Contents

1. Hodge Theory

By a complex variety over we will mean a separated scheme of finite type over $\mathbb{C}$. The set of $\mathbb{C}$-valued points $X(\mathbb{C})$ has a natural structure of a complex manifold. By abuse of notation we will denote it simply by X. It will always be clear from the context whether we consider X as an abstract variety or as a complex manifold. In this section, we will consider a smooth projective complex variety X of dimension d. We want to extract as much information as we can from the cohomology of X. For most of the material of this section the reader can consult [GH94] of [Wel80].

First, we can consider the topological space underlying X. To it we can associate our first invariant: the singular or Betti cohomology, $H^*_{\text{sing}}(X,\mathbb{Z})$. It is a finite graded $\mathbb{Z}$-algebra. The definition of Betti cohomology only uses the topology of X and not the complex or algebraic structures. Thus it is a very crude invariant of the variety. For instance Betti cohomology can distinguish between $\mathbb{P}^1$ and an elliptic curve, but cannot tell one elliptic curve from another.

The next tool we can use is de Rham cohomology. We consider now the differential manifold underlying X. To it we can associate a tangent bundle TX and a cotangent bundle T^*X. This is a differentiable bundle. The differentiable sections of the exterior power $\bigwedge^n T^*X$ are called differential forms of degree n. The space of differential forms of degree n is denoted $A^n_{\mathbb{R}}(X)$. There is a differential $\mathrm{d}\colon A^n_{\mathbb{R}}(X) \longrightarrow A^{n+1}_{\mathbb{R}}(X)$ turning $A^*_{\mathbb{R}}(X) = \bigoplus_n A^n_{\mathbb{R}}(X)$ into a complex. The cohomology of this complex is the (differentiable) de Rham cohomology of X, $H^*_{\text{dR}}(X,\mathbb{R})$. There is a comparison isomorphism

$$H^*_{\text{sing}}(X,\mathbb{Z}) \otimes \mathbb{R} \simeq H^*_{\text{dR}}(X,\mathbb{R})$$

Still, de Rham cohomology is a crude invariant, because it only uses the structure of differentiable manifold of X, hence cannot tell one elliptic curve from another. But de Rham cohomology paves the road to include the algebraic structure and the complex structure in cohomology.

Fist we see how to use the algebraic structure. Let $K \subset \mathbb{C}$ be a subfield, for instance $K = \mathbb{Q}$, a number field, or $K = \mathbb{R}$. Assume that X_K is a smooth projective variety defined over K such that X is the complex variety associated to X_K. On X_K we can consider the algebraic de Rham complex $\Omega^*_{X_K}$. Then, the algebraic de Rham cohomology is defined as the (hyper-)cohomology of the algebraic de Rham complex

$$H^*_{\text{dR}}(X_K, K) = R\Gamma(X_K, \Omega^*_{X_K}).$$

There are comparison isomorphism

$$H^*_{\text{dR}}(X_K, K) \underset{K}{\otimes} \mathbb{C} \simeq H^*_{\text{dR}}(X,\mathbb{R}) \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq H^*_{\text{sing}}(X,\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}.$$

Imagine, for instance that $K = \mathbb{Q}$ and $X_{\mathbb{Q}}$ is a variety defined over $\mathbb{Q}$. Then $H^*_{\mathrm{dR}}(X,\mathbb{R}) \underset{\mathbb{R}}{\otimes} \mathbb{C}$ has two different $\mathbb{Q}$ structures. One coming from $H^*_{\mathrm{dR}}(X_{\mathbb{Q}},\mathbb{Q})$ and the other coming from $H^*_{\mathrm{sing}}(X,\mathbb{Q})$. These two rational structures are different and its comparison leads to the definition of "periods". The ubiquitous example of a period is $2\pi i$ and its powers (see exercises).

Even, when $K = \mathbb{C}$ we can extract a lot of information from the comparison between singular and de Rham cohomology, because (algebraic) de Rham cohomology, being the hyper-cohomology of a complex of sheaves, comes equipped with a filtration.

Definition 1.1. The Hodge filtration of the complex $\Omega^*_{X_K}$ is the decreasing filtration

$$F^*\Omega^*_{X_K} = \bigoplus_{n\geq p} \Omega^*_{X_K}.$$

The Hodge filtration induces a filtration, also called the Hodge filtration in cohomology.

In fact (as we will see later), the comparison isomorphism between singular and de Rham cohomology and the Hodge filtration are enough to determine an elliptic curve.

