the topos, and we apply an *unpointed* localization functor, which localizes *all* components of a space.

(iv) implies (i): We start with the remark, that if $f : X \to Y$ is a morphism of π_0-connected simplicial sheaves, then $L_f Z$ is π_0-connected for any π_0-connected simplicial sheaf Z. This can be seen by the direct construction of L_f e.g. in [GJ98, Section 4]. The functor L_f is constructed as transfinite composition of pushouts

$$\begin{tikzcd} X \times \Delta^n \cup_{X \times \partial\Delta^n} Y \times \partial\Delta^n \arrow[r] \arrow[d] & L_f^\alpha Z \arrow[d] \\ Y \times \Delta^n \arrow[r] & Z', \end{tikzcd}$$

where $L_f^{\alpha+1}Z = \mathrm{Ex}_T^\infty Z'$ and Ex_T^∞ is a functorial fibrant replacement for the simplicial model structure on $\Delta^{op}\mathcal{S}hv(T)$. These pushouts can be computed at the points, and in particular $x^*(L_f^\alpha Z)$ and $x^*(Y \times \Delta^n)$ are connected simplicial sets for any point x of the topos. The pushout Z' of these simplicial sets is again connected, and a fibrant replacement does not change the homotopy types at the points. Therefore $L_f Z$ is π_0-connected for any π_0-connected simplicial sheaf Z, in particular for X and Y.

Now we can proceed with the argument from [BF03, Theorem 2.1]: From Proposition 2.11 we know that $L_f(X \vee Y) \simeq L_f(L_f X \vee L_f Y)$. Define $W = A_{L_f}X \vee A_{L_f}Y$. By assumption and the formula above, $L_f W \simeq *$, so any L_f-local space is also W-local.

Letting $g : W \to *$, we have $* \simeq L_g W \simeq L_g(L_g A_{L_f}X \vee L_g A_{L_f}Y)$, the latter weak equivalence again from Proposition 2.11. From the definition of locality, $L_g Z \simeq *$ if and only if $Z \to *$ is an g-local weak equivalence if and only if $\mathbf{Hom}_*(Z, P) \simeq *$ for any g-local space P. Note that this uses the π_0-connectedness of Z. It follows that

$$\mathbf{Hom}_*(L_g A_{L_f}X \vee L_g A_{L_f}Y, L_g A_{L_f}X) \simeq *.$$

Because the morphisms $*, \mathrm{id} \vee * : L_g A_{L_f}X \vee L_g A_{L_f}Y \to L_g A_{L_f}X$ are then homotopic, this implies $L_g A_{L_f}X \simeq *$. The same holds for $A_{L_f}Y$. Consider the commutative diagram

$$\begin{tikzcd} L_g X \arrow[r, "L_g f"] \arrow[d] & L_g Y \arrow[d] \\ L_g L_f X \arrow[r, "L_g L_f f"'] & L_g L_f Y. \end{tikzcd}$$

Both vertical arrows are homotopy equivalences, since their homotopy fibres are contractible, cf. Corollary 3.6. $L_f f$ is a homotopy equivalence, because it is an f-local equivalence of f-local spaces, cf. [Hir03, Theorem 3.2.18]. Therefore, $L_g L_f f$ is a weak equivalence and by 2-out-of-3 also $L_g f$.

Thus every W-local space is L_f-local and the two localization functors agree up to homotopy. □

Remark 4.8. *It seems plausible that the implication from (iv) to (i) also holds under weaker connectedness assumptions than used above. In view of* [MV99, Corollary 2.3.22], *we know that for an interval I on site, the localization $L_I X$ of a π_0-connected X is again π_0-connected. The interval $\mathbb{A}^1$ in Sm_S is not π_0-connected, so the π_0-connectedness of the spaces X and Y in Theorem 4.7 is certainly not necessary.*

5. Characterizing Local Fibrations

We now finally give the criterion for a fibre sequence to be preserved by a nullification. There is one preparatory lemma we need, which is a sheaf version of [BF03, Lemma 3.2]. All we need for the simplicial set proof of this lemma to work also for simplicial sheaves is Theorem 4.7.

