Contents

1. Hodge complexes

Once we have seen what are the Hodge structures and what we can do with them, following Beilinson [Bei83], we want to refine a little bit the assigment that to each complex variety X associates a collection of mixed Hodge structures. Namely, we want to associate to X an element of the derived category of mixed Hodge structures.

Definition 1.1. An unshifted A-Hodge complex is a diagram

$$\begin{tikzcd} & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, W) & \\ \mathcal{F}_A \arrow[ur, "\varphi_1"] & & (\mathcal{F}_{\mathbb{Q}}, W) \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}}, W, F) \arrow[ul, "\psi_2"'] \end{tikzcd}$$

where $\mathcal{F}_A$ is a complex of A-modules, $\mathcal{F}'_{\mathbb{Q}}$ of $A \otimes \mathbb{Q}$ modules, $(\mathcal{F}_{\mathbb{Q}}, W)$ is a filtered complex of $A \otimes \mathbb{Q}$ modules, $(\mathcal{F}'_{\mathbb{C}}, W)$ of $\mathbb{C}$ modules and $(\mathcal{F}'_{\mathbb{C}}, W, F)$ is a bifiltered complex of $\mathbb{C}$ modules. The arrows $\mathcal{F}_A \otimes \mathbb{Q} \to \mathcal{F}'_{\mathbb{Q}}$, $\mathcal{F}_{\mathbb{Q}} \to \mathcal{F}'_{\mathbb{Q}}$, $(\mathcal{F}_{\mathbb{Q}}, W) \otimes \mathbb{C} \to (\mathcal{F}'_{\mathbb{C}}, W)$ and $(\mathcal{F}_{\mathbb{C}}, W) \to (\mathcal{F}'_{\mathbb{C}}, W)$ are (filtered) quasi-isomorphisms.

This diagram should satisfy

(1) The cohomology groups $H^i(\mathcal{F}_A)$ are finitely generated A modules and only a finite number of them are different from zero.
(2) For any $n \in \mathbb{Z}$ consider the filtered complex $(\operatorname{Gr}^W_n \mathcal{F}_{\mathbb{C}}, \operatorname{Gr}^W_n F)$. In this complex the differential is strictly compatible with the filtration.
(3) The induced Hodge filtration together with the isomorphism $H^i(\operatorname{Gr}^W_n \mathcal{F}_{\mathbb{Q}}) \to H^i(\operatorname{Gr}^W_n \mathcal{F}_{\mathbb{C}})$ defines a pure $A \otimes \mathbb{Q}$-Hodge structure of weight $n+i$.

The weight filtration that appears in the definition of unshifted A-Hodge complex is the one that is simpler to write down for smooth complex varieties, nevertheless it has some drawbacks. First, the weight filtration we want in cohomology is not exactly the one induced by the weight filtration in the complex, but it has to be shifted according to the degree, and second, the spectral sequence associated to the weight filtration degenerates at E_2. In some cases the following alternative definition is simpler to use.

Definition 1.2. An A-Hodge complex is a diagram

$$\begin{tikzcd} & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, \hat{W}) & \\ \mathcal{F}_A \arrow[ur, "\varphi_1"] & & (\mathcal{F}_{\mathbb{Q}}, \hat{W}) \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}}, \hat{W}, F) \arrow[ul, "\psi_2"'] \end{tikzcd}$$

as in Definition 1.1, satisfying the same properties except that the $A \otimes \mathbb{Q}$-Hodge structure induced in $H^i(\mathrm{Gr}_n^W \mathcal{F}_{\mathbb{Q}})$ pure of weight n.

Given a filtered complex (K^*, W), the decalé filtration is defined as

$$\mathrm{Dec}(W)_n K^i = \{x \in W_{n-i} K^i \mid \mathrm{d}\, x \in W_{n-i-1} K^{i-1}\}.$$

In the exercises we will see the main properties of the decalé filtration.

One can go from unshifted A-Hodge complexes to A-Hodge complexes using the following result.

Lemma 1.3. *Let*

$$\begin{tikzcd}
 & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, W) & \\
\mathcal{F}_A \arrow[ur, "\varphi_1"] & & (\mathcal{F}_{\mathbb{Q}}, W) \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}}, W, F) \arrow[ul, "\psi_2"']
\end{tikzcd}$$

be an unshifted A-Hodge complex. Then

$$\begin{tikzcd}
 & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, \mathrm{Dec}(W)) & \\
\mathcal{F}_A \arrow[ur, "\varphi_1"] & & (\mathcal{F}_{\mathbb{Q}}, \mathrm{Dec}(W)) \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}}, \mathrm{Dec}(W), F) \arrow[ul, "\psi_2"']
\end{tikzcd}$$

is an A-Hodge complex.

We will usually employ the letter W for the weight filtration of a unshifted A-Hodge complex and $\hat{W}$ for the weight filtration of an A-Hodge complex to indicate that it is a decalé filtration.