We next show how to introduce the complex structure into the picture. The complex structure of X is given by an operator $J\colon TX \to TX$ that satisfies $J^2 = -\operatorname{Id}$. This operator induces a dual operator, also denoted $J\colon T^*X \to T^*X$. We can decompose $TX\otimes\mathbb{C}$ and $T^*X\otimes\mathbb{C}$ into eigenspaces for this operator:

$$TX\otimes\mathbb{C} = T'X \oplus T''X, \qquad T^*X\otimes\mathbb{C} = T^{*\prime}X \oplus T^{*\prime\prime}X,$$

where $T'X$ and $T^{*\prime}X$ are the subspaces where J acts by multiplication by i. They are called the holomorphic part. While $T''X$ and $T^{*\prime\prime}X$ are the subspaces where J acts by multiplication by $-i$. They are called the anti-holomorphic part. From them we obtain decompositions

$$\bigwedge^n T^*X\otimes\mathbb{C} = \bigoplus_{p+q=n} \bigwedge^p T^{*\prime}X \otimes \bigwedge^q T^{*\prime\prime}X.$$

Hence a bigrading

$$A^n_{\mathbb{C}}(X) := A^n_{\mathbb{R}}(X)\otimes\mathbb{C} = \bigoplus_{p+q=n} A^{p,q}(X).$$

Since $A^n_{\mathbb{C}}(X)$ is defined as the tensor product of a real space by $\mathbb{C}$ it has an induced complex conjugation. This bigrading and the conjugation are compatible: $\overline{A^{p,q}(X)} = A^{q,p}(X)$. We can decompose $\mathrm{d}\colon A^n_{\mathbb{C}}(X) \longrightarrow A^{n+1}_{\mathbb{C}}(X)$ into $\mathrm{d} = \partial + \overline{\partial}$ where ∂ is bihomogeneous of bidegree $(1,0)$ and $\overline{\partial}$ is bihomogeneous of bidegree $(0,1)$.

If we denote by $\mathscr{A}^n_{X,\mathbb{C}}$ the sheaf of smooth sections of $\bigwedge^n T^*X\otimes\mathbb{C}$ and by $\mathscr{A}^{p,q}_X$ the sheaf of smooth sections of $\bigwedge^p T^{*\prime}X\otimes\bigwedge^q T^{*\prime\prime}X$ they are acyclic sheaves. The operators d, ∂ and $\overline{\partial}$ are local in the sense that there are operators

$$\begin{aligned}
\mathrm{d}\colon \mathscr{A}^n_{X,\mathbb{C}} &\longrightarrow \mathscr{A}^{n+1}_{X,\mathbb{C}}\\
\partial\colon \mathscr{A}^{p,q}_X &\longrightarrow \mathscr{A}^{p+1,q}_X\\
\overline{\partial}\colon \mathscr{A}^{p,q}_X &\longrightarrow \mathscr{A}^{p,q+1}_X
\end{aligned}$$

Moreover, the Poincaré lemma implies that, if $\mathbb{C}_X$ is the constant sheaf $\mathbb{C}$ on X and Ω_X^p is the sheaf of holomorphic p-forms on X, then the sequences

$$0 \to \underline{\mathbb{C}}_X \to \mathscr{A}^0_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \mathscr{A}^1_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \mathscr{A}^{2d}_{X,\mathbb{C}},$$
$$0 \to \Omega^p_X \to \mathscr{A}^{p,0}_X \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1}_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\overline{\partial}} \mathscr{A}^{p,d}_{X,\mathbb{C}},$$

are exact. Thus, if we denote

$$H^{p,q}_{\overline{\partial}}(X) = H^q(A^{p,*}(X), \overline{\partial}),$$

then

$$H^n_{\mathrm{dR}}(X, \mathbb{C}) = H^n(X, \mathbb{C}_X), \quad H^{p,q}_{\overline{\partial}}(X) = H^q(X, \Omega^p_X).$$

We cannot rush and use the bigrading in the de Rham complex to induce a bigrading in cohomology because the differential d is not a bihomogeneous operator. We need a little more work. Assume that we have a smooth hermitian metric on X. That is, a positive definite hermitian inner product

$$\langle\cdot,\cdot\rangle\colon T'_zX \otimes T''_zX \longrightarrow \mathbb{C},$$

that varies smoothly with z. Such hermitian metric is given in local coordinates by $\sum h_{\mu,\nu}\,\mathrm{d}\,z_\mu \otimes \mathrm{d}\,\overline{z}_\nu$. We associate to it a diferential form of bidegree $(1,1)$ given by

$$\omega = \frac{i}{2}\sum h_{\mu,\nu}\,\mathrm{d}\,z_\mu \wedge \mathrm{d}\,\overline{z}_\nu.$$