Lemma 5.1. *Let T be a Grothendieck site, and let L_f be a nullification functor on $\Delta^{op}Shv(T)$. Consider the following diagram of simplicial sheaves, in which the square is a homotopy pullback and the lower sequence is a fibre sequence with local base C:*

$$\begin{tikzcd} D \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] & \\ A \arrow[r] & B \arrow[r, "r"] & C \end{tikzcd}$$

Then there is a homotopy equivalence of homotopy fibres

$$\mathrm{hofib}(Lq) \simeq \mathrm{hofib}(Lp),$$

and p is preserved by L if and only if q is.

Proof. Consider the following diagram:

$$\begin{tikzcd} D \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r] \arrow[d] & B \arrow[d, "r"] \\ * \arrow[r] & C \end{tikzcd}$$

As $A \to B \to C$ is a fibre sequence, the lower square is a homotopy pullback. By the homotopy pullback lemma [GJ99, Lemma II.8.22], the sequence $D \to E \xrightarrow{r \circ p} C$ is also a fibre sequence. Since L_f is a nullification and C is f-local, the two fibre sequences r and $r \circ p$ are preserved by L, cf. Theorem 4.7. Then we consider the localized diagram

$$\begin{tikzcd}
L_f D \arrow[r] \arrow[d, "L_f q"'] & L_f E \arrow[d, "L_f p"] \\
L_f A \arrow[r] \arrow[d] & L_f B \arrow[d, "L_f r"] \\
* \arrow[r] & L_f C
\end{tikzcd}$$

By the pullback lemma and properness of the (non-local!) model structure, the upper square is a homotopy pullback. This proves the assertion $\mathrm{hofib}(L_f q) \simeq \mathrm{hofib}(L_f p)$. Also, $\mathrm{hofib}(p) \simeq \mathrm{hofib}(q)$, because the non-localized upper square was assumed to be a homotopy pullback, and this implies that p is preserved by L precisely when q is. $\square$

The following theorem provides a criterion for locality of fibre sequences which appeared in [BF03, Theorem 4.1] for the case of simplicial sets resp. topological spaces. Although the statement below is more general and applies to general model categories of simplicial sheaves, the proof is still almost the original one. Note that we need the hypothesis that $f : X \to Y$ is null-homotopic in order to apply the previous lemmas.

Theorem 5.2. *Let T be a site, and let $f : X \to Y$ be a* null-homotopic *morphism of simplicial sheaves in $\Delta^{op}\mathcal{S}hv(T)$. Furthermore, let $F \to E \xrightarrow{p} B$ be a fibre sequence.*

Let $L_f F \to \overline{E} \to B$ be a fibrewise localization of p, and let $L_f F \to E' \to A_{L_f} B$ be the pullback of this fibrewise localization of p to $A_{L_f} B = \mathrm{hofib}(B \to L_f B)$. Then the following are equivalent:

(i) The fibre sequence p is an f-local fibre sequence,
(ii) the fibre sequence p is preserved by L_f,
(iii) $E' \simeq L_f F \times A_{L_f} B$ and therefore there is an f-local weak equivalence $E' \simeq L_f F$, and
(iv) the following composition of morphisms is null-homotopic:

$$A_{L_f} B \to B \xrightarrow{p} B^f F \to B^f L_f F.$$

The first morphism in the above composition is the natural inclusion of the homotopy fibre $A_{L_f} B = \mathrm{hofib}(B \to L_f B)$ into B, then follows the morphism $B \to B^f F$ classifying the fibre sequence p, and finally the morphism $B^f F \to B^f L_f F$ induced by fibrewise localization.

Proof. (i) and (ii) are equivalent by Proposition 4.2. (iii) and (iv) are equivalent, as (iv) describes via morphisms on classifying spaces what the construction in (iii) does on the level of fibre sequences.