It is possible to see that the category of A-Hodge complexes up to homotopy is a triangulated category and one can define the corresponding derived category, that we denote $D(\text{A-}\mathcal{HC})$, by inverting quasi-isomorphism. Given a complex of mixed A-Hodge structures $H^* = (H^*_{\mathbb{Z}}, (H^*_{\mathbb{Q}}, \widetilde{W}), (H^*_{\mathbb{C}}, \widetilde{W}, \widetilde{F}))$ we can define an A-Hodge complex by writing

$$\mathcal{F}_A = H^*_A, \quad \mathcal{F}_{\mathbb{Q}} = \mathcal{F}'_{\mathbb{Q}} = H^*_{\mathbb{Q}}, \quad \mathcal{F}_{\mathbb{C}} = \mathcal{F}'_{\mathbb{C}} = H^*_{\mathbb{C}}$$

with filtrations $F = \widetilde{F}$ and $\hat{W} = \widetilde{W}$. This induces a functor $D^b(\text{A-}\mathcal{MHS}) \to D(\text{A-}\mathcal{HC})$. In particular the Tate Hodge structures $A(n)$ determine elements of $D(\text{A-}\mathcal{HC})$ that are also denoted by $A(n)$.

Theorem 1.4 (Beilinson)**.** *The functor $D^b(\text{A-}\mathcal{MHS}) \to D(\text{A-}\mathcal{HC})$ is an equivalence of categories.*

This theorem induces a cohomological t-structure on $D(\text{A-}\mathcal{HC})$, whose heart is A-$\mathcal{MHS}$, namely, the natural t-structure of $D^b(\text{A-}\mathcal{MHS})$ as the bounded derived category of an abelian category.

In case that A is already a field, hence $A \otimes \mathbb{Q} = A$, we can simplify the definition of A-Hodge complex because the two leftmost complexes do not carry any information. In fact we have the following immediate result.

Lemma 1.5. *Assume that A is a field and let*

$$\mathbf{H} = \left(\begin{tikzcd}[column sep=small] & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, \hat{W}) & \\ \mathcal{F}_A \arrow[ur, "\varphi_1"] & & (\mathcal{F}_{\mathbb{Q}}, \hat{W}) \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}}, \hat{W}, F) \arrow[ul, "\psi_2"'] \end{tikzcd} \right)$$

be an A-Hodge complex. Then, in $D^b(\text{A-}\mathcal{MHS})$, there is a canonical isomorphism of A-Hodge complexes between $\mathbf{H}$ and

$$\left(\begin{tikzcd}[column sep=small] & \mathcal{F}_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, \hat{W}) & \\ \mathcal{F}_{\mathbb{Q}} \arrow[ur, "\mathrm{Id}"] & & (\mathcal{F}_{\mathbb{Q}}, \hat{W}) \arrow[ul, "\mathrm{Id}"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}}, \hat{W}, F) \arrow[ul, "\psi_2"'] \end{tikzcd} \right)$$

Thus, when A is a field we will use the following alternative definition of unshifted A-Hodge complex and similarly for A-Hodge complexes.

Definition 1.6. Assume that A is a field. we will say that a diagram

$$\begin{tikzcd}[column sep=small] & (\mathcal{F}'_{\mathbb{C}}, W) & \\ (\mathcal{F}_{\mathbb{Q}}, W) \arrow[ur, "\varphi"] & & (\mathcal{F}_{\mathbb{C}}, W, F) \arrow[ul, "\psi"'] \end{tikzcd}$$

is an unshifted A-Hodge complex if the diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{F}_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, W) & \\ \mathcal{F}_{\mathbb{Q}} \arrow[ur, "\mathrm{Id}"] & & (\mathcal{F}_{\mathbb{Q}}, W) \arrow[ul, "\mathrm{Id}"'] \arrow[ur, "\varphi"] & & (\mathcal{F}_{\mathbb{C}}, W, F) \arrow[ul, "\psi"'] \end{tikzcd}$$

satisfies definition 1.1.

To each complex variety X we can associate an A-Hodge complex $\mathbf{H}(X)$. The assignment $X \to \mathbf{H}(X)$ is a functor from the category of complex varieties to $D(\text{A-}\mathcal{HC})$. In this section we will give the A-Hodge complex associated to a smooth proper complex variety and in the next talk, that of a smooth complex variety (non-necessarily proper). For singular varieties we need to use hyper-resolutions and we will not cover it in this notes.

Definition 1.7. Given a complex $(K^* \,\mathrm{d})$, the *canonical filtration* τ is defined by

$$\tau_n K^r = \begin{cases} K^n, & \text{if } r < n, \\ \operatorname{Ker} \mathrm{d}, & \text{if } r = n, \\ 0, & \text{if } r > n. \end{cases}$$

Observe that the canonical filtration is the decalé of the trivial filtration T that satisfies $T_{-1}K = 0$ and $T_0 K = K$.