The hermitian metric on $T'X$ induces hermitian metrics, also denoted $\langle\cdot,\cdot\rangle$, on all the bundles $\bigwedge^p T^{*\prime}X \otimes \bigwedge^q T^{*\prime\prime}X$ and a volume form $\mathrm{vol} = \frac{1}{n!}\omega^{\wedge n}$. Thus we obtain an hermitian metric on each space $A^{p,q}(X)$ given by

$$(\varphi,\psi) = \int_X \langle\varphi,\psi\rangle\,\mathrm{d}\,\mathrm{vol}\,.$$

Then there exist formal adjoints of d and $\overline{\partial}$, denoted by d^* and $\overline{\partial}^*$ respectively and determined by

$$(\varphi, \mathrm{d}\,\psi) = (\mathrm{d}^*\,\varphi, \psi), \qquad (\varphi, \overline{\partial}\psi) = (\overline{\partial}^*\varphi, \psi).$$

From them we can define the Laplacian operators

$$\Delta_{\mathrm{d}} = \mathrm{d}\circ\mathrm{d}^* + \mathrm{d}^*\circ\mathrm{d}\colon A^n_{\mathbb{C}}(X) \longrightarrow A^n_{\mathbb{C}}(X)$$
$$\Delta_{\overline{\partial}} = \overline{\partial}\circ\overline{\partial}^* + \overline{\partial}^*\circ\overline{\partial}\colon A^{p,q}(X) \longrightarrow A^{p,q}(X).$$

It is easy to see that both Laplacian operators are self-adjoint and satisfy

$$\ker\Delta_{\mathrm{d}} = \ker\mathrm{d}\cap\ker\mathrm{d}^*, \qquad \ker\Delta_{\overline{\partial}} = \ker\overline{\partial}\cap\ker\overline{\partial}^*$$

The spaces of harmonic forms are defined as

$$\mathscr{H}^n_{\Delta_{\mathrm{d}}} = \ker\Delta_{\mathrm{d}}\colon A^n_{\mathbb{C}}(X) \longrightarrow A^n_{\mathbb{C}}(X)$$
$$\mathscr{H}^{p,q}_{\Delta_{\overline{\partial}}} = \ker\Delta_{\overline{\partial}}\colon A^{p,q}(X) \longrightarrow A^{p,q}(X).$$

The main theorem of Hodge theory is the following

Theorem 1.2 (Hodge)**.** *For each integer n, there are differential operators*

$$H_{\Delta_{\mathrm{d}}}\colon A^n_{\mathbb{C}}(X) \longrightarrow A^n_{\mathbb{C}}(X)$$
$$G_{\Delta_{\mathrm{d}}}\colon A^n_{\mathbb{C}}(X) \longrightarrow A^n_{\mathbb{C}}(X)$$

such that

(1) $H_{\Delta_{\mathrm{d}}}(A^n_{\mathbb{C}}(X)) = \mathscr{H}^n_{\Delta_{\mathrm{d}}}$ *and* $\dim \mathscr{H}^n_{\Delta_{\mathrm{d}}} < \infty$.
(2) $\Delta_{\mathrm{d}} \circ G_{\Delta_{\mathrm{d}}} + H_{\Delta_{\mathrm{d}}} = G_{\Delta_{\mathrm{d}}} \circ \Delta_{\mathrm{d}} + H_{\Delta_{\mathrm{d}}} = \mathrm{Id}$.
(3) $A^n_{\mathbb{C}}(X) = \mathscr{H}^n_{\Delta_{\mathrm{d}}} \oplus \Delta_{\mathrm{d}} \circ G_{\Delta_{\mathrm{d}}}(A^n_{\mathbb{C}}(X)) = \mathscr{H}^n_{\Delta_{\mathrm{d}}} \oplus G_{\Delta_{\mathrm{d}}} \circ \Delta_{\mathrm{d}}(A^n_{\mathbb{C}}(X))$ *and the decomposition is orthogonal.*

There is a theorem completely analogous for the operator $\Delta_{\overline{\partial}}$.

The first consequence of Hodge theorem is that we can use harmonic forms to compute cohomology.

Corollary 1.3. *The inclusion* $\mathscr{H}^n_{\Delta_{\mathrm{d}}} \to A^n_{\mathbb{C}}(X)$ *and* $\mathscr{H}^{p,q}_{\Delta_{\overline{\partial}}} \to A^{p,q}(X)$ *induce isomorphisms*

$$\mathscr{H}^n_{\Delta_{\mathrm{d}}} \to H^n_{\mathrm{dR}}(X, \mathbb{C}), \qquad \mathscr{H}^{p,q}_{\Delta_{\overline{\partial}}} \to H^{p,q}_{\overline{\partial}}(X).$$

We now add the hypothesis that the metric is Kähler, which means that $\mathrm{d}\,\omega = 0$. This is possible because X is a projective variety, and every projective complex variety has a Kähler metric obtained as the inverse image by a projective embedding of the Fubini-Study metric. The fact that the metric is Kähler has many important consequences. The first is