We write $L = L_f$ omitting f from the notation. We know from Lemma 4.5 that p is preserved by L if and only if $\overline{p}$ is. Applying Lemma 5.1 to the diagram

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d, "q"'] & \overline{E} \arrow[d, "\overline{p}"] & \\
A_L B \arrow[r, "d_B"'] & B \arrow[r, "l_B"'] & LB
\end{tikzcd}$$

we see that $\overline{p}$ is preserved if and only if q is. Since LA_LB is contractible, q is preserved if and only if there is a map of fibre sequences:

$$\begin{tikzcd} LF \arrow[r] \arrow[d, "="'] & E_1 \arrow[r] \arrow[d] & A_LB \arrow[d] \\ LF \arrow[r, "\simeq"'] & LE_1 \arrow[r] & * \end{tikzcd}$$

Such a morphism of fibre sequences can only exist if the upper fibre sequence is trivial, i.e. there is a weak equivalence $E_1 \simeq LF \times A_LB$. By the Dold theorem 5.4 discussed in the next paragraph, the fibre sequence q is equivalent to a trivial one if and only if the inclusion of the fibre $LF \hookrightarrow E_1$ has a left homotopy inverse. Therefore q is preserved by L if and only if q is equivalent to a trivial fibre sequence. □

We can even prove a more general result describing sharp conditions under which a homotopy pullback of simplicial sheaves descends to a nullification model structure.

Corollary 5.3. *Let T be a site and let a homotopy pullback of simplicial sheaves in $\Delta^{op}\mathcal{S}hv(T)$ be given:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] & Y \arrow[d, "q"] \\ Z \arrow[r, "p"'] & W. \end{tikzcd}$$

Furthermore, let $f : A \to B$ be a null-homotopic *morphism of simplicial sheaves in $\Delta^{op}\mathcal{S}hv(T)$. Assume that the f-local model structure is proper. Then the above homotopy pullback is an f-local homotopy pullback if and only if the fibrewise localization of the fibre sequence* $\mathrm{hofib}\, q \to Y \to W$ *splits over $A_{L_f}Z$ if and only if the fibrewise localization of the fibre sequence* $\mathrm{hofib}\, p \to Z \to W$ *splits over $A_{L_f}Y$.*

Proof. The proof is similar to the one of Theorem 5.2. Assume the homotopy pullback is f-local. By properness, it suffices to check one factorization, i.e. one of the conclusions, cf. [GJ99, Lemma II.8.18]. Since the diagram is a homotopy pullback, we have a fibre sequence $\mathrm{hofib}\, q \to Y \to W$. We also consider the local fibre sequence $\mathrm{hofib}\, Lq \to LY \to LW$. It is not necessary that both agree, i.e. that L preserves the fibre sequence q. We pull these two fibre sequences back along $p : Z \to W$. Since we assumed the diagram is a local homotopy pullback, the pullback of $Lq : LY \to LW$ is locally weakly equivalent to $j : X \to Z$, which is the pullback of the fibre sequence q. Therefore the pullback of the fibre sequence q along $Z \to W$ is preserved by L, and the equivalence of (ii) and (iii) in Theorem 5.2 implies that the pullback of q along $Z \to W$ splits over the non-local part of Z. This is what we wanted to prove.

Assume that the fibrewise localization of $\mathrm{hofib}\, q \to Y \to W$ splits over $A_{L_f}Z$. Then the pullback of $\mathrm{hofib}\, q \to Y \to W$ to Z is an f-local fibre sequence by Theorem 5.2. This implies that after replacing q by a local fibration $\tilde{Y} \to W$, there is a local weak equivalence $\tilde{Y} \times_W Z \simeq X$. Therefore the homotopy pullback above is also a local homotopy pullback. □

5.1. **The Dold Theorem:** For the proof of Theorem 5.2, we needed a result characterizing trivial fibre sequences. This was originally proved by Dold [Dol63], and we show that it also holds in categories of simplicial sheaves. This is again proved by looking at the points of the topos.

Proposition 5.4. *Any fibre sequence $F \to E \to B$ in a category $\Delta^{op}\mathcal{S}hv(T)$ of simplicial sheaves is trivial in the sense that there is a weak equivalence $E \simeq F \times B$, if and only if the inclusion of the fibre admits a left homotopy inverse.*

Proof. One direction is easy: Let $F \to E \to B$ be a fibre sequence which is equivalent to a trivial one. We consider a nice model, i.e. a fibration $p : E \to B$ of fibrant objects and a weak equivalence $F \to p^{-1}(*)$, and we assume F fibrant. By the assumption that p is trivial, there is a homotopy equivalence $E \to F \times B$, which composed with the projection $F \times B \to F$ provides the left homotopy inverse of $i : F \to E$.