Theorem 1.4. *The fact that the metric is Kähler implies the equality*

$$\Delta_{\mathrm{d}} = 2\Delta_{\overline{\partial}}.$$

As a consequence we deduce that the operator Δ_{d} is bihomogeneous and we obtain the Dolbeault decomposition

$$H^n_{\mathrm{dR}}(X, \mathbb{C}) \simeq \mathscr{H}^n_{\Delta_{\mathrm{d}}} = \bigoplus_{p+q=n} \mathscr{H}^{p,q}_{\Delta_{\overline{\partial}}} \simeq \bigoplus_{p+q=n} H^q(X, \Omega^p_X). \tag{1.1}$$

We will denote by $H^{p,q}(X, \mathbb{C}) = H^q(X, \Omega^p_X)$. In the exercises we will see different ways to look at this decomposition.

Observe that $H^q(X, \Omega^p_X)$ is the term $E_1^{p,q}$ of the spectral sequence associated to the Hodge filtration of $\Omega^p_{X_{\mathbb{C}}}$ (by the GAGA principle, it does not matter if we consider holomorphic or algebraic differentials).

Thus, as a Corollary of Hodge Theorem we can obtain the following.

Corollary 1.5. *Let X be a smooth projective variety over $\mathbb{C}$. Then the spectral sequence associated to the Hodge filtration degenerates at the term E_1. Moreover the induced filtration in cohomology satisfies*

$$F^p H^n_{\mathrm{dR}}(X, \mathbb{C}) \simeq \bigoplus_{p' \geq n} H^{n-p'}(X, \Omega^{p'}_X).$$

Let now E be an elliptic curve over $\mathbb{C}$. Then $H^1_{\mathrm{sing}}(E, \mathbb{Z}) \simeq \mathbb{Z}^2$ and $H^{0,1}(E, \mathbb{C}) \cong \mathbb{C}$. We consider the projection

$$\pi \colon H^1(E, \mathbb{C}) \to H^1(E, \mathbb{C})/F^1 H^1(E, \mathbb{C}) \simeq H^{0,1}(E, \mathbb{C}) \simeq \mathbb{C}.$$

The image of $H^1_{\mathrm{sing}}(E, \mathbb{Z})$ under this projection is a lattice in $H^{0,1}(E, \mathbb{C})$ and we can recover the elliptic curve as

$$E \simeq H^{0,1}(E, \mathbb{C})/\pi(H^1_{\mathrm{sing}}(E, \mathbb{Z})).$$

Thus, in contrast with plain Betti cohomology or plain de Rham cohomology, the interplay between singular cohomology and the Hodge filtration of Betti cohomology contains enough information to recover the elliptic curve.

Summing up, the de Rham cohomology groups of a smooth projective complex variety X have two structures. An integral structure coming from topology and a bigrading coming from the complex structure. It is the interplay between both structures what gives us a very rich theory.

The intersection product induces another important, although subtle, property of the cohomology of a smooth projective variety called the polarizability.

2. Hodge structures

In this section we will recall the definition of Hodge structures and mixed Hodge structures. Most of the material in this section comes from [Del71] and [Del75].

We first axiomatize the properties of the cohomology of a smooth projective complex variety.

Definition 2.1. Let $A \subset \mathbb{R}$ be a noetherian subalgebra such that $A \otimes \mathbb{Q}$ is a field (for instance $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$). A *pure A-Hodge structure of weight n* is a triple $H = (H_A, (H_\mathbb{C}, F), \alpha)$ where H_A is an A-module of finite type, $(H_\mathbb{C}, F)$ is a complex vector space with a finite decreasing filtration F and α is an isomorphism $H_A \otimes \mathbb{C} \to H_\mathbb{C}$ that satisfy the property that, for all $p \in \mathbb{Z}$,

$$H_\mathbb{C} = F^p H_\mathbb{C} \oplus \overline{F^{n-p+1} H_\mathbb{C}},$$

where the complex conjugation is obtained from the real structure of $H_\mathbb{C}$ induced by α. A morphism $f\colon H = (H_A, (H_\mathbb{C}, F), \alpha) \to H' = (H'_A, (H'_\mathbb{C}, F), \alpha')$ of pure A-Hodge structures is a pair $(f_A, f_\mathbb{C})$, with $f_A\colon H_A \to H'_A$ a morphism, $f_\mathbb{C}\colon H_\mathbb{C} \to H'_\mathbb{C}$ a filtered morphism such that $\alpha' \circ f_A = f_\mathbb{C} \circ \alpha$.