For the converse, let a fibration of fibrant objects $p : E \to B$ be given, together with a morphism $l : E \to F = p^{-1}(*)$. This yields a morphism $l \times p : E \to F \times B$. We assume that l is left homotopy inverse to $i : p^{-1}(*) \to E$ and we want to show that $l \times p$ is a weak equivalence. Assume for now that T has enough points. By definition of the injective model structure, it suffices to check this on points. Let x be any point of the topos, and consider $x^*(F) \to x^*(E) \to x^*(B)$. This is a fibre sequence by [MV99, Lemma 2.1.20]. Moreover, the morphism $x^*(l) : x^*(E) \to x^*(F)$ is a left homotopy inverse of $x^*(p)$. By the usual Dold theorem [Dol63, Theorem 6.1] the morphism $x^*(l \times p) = x^*(l) \times x^*(p)$ is a weak equivalence of simplicial sets. Therefore, $l \times p$ induces a weak equivalence for any point x of T and thus is a local weak equivalence.

More generally, since we allowed ourselves to choose a nice model, a weak equivalence can be check on sections: If $(l \times p)(U) : E(U) \to (F \times B)(U)$ is a weak equivalence for any $U \in T$, and all of E, B and F are fibrant, then $l \times p$ is a weak equivalence [MV99, Lemma 2.1.10]. This again follows from the usual Dold theorem for simplicial sets. The latter argument of course works for any topos, even the ones without points. □

5.2. **Pullback Stability:** In the following $L = L_W$ will be a nullification functor. The following is an immediate corollary to Theorem 5.2. Again the proof follows the proof of [BF03, Theorem 0.2].

Proposition 5.5. *Let T be a Grothendieck site, and let $L = L_W$ be a nullification functor on $\Delta^{op}\mathcal{S}hv(T)$. If a pointed fibre sequence $F \to E \xrightarrow{p} B$ of simplicial sheaves is preserved by L, then any pointed fibre sequence p_1 induced from p by pullback along $f : X \to B$ is also preserved by L.*

Proof. Let $F \to E \xrightarrow{p} B$ be a fibre sequence, and let $f : X \to B$ be a morphism of simplicial sheaves. Then the fibre sequence p is classified by a morphism $p : B \to B^f F$, and the pullback of p along $f : X \to B$ is classified by the composition $X \to B \to B^f F$. By Theorem 5.2, it suffices to show that if the composition

$$A_L B \to B \to B^f F \to B^f LF$$

is null-homotopic, then so is the composition

$$A_L X \to X \to B \to B^f F \to B^f LF.$$

This follows since we have a commuting square

$$\begin{tikzcd} A_L X \arrow[r] \arrow[d] & A_L B \arrow[d] \\ X \arrow[r] & B, \end{tikzcd}$$

which is obtained by taking the homotopy fibres of the coaugmentation morphisms $X \to LX$ resp. $B \to LB$. □

It is also possible under some connectedness assumptions to prove this result in a way similar to [BF03, Proposition 3.1].

6. Nullifications and Properness

Denis-Charles Cisinski has kindly pointed out to me that (iii) in the above Theorem 4.7 holds if we know that both the original and the local model category are proper. This basically raises the question of the relation between nullification functors L_f and properness of the f-local model category, which we partially answer below.

Cisinski's argument that Theorem 4.7 (iii) follows from properness runs something like this: Consider a local and fibrant space B and a simplicial fibre sequence $F \to E \xrightarrow{p} B$ over it. We want to show that F is the f-local homotopy fibre of p. Using properness of the f-local model structure, it suffices to take a factorization of $* \to B$ as an f-local trivial cofibration $c : * \to \tilde{B}$ and an f-local fibration $q : \tilde{B} \to B$, and to prove that the induced morphism $d : F \to \tilde{B} \times_B E$ is an f-local weak equivalence, cf. [GJ99, Section II.8]. The situation is depicted in the following diagram:

$$\begin{tikzcd} F \arrow[r, "d"] \arrow[d] & \tilde{B} \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r, "c"'] & \tilde{B} \arrow[r, "q"'] & B \end{tikzcd}$$