If H is a pure A-Hodge structure of weight n, then, for $p \in \mathbb{Z}$, we write

$$H^{p,n-p} = F^p H_\mathbb{C} \cap \overline{F^{n-p} H_\mathbb{C}}.$$

Definition 2.2. Let H be a pure Hodge structure of weight n. A *polarization* of H is a bilinear form Q on H_A that, when extended to $H_\mathbb{C}$ by linearity, satisfies

$$\begin{aligned} Q(\varphi, \psi) &= (-1)^n Q(\psi, \varphi) \\ Q(\varphi, \psi) &= 0, \text{ if } \varphi \in H^{p,q},\ \psi \in H^{p',q'}, p \neq q' \\ (i)^{p-q} Q(\varphi, \overline{\varphi}) &> 0, \text{ if } \varphi \in H^{p,q},\ \varphi \neq 0. \end{aligned}$$

A pure Hodge structure of weight n is *polarizable* if it admits a polarization.

These groups satisfy $H^{q,p} = \overline{H^{p,q}}$. The Hodge decomposition of H is the direct sum decomposition

$$H_\mathbb{C} = \bigoplus_p H^{p,n-p}.$$

Example 2.3. The trivial A Hodge structure $A(0)$ is the pure A-Hodge structure of weight 0 with $A(0)_A = A$, $A(1)_\mathbb{C} = \mathbb{C}$, $F^0 A(0)_\mathbb{C} = A(0)_\mathbb{C}$, $F^1 A(0)_\mathbb{C} = 0$ and $\alpha(1) = 1$. The Tate A-Hodge structure $A(1)$ is the pure A-Hodge structure of weight -2 with $A(1)_A = A$, $A(1)_\mathbb{C} = \mathbb{C}$, $F^{-1} A(1)_\mathbb{C} = A(1)_\mathbb{C}$, $F^0 A(1)_\mathbb{C} = 0$ and $\alpha(1) = 2\pi i$. Thus

$$A(0)^{p,q} = \begin{cases} \mathbb{C}, & \text{if } p = q = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{while} \quad A(1)^{p,q} = \begin{cases} \mathbb{C}, & \text{if } p = q = -1, \\ 0, & \text{otherwise.} \end{cases}$$

The category of A-Hodge structures, denoted $A\mathcal{HS}$ is the category whose objects are finite direct sums of pure Hodge structures and morphisms are direct sums of morphisms of pure Hodge structures.

The category $A\mathcal{HS}$ is an *abelian* category. This is not completely obvious because the category of filtered vector spaces is not an abelian category. In the exercises there will be a related discussion.

The category of polarizable A-Hodge structures will be denoted by $A\mathcal{PHS}$. If we assume that A is a field, the category $A\mathcal{PHS}$ is semisimple. That is, there are no non-trivial extensions.

If H and H' are pure A-Hodge structures of weight n and n' respectively, then the tensor product $H \otimes H'$ is a pure A-Hodge structures of weight $n+n'$, while $\underline{\mathrm{Hom}}(H,H')$ is again a pure Hodge structure, this time of weight $n'-n$. In particular, the dual $H^\vee$ has weight $-n$. One can see that the category of Hodge structures is not only an abelian but, in fact it is a Tannakian category.

Notation 2.4. For $n \geq 1$ we will denote $A(n) = A(1)^{\otimes n}$ and for $n \leq -1$ we denote $A(n) = (A(1)^\vee)^{\otimes -n}$. More generally, if H is a pure A-Hodge structure, we will write $H(n) = H \otimes A(n)$. The Hodge structure $A(-1)$ is called the Lefschetz Hodge structure.

From the discussion of the previous section we derive

Theorem 2.5. *Let X be a smooth projective variety. Then, for each $n \geq 0$, $H^n(X,\mathbb{C})$ has a pure $\mathbb{Z}$-Hodge structure of weigh n. Moreover, this Hodge structure is polarizable.*

We can ask ourselves what happens if the variety is non proper or singular. The first difficulty we encounter is that we can not use directly differential forms to compute the Hodge structure of a non-proper complex variety. We can see this in the simplest of examples. Consider the affine line $\mathbb{A}^1_{\mathbb{C}}$. Since the associated topological space is contractible we know that

$$H^n_{\mathrm{sing}}(\mathbb{A}^1_{\mathbb{C}},\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ 0, & \text{otherwise.} \end{cases} \qquad H^n_{\mathrm{dR}}(\mathbb{A}^1_{\mathbb{C}},\mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } n=0, \\ 0, & \text{otherwise.} \end{cases}$$

By contrast $H^0(\mathbb{A}^1_{\mathbb{C}},\Omega^0_{\mathbb{A}^1_{\mathbb{C}}})$ is the space of holomorphic functions on $\mathbb{A}^1_{\mathbb{C}}$ that is infinite dimensional. Thus the Dolbeault decomposition does not hold in the case of non-proper varieties. The main idea of Deligne is that, in order to have an analogue of the Dolbeault decomposition for non-proper varieties one has to use differential forms with *logarithmic singularities along the boundary.* We will see more on logarithmic singularities in later sections.