Since $\tilde{B}$ is f-locally contractible but also local, it is contractible, i.e. c is a (non-local) weak equivalence. Since the outer square is a (non-local) homotopy pullback by definition, the right one as well, so the left square is a (non-local) homotopy pullback. Then the (non-local) weak equivalence $* \to \tilde{B}$ pulls back to an (non-local) and therefore also f-local weak equivalence $d : F \to \tilde{B} \times_B E$. This follows from properness of the non-local model structure. The f-local homotopy fibre of p is $\tilde{B} \times_B E$ and we have shown it f-locally is the same as F. Therefore $F \to E \to B$ is an f-local fibre sequence.

Under suitable assumptions on connectedness of f, we now obtain the following corollary. Probably this can also be obtained under weaker assumptions, one of the most interesting cases being a morphism of representable objects $f : U_1 \to U_2$ in T.

Corollary 6.1. *Let T be a Grothendieck site, and let $f : X \to Y$ be a morphism of π_0-connected simplicial sheaves in $\Delta^{op}\mathcal{S}hv(T)$. If the f-local model structure on $\Delta^{op}\mathcal{S}hv(T)$ is right proper, then there exists a simplicial sheaf W such that the localization functors L_f and L_W are equivalent.*

Proof. Cisinski's argument above shows that if the f-local model structure is proper, then (iii) in Theorem 4.7 holds. By Theorem 4.7, the localization has to be equivalent to L_W for some simplicial sheaf W. □

Using the characterization of properness due to Cisinski in [Cis02], we can also prove the other direction:

Corollary 6.2. *Let T be a Grothendieck site, and let $f : X \to Y$ be a morphism of simplicial sheaves on T. If f is null-homotopic, then the local model structure $L_f\Delta^{op}\mathcal{S}hv(T)$ is proper.*

Proof. Using [Cis02, Theorem 4.8], it suffices to show that for a fibration $p : E \to B$ with fibrant base B, for any f-local weak equivalence $i : X \to Y$ and every morphism $u : Y \to B$, the morphism $j : E \times_B X \to E \times_B Y$ is also an f-local weak equivalence.

By Theorem 4.7, the fibre sequence $\mathrm{hofib}(p) \to E \to B$ is preserved by f-localization. By Proposition 5.5, the same also holds for the pulled-back fibre sequences $\mathrm{hofib}(p) \to E \times_B Y \to Y$ and $\mathrm{hofib}(p) \to E \times_B X \to X$. We therefore obtain a commutative ladder of fibre sequences:

$$\begin{array}{ccccc} L_f\,\mathrm{hofib}(p) & \longrightarrow & L_f(E \times_B X) & \longrightarrow & L_fX \\ \downarrow & & \downarrow & & \downarrow \\ L_f\,\mathrm{hofib}(p) & \longrightarrow & L_f(E \times_B Y) & \longrightarrow & L_fY. \end{array}$$

It is clear that the morphism on the fibres is a weak equivalence, and since i is an f-local weak equivalence, $L_fi : L_fX \to L_fY$ is also a weak equivalence. From Lemma 4.6, we conclude that $L_f(E \times_B X) \to L_f(E \times_B Y)$ is a weak equivalence. Therefore j is an f-local weak equivalence.

Note that all the results cited from Section 4 do not use properness of the local model structure, only properness of $\Delta^{op}\mathcal{S}hv(T)$. □

This provides another proof of the properness results from [MV99, Theorem 2.3.2] resp. [Jar00, Theorem A.5].

7. Application: Fibrations in $\mathbb{A}^1$-Homotopy Theory

In this section, we apply the localization theory developed earlier to discuss fibrations in $\mathbb{A}^1$-homotopy theory. Hence we specialize to the site $T = \mathrm{Sm}_k$ of smooth schemes over a field k equipped with the Zariski or Nisnevich topology. We obtain a model structure of simplicial sheaves $\Delta^{op}\mathcal{S}hv(\mathrm{Sm}_k)$, and apply a Bousfield localization to the scheme $\mathbb{A}^1_S$ considered as constant representable simplicial sheaf. More details on the construction of $\mathbb{A}^1$-homotopy theory can be found in [MV99].

Now recall from [Wen09], that for each simplicial sheaf F, there is a classifying space of locally trivial fibre sequences with fibre F. We denote this space by $B\,\mathrm{hAut}_\bullet(F)$, since [Wen09, Theorem 5.8] shows that this space can be constructed as the classifying space of the simplicial sheaf of monoids $\mathrm{hAut}_\bullet(F)$ of homotopy self-equivalences of F. We assume here that the fibre sequences considered are locally trivial in the Nisnevich topology.

The main general result is the following.

Theorem 7.1. *Let X be a cofibrant and $\mathbb{A}^1$-local fibrant simplicial sheaf on Sm_k. Then $B\,\mathrm{hAut}_\bullet(X)$ is $\mathbb{A}^1$-local if and only if the sheaf of homotopy self-equivalence groups $\pi_0(\mathrm{hAut}_\bullet(X))$ is strongly $\mathbb{A}^1$-invariant.*

Proof. (i) We first prove that the simplicial sheaf of monoids of homotopy self-equivalences $\mathrm{hAut}_\bullet(X)$ is fibrant and $\mathbb{A}^1$-local.

By [MV99, Lemma I.1.8], there is a fibration

$$\mathbf{Hom}(X,Y)\to Y$$

if Y is fibrant. Thus $\mathbf{Hom}(X,X)$ is fibrant if X is fibrant. The simplicial set $\mathrm{hAut}_\bullet(X)(U)$ is a union of connected components of $\mathbf{Hom}(X,X)(U)$. By 2-out-of-3 for weak equivalences a morphism $f: X\times U\to X\times U$ is a weak equivalence if it is homotopic to a morphism $f': X\times U\to X\times U$ which is a weak equivalence. Therefore $\mathrm{hAut}_\bullet(X)(U)$ consists exactly of the union of the components of $\mathbf{Hom}(X,X)(U)$ which contain weak equivalences.

Consider now the commutative diagram

$$\begin{tikzcd}
\mathrm{hAut}_\bullet(X)(U) \arrow[r] \arrow[d] & \mathrm{hAut}_\bullet(X)(U\times\mathbb{A}^1) \arrow[d] \\
\mathbf{Hom}(X,X)(U) \arrow[r] & \mathbf{Hom}(X,X)(U\times\mathbb{A}^1).
\end{tikzcd}$$

The vertical arrows are the inclusions as described above, and the lower horizontal morphism is a weak equivalence of simplicial sets since we noted that $\mathbf{Hom}(X,X)$ is $\mathbb{A}^1$-local. In particular, the lower morphism induces a bijection on the connected components. This bijection restricts to a bijection between the components consisting of weak equivalences: first, any morphism $f: X\times U\to X\times U$ is a retract of $f\times\mathrm{id}: X\times U\times\mathbb{A}^1\to X\times U\times\mathbb{A}^1$, therefore the preimage of a component in $\mathrm{hAut}_\bullet(X)(U\times\mathbb{A}^1)$ is in $\mathrm{hAut}_\bullet(X)(U)$. Similarly, if f is a weak equivalence, then $f\times\mathrm{id}$ is a weak equivalence. But then the morphism $\mathrm{hAut}_\bullet(X)(U)\to\mathrm{hAut}_\bullet(X)(U\times\mathbb{A}^1)$ is a weak equivalence because it is a bijection on connected components, and the connected components are connected components of the mapping spaces $\mathbf{Hom}(X,X)$, where we have a weak equivalence. This implies that $\mathrm{hAut}_\bullet(X)$ is $\mathbb{A}^1$-local if X is $\mathbb{A}^1$-local.