We next see an example of a singular variety. Let X be a smooth projective curve and $x,y \in X$ closed points. We can form a projective singular curve X' by identifying x and y. Let $\pi\colon X \to X'$ be the projection. Denote $Y = \mathrm{Spec}(\mathbb{C})$ and let $i,j\colon Y \to X$ be the immersions of Y on X determined by x and y. Then there is a long exact sequence of cohomology

$$0 \to H^0(X') \xrightarrow{H^0(\pi)} H^0(X) \xrightarrow{H^0(i)-H^0(j)} H^0(Y) \xrightarrow{\partial} H^1(X') \xrightarrow{H^1(\pi)} H^1(X) \to 0.$$

Since $H^0(\pi)$ is an isomorphism we obtain a short exact sequence

$$0 \to H^0(Y) \xrightarrow{\partial} H^1(X') \xrightarrow{H^1(\pi)} H^1(X) \to 0.$$

Since $H^0(Y)$ is a pure Hodge structure of weight 0 and $H^1(X)$ is a pure Hodge structure of weight one, we observe that $H^1(X')$ is a mixture of Hodge structures of weight zero and one. More precisely, $H^1(X')$ has an increasing filtration W, the *weight filtration*, with

$$\operatorname{Gr}_0^W H^1(X') = H^0(Y), \quad \operatorname{Gr}_1^W H^1(X') = H^1(X),$$

pure Hodge structures of weight 0 and 1 respectively.

Similarly one can show that the cohomology of an open variety can be constructed by mixing pure pieces of different weights. This leads to the notion of mixed Hodge structures.

Definition 2.6. Let $A \subset \mathbb{R}$ be a subalgebra. A *mixed A-Hodge structure* is a 5-tuple $H = (H_A, (H_{\mathbb{Q}}, W), (H_{\mathbb{C}}, W, F), \alpha, \beta)$ where H_A is an A-module of finite type, $(H_{\mathbb{Q}}, W)$ is an $A \otimes \mathbb{Q}$ vector space with a finite increasing filtration, $(H_{\mathbb{C}}, W, F)$ is a complex vector space with a finite increasing filtration W and a finite decreasing filtration F, β is an isomorphism $H_A \otimes \mathbb{Q} \to H_{\mathbb{Q}}$ and α is a filtered isomorphism $(H_{\mathbb{Q}} \otimes \mathbb{C}, W) \to (H_C, W)$ such that, for each $n \in \mathbb{Z}$ the triple

$$(\operatorname{Gr}_n^W H_{\mathbb{Q}}, (\operatorname{Gr}_n^W H_{\mathbb{C}}, F), \operatorname{Gr}_n^W \alpha)$$

is a pure $A \otimes \mathbb{Q}$-Hodge structure of weight n. A morphism $f\colon H, H'$ of mixed Hodge structures is a triple of morphisms $(f_A, f_{\mathbb{Q}}, f_{\mathbb{C}})$ that preserve the filtrations and commute with the isomorphisms.

Definition 2.7. Given a morphism of filtered objects $f\colon (H, F) \to (H', F)$, we say that f is *strictly compatible with the filtration* if $\operatorname{Img}(f) \cap F^p = \operatorname{Img}(f|_{F^p})$.

The basic property of Mixed Hodge structures is given by the following result by Deligne.

Theorem 2.8 (Deligne)**.** *Let $f\colon H \to H'$ be a morphism of mixed A-Hodge structures. Then the morphisms $f_{\mathbb{Q}}\colon (H_{\mathbb{Q}}, W) \to (H'_{\mathbb{Q}}, W)$ and $f_{\mathbb{C}}\colon (H_{\mathbb{C}}, W, F) \to (H'_{\mathbb{C}}, W, F)$ are strictly compatible with the filtrations. Moreover f is an isomorphism if and only if $f_A\colon H_A \to H'_A$ is an isomorphism.*

The category of mixed A-Hodge structures is again an abelian tensor category (in fact again a Tannakian category). Each pure A-Hodge structure $H = (H_A, H_{\mathbb{C}}, \alpha)$ of weight n gives a mixed Hodge structure by writing $H_{\mathbb{Q}} = H_A \otimes \mathbb{Q}$, $W_{n-1} H_{\mathbb{Q}} = 0$, $W_n H_{\mathbb{Q}} = H_{\mathbb{Q}}$ and $\beta = \mathrm{Id}$. In particular we also denote by $A(n)$ the mixed Hodge stucture given by the pure Hodge structure $A(n)$ and for any mixed Hodge structure H, we write $H(n) = H \otimes A(n)$.