(ii) By [Mor06b, Theorem 3.46], $B\,\mathrm{hAut}_\bullet(X)$ is $\mathbb{A}^1$-local if and only if the sheaf of groups $\pi_0 L_{\mathbb{A}^1}\Omega B\,\mathrm{hAut}_\bullet X$ is strongly $\mathbb{A}^1$-invariant. The theorem follows, if we can prove that the obvious morphism

$$\mathrm{hAut}_\bullet X\to\Omega B\,\mathrm{hAut}_\bullet X\to L_{\mathbb{A}^1}\Omega B\,\mathrm{hAut}_\bullet X$$

induces an isomorphism of sheaves of groups π_0. But the obvious morphism

$$\mathrm{hAut}_\bullet X\to\Omega B\,\mathrm{hAut}_\bullet X$$

is already a weak equivalence of simplicial sheaves, because the stalks of $\mathrm{hAut}_\bullet X$ are monoids of homotopy self-equivalences of simplicial sets which are group-like. Therefore, the morphism induces weak equivalences on the stalks, cf. [Rud98, Corollary IV.1.68]. Therefore, $\Omega B\,\mathrm{hAut}_\bullet X$ is already $\mathbb{A}^1$-local, hence the localization $\Omega B\,\mathrm{hAut}_\bullet X \to L_{\mathbb{A}^1}\Omega B\,\mathrm{hAut}_\bullet X$ is a simplicial weak equivalence. □

This result has the following consequence:

Corollary 7.2. *Let X be a cofibrant, fibrant and $\mathbb{A}^1$-local simplicial sheaf on Sm_k for which $\mathrm{hAut}_\bullet(X)$ is strongly $\mathbb{A}^1$-invariant.*

Then we have the following statements:

(i) The universal fibre sequence

$$X \to B(*, \mathrm{hAut}_\bullet X, X) \to B\,\mathrm{hAut}_\bullet X$$

is $\mathbb{A}^1$-local.

(ii) Any Nisnevich locally trivial fibre sequence $F \to E \to B$ such that F has the $\mathbb{A}^1$-homotopy type of X is also $\mathbb{A}^1$-local.

(iii) Denoting by $\mathcal{H}^{\mathbb{A}^1}(Y,X)$ the pointed set of Nisnevich locally trivial fibre sequences over Y with fibre X up to $\mathbb{A}^1$-equivalence, we have a natural bijection

$$\mathcal{H}^{\mathbb{A}^1}(-,X) \cong [-, B\,\mathrm{hAut}_\bullet X]_{\mathbb{A}^1}$$

Proof. (i) The localization criterion Theorem 4.7 and Proposition 3.4 implies that the universal fibre sequence is $\mathbb{A}^1$-local if (a simplicial fibrant replacement of) the classifying space $B\,\mathrm{hAut}_\bullet X$ is $\mathbb{A}^1$-local. But $B\,\mathrm{hAut}_\bullet X$ is local since the conditions of Theorem 7.1 are satisfied.

(ii) follows from Proposition 5.5. Any Nisnevich locally trivial fibre sequence is a pullback of the universal fibre sequence with fibre F along some morphism $B \to B\,\mathrm{hAut}_\bullet F$. But from (i) it follows that the universal fibre sequence over $B\,\mathrm{hAut}_\bullet L_{\mathbb{A}^1}F \simeq B\,\mathrm{hAut}_\bullet X$ is $\mathbb{A}^1$-local.

For (iii) we first note that [Wen09, Theorem 5.8] yields a bijection

$$\mathcal{H}(-,X) \cong [-, B\,\mathrm{hAut}_\bullet X].$$

Since $B\,\mathrm{hAut}_\bullet X$ is $\mathbb{A}^1$-local, we also have a bijection

$$[-, B\,\mathrm{hAut}_\bullet X] \cong [-, B\,\mathrm{hAut}_\bullet X]_{\mathbb{A}^1}.$$

On the other hand, since $B\,\mathrm{hAut}_\bullet X$ is $\mathbb{A}^1$-local, the classifying morphism $B \to B\,\mathrm{hAut}_\bullet X$ factors up to homotopy through a morphism $L_{\mathbb{A}^1}B \to B\,\mathrm{hAut}_\bullet X$. By Theorem 4.7, we can hence assume that the fibre sequence classified by this consists of $\mathbb{A}^1$-local spaces. Since a morphism between local spaces is an $\mathbb{A}^1$-weak equivalence if and only if it is a simplicial weak equivalence, the two equivalence notions for fibre sequences coincide, and we have the final bijection $\mathcal{H}^{\mathbb{A}^1}(-,X) \cong \mathcal{H}(-,X)$. □

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