In contrast with the category of pure Hodge structures, it has many interesting extensions. For instance, let $H = (H_{\mathbb{Z}}, H_{\mathbb{C}}, \alpha_H)$ be a pure $\mathbb{Z}$-Hodge structure of odd weight $2n-1$ such that $H_{\mathbb{Z}}$ is torsion free. We next compute $\operatorname{Ext}^1(\mathbb{Z}(-n), H)$. Assume that there is a short exact sequence of mixed Hodge structures

$$0 \to H \longrightarrow M \longrightarrow \mathbb{Z}(-n) \to 0,$$

where $M = (M_{\mathbb{Z}}, M_{\mathbb{Q}}, M_{\mathbb{C}}, \alpha_M, \beta_M)$. We choose an splitting $M_{\mathbb{Z}} = H_{\mathbb{Z}} \oplus \mathbb{Z}$. We can identify $M_{\mathbb{Q}}$ with $M_{\mathbb{Z}} \otimes \mathbb{Q}$, so we may assume that $\beta_M = \mathrm{Id}$, the previous splitting determines an splitting $M_{\mathbb{Q}} = H_{\mathbb{Q}} \oplus \mathbb{Q}$. Then we can identify $W_{2n-1} M_Q = H_Q$ and $\operatorname{Gr}_{2n}^W M_{\mathbb{Q}} = \mathbb{Q}$. Now we can identify $F^{n+1} M_{\mathbb{C}} = F^{n+1} H_{\mathbb{C}}$ and there is an isomorphism $F^n M_{\mathbb{C}} \simeq F^n H_{\mathbb{C}} \oplus \mathbb{C}$. We choose an splitting $F^n M_{\mathbb{C}} \simeq F^n H_{\mathbb{C}} \oplus \mathbb{C}$, that induces an splitting $M_{\mathbb{C}} = H_{\mathbb{C}} \oplus \mathbb{C}$. In order to determine the extension $M_{\mathbb{C}}$ it only remains to write the isomorphism α_M in the above splittings. Since, necessarily $\alpha_M \mid_{H_{\mathbb{Q}}\otimes\mathbb{C}} = \alpha_H$, the only freedom we have is to choose $\alpha_M(1)$ for $1 \in \mathbb{Q}$, but also we know that $\alpha_M(1) = (2\pi i)^{-n} + c_M$ with $c_M \in H_{\mathbb{C}}$. It remains to see the effect of the choice of splittings in the c_M. If we change the splitting of $M_{\mathbb{Z}}$ we add to c_M an element of $\alpha_H(H_{\mathbb{Z}})$, while if we change the splitting of $F^n M_{\mathbb{C}}$ we add to c_M an element of $F^n H_C$. Therefore we deduce that

$$\operatorname{Ext}^1(\mathbb{Z}(-n), H) = H_{\mathbb{C}}/(\alpha_H(H_{\mathbb{Z}}) + F^n H_{\mathbb{C}}). \tag{2.1}$$

It is easy to see that the right hand side of equation (2.1) is a compact complex torus that is called the intermediate Jacobian of H and denoted $J(H)$.

The main theorem relating mixed Hodge structures and algebraic varieties is the following.

Theorem 2.9 (Deligne). *Let X be a variety over $\mathbb{C}$.*

(1) *For each integer n we can associate to X a mixed Hodge structures*
$$H^n(X) = (H^n_{\text{sing}}(X,\mathbb{Z}), (H^n_{\text{sing}}(X,\mathbb{Q}), W), (H^n_{\text{dR}}(X), W, F), \alpha, \beta),$$
where β is the change of coefficients isomorphism and α is the comparison isomorphism between Betti and de Rham cohomology. The cohomology with compact support has also a mixed Hodge structure denoted $H^n_c(X)$.

(2) *If $f\colon X \to Y$ is a morphism of complex varieties, there are induced morphism of mixed Hodge structures $f^*\colon H^n(Y) \to H^n(X)$. If moreover, f is proper and of pure relative dimension e, there is a Gysin morphism of mixed Hodge structures $f_*\colon H^{n+2e}(X)(e) \to H^n(Y)$.*

(3) *If $Z \subset X$ is a closed subvariety and $U = X \setminus Z$. Then there are long exact sequences of mixed $\mathbb{Z}$-Hodge structures*
$$\dots \longrightarrow H^n_Z(X) \longrightarrow H^n(X) \longrightarrow H^n(U) \longrightarrow \dots$$
$$\dots \longrightarrow H^n_c(U) \longrightarrow H^n(X) \longrightarrow H^n(Z) \longrightarrow \dots$$

An important property of the mixed Hodge structures is the following

Proposition 2.10. *Let M and N be mixed Hodge structures. Then, for all $i > 1$, $\operatorname{Ext}^i_{A\,\mathcal{MHS}}(M,N) = \operatorname{Ext}^i_A(M_A, N_A)$. In particular, if $A = \mathbb{Z}$ or is a field, then $\operatorname{Ext}^i_{A\,\mathcal{MHS}}(M,N) = 0$, for $i > 1$.*

Corollary 2.11. *Let $A = \mathbb{Z}$ or a field. If $M \in D^b(A\,\mathcal{MHS})$, then there is an isomorphism $M \cong \bigoplus_i H^i(M)[-i]$.*

We end this section discussing a geometric theorem that can be proved using Hodge theory.

An algebraic variety is said to be *rational* if it is birational to the projective space. A variety is unirational if it can be dominated by a rational variety. For curves, unirational implies rational and for surfaces over a field of characteristic zero unirational implies rational. The first example of a unirational variety over a field of characteristic zero that is not rational was given by Clemens and Griffiths [CG72]. The cubic threefold is a hyper-surface of degree three of $\mathbb{P}^4$. It is known to be unirational.

Theorem 2.12 (Clemens-Griffiths). *The smooth cubic threefold is not rational.*

Sketch of proof. Let X be a smooth cubic threefold. The Hodge decomposition of the cohomology of X is the following

$$\dim H^{p,q}(X) = \begin{cases} 1, & \text{if } (p,q) = (0,0),(1,1),(2,2),(3,3), \\ 5, & \text{if } (p,q) = (1,2),(2,1), \\ 0, & \text{otherwise.} \end{cases}$$

Thus the interesting part of the cohomology lives in degree 3. The intermediate Jacobian of X, $J(X)$, is the intermediate Jacobian of $H^3(X)$. It is a complex torus of dimension $g = 5$. The intersection product in cohomology induces a polarization in $H^3(X)$ that gives to $J(X)$ the structure of a principally polarized abelian variety. In particular it has an associated Theta divisor Θ.

If X were rational, by the weak factorization theorem, it would be related to $\mathbb{P}^3$ by a collection of blow-ups and Blow-downs. By studying the effect of a blow-up on the intermediate Jacobian, one concludes that, if X were rational, $J(X)$ would be zero or isomorphic (as principally polarized abelian variety) to the Jacobian of a curve. It is known that if A is a principally polarized abelian variety of dimension g, and isomorphic to the Jacobian of a curve, then the dimension of the singular locus of Θ is $g-3$ of $g-4$.

The key point of Griffiths and Clemens argument is to show that, in the case of the cubic threefold, the singularities of Θ are isolated points. Thus $\dim \operatorname{sing} \Theta = 0 < g-4$. Thus $J(X)$ and not be the Jacobian of a curve, hence X is not rational $\square$

3. Exercises

Exercises

(1) Finite dimensional Hodge theory. Let (A^*, d) be a bounded complex of finite dimensional real vector spaces. Assume that each piece A^* is provided with an Euclidean product. Prove Theorem 1.2 and Corollary 1.3 in this case.

(2) Let k be a field and $\mathcal{C}$ the category of filtered k-vector spaces.
   (a) Give an example of a morphism f in $\mathcal{C}$ such that $\mathrm{Img}(f) \neq \mathrm{Coimg}(f)$. Conclude that $\mathbb{C}$ is not an abelian category.
   (b) Show that, if f is a morphism in $\mathcal{C}$ that is strictly compatible with the filtrations, then $\mathrm{Img}(f) = \mathrm{Coimg}(f)$.
   (c) Show that, if $f\colon H \to H'$ is a morphism of pure Hodge structures of weight n, then it is strictly compatible with the Hodge filtration.
   (d) Let (A^*, d, F) be a filtered bounded complex of finite dimensional vector spaces, with F a bounded decreasing filtration. Prove that the differential d is strictly compatible with the filtration F if and only if the spectral sequence associated to F degenerates at E_1.

(3) On the projective line $X = \mathbb{P}^1$, with absolute coordinate z, consider the differential form

$$\omega = \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{(1 + z\overline{z})^2}.$$

   (a) Show that ω is smooth on the whole X and represents a generator of $H^1(X, \Omega^1_{X_\mathbb{Q}})$. (Hint: Use first, that in Čech cohomology, a generator of $H^1(X, \Omega^1_{X_\mathbb{Q}})$ is given by $\mathrm{d}\, z/z$ in $\mathbb{P}^1 \setminus \{0, 1\}$ and second, the Čech complex of the Dolbeault resolution of $\Omega^1_{X_\mathbb{C}}$.)
   (b) Show that $\int_X \omega = 2\pi i$ and conclude that the Hodge structure of the cohomology group $H^2(X)$ is the Lefschetz Hodge structure.

(4) Compute $Ext^1(\mathbb{Z}(i), \mathbb{Z}(j))$.

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