Lemma 1.8. *Let $C^* \to K^*$ be a quasi-isomorphim, then $(C^*, \tau) \to (K^*, \tau)$ is a filtered quasi-isomorphism.*

Let X be a smooth proper complex variety. Then the diagram

$$\begin{tikzcd}[column sep=small]
 & S^*(X, A\otimes\mathbb{Q}) & & (S^*(X,\mathbb{C}),\tau) & \\
S^*(X,A) \arrow[ur] & & (S^*(X, A\otimes\mathbb{Q}),\tau) \arrow[ul] \arrow[ur] & & (A^*_{\mathbb{C}}(X),\tau,F), \arrow[ul]
\end{tikzcd}$$

is an A-Hodge complex, where $S^*(X,R)$ is the complex of smooth singular cochains with R coefficients, the filtration F is the Hodge filtration on the complex of differential forms, τ is the canonical filtration, and the arrows are the obvious morphisms. This A-Hodge complex is denoted $\mathbf{H}(X)$. The unshifted A-Hodge complex is obtained by using the trivial filtration in place of the canonical filtration.

From the explicit description of $\mathbf{H}(X)$ we can derive

Lemma 1.9. *If X is an irreducible smooth proper complex variety, then there is a canonical map $A(0) \to \mathbf{H}(X)$.*

2. Absolute Hodge cohomology

Assume that we have a triangulated category D. From an object $C \in D$ we want to extract some "cohomology", from which we can extract information more easily. There are two processes to do so. First, if we can identify D as the derived category of an abelian category $\mathcal{A}$, then D has a t-structure with heart $\mathcal{A}$. Hence we can define the cohomology $H^n(C) = t_{\leq n}t_{\geq n}C \in \mathcal{A}$. This is called the geometric cohomology.

For the second method, assume that we have a distinguished object $R \in D$, then we can compute $\mathrm{RHom}(R, C) \in D^b(\mathcal{A})$, where $\mathcal{A}$ is the abelian category of abelian groups, maybe with some extra structure depending on the context. Since the category $D^b(\mathcal{A})$ comes with a t-structure, we can define the *arithmetic cohomology groups* of C with coefficients in R as $H^*(\mathrm{RHom}_D(R, C))$.

The arithmetic cohomology groups have several advantages,

(1) They may contain more information as we will see in the case of Hodge structures.
(2) It may be the case that we do not have at our disposal a t-structure. For instance, we do not know a good t-structure on the derived category of mixed motives. Nevertheless we can define motivic cohomology as a kind of arithmetic cohomology.

We will apply these ideas to the category of A-Hodge complexes. To each complex variety X we can associate an element on the derived category of A-Hodge complex, A-$\mathcal{HC}(X)$. Since the derived category of A-Hodge complexes is equivalent to the derived category of mixed A-Hodge structures, it has a t-structure whose heart is the category of mixed A-Hodge structures A-$\mathcal{MHS}$, the obtained cohomology, denoted $H^n(X, A) \in$ A-$\mathcal{MHS}$, is the usual cohomology groups with their mixed A-Hodge structure.

We now apply the second method. The category A-$\mathcal{HC}$ has a distinguished object $A(0)$.

Definition 2.1. The absolute cohomology groups of X are defined as

$$H^n_{\mathcal{AH}}(X, A(j)) = H^n(\mathrm{RHom}(A(0), \mathbf{H}(X)(j))).$$

Let us see an example of what kind of beast we have defined. To fix ideas, put $A = \mathbb{Z}$, and assume that X is smooth and projective. By the Proposition 2.10 from last lecture, we know that, if M is a mixed $\mathbb{Z}$-Hodge structure, then $\mathrm{Ext}^i(\mathbb{Z}(0), M) = 0$ for $i > 1$. Therefore, the spectral sequence associated to RHom:

$$E_2^{p,q} = \mathrm{Ext}^p(\mathbb{Z}(0), H^q(X)(j)) \Longrightarrow H^{p+q}_{\mathcal{AH}}(X, \mathbb{Z}(j))$$

only contains terms with $p = 0, 1$. Therefore, it degenerates at the term E_2 and there are short exact sequences

(2.1)
$$0 \to \mathrm{Ext}^1(\mathbb{Z}(0), H^{n-1}(X)(j)) \to H^n_{\mathcal{AH}}(X, \mathbb{Z}(j)) \to \mathrm{Hom}(\mathbb{Z}(0), H^n(X)(j)) \to 0.$$

In the case $n = 2j$ we obtain that

$$\mathrm{Hom}(\mathbb{Z}(0), H^{2j}(X)(j)) = \alpha(H^{2j}(X, \mathbb{Z})) \cap H^{j,j}(X, \mathbb{C}),$$

where $\alpha\colon H^{2j}(X, \mathbb{Z}) \otimes \mathbb{C} \to H^{2j}(X, \mathbb{C})$ is the comparison isomorphism between Betti and de Rham cohomology, multiplied by by $(2\pi i)^j$. This group is called the group of Hodge cycles. While

$$\mathrm{Ext}^1(\mathbb{Z}, H^{2j-1}(X)(j)) = J(H^{2j-1}(X)(j)) =: J^j(X)$$

is the j-th intermediate Jacobian of X.

Now we want to construct a complex that computes absolute Hodge cohomology. Given a diagram

$$(2.2) \qquad \mathcal{D} = \left(\begin{tikzcd}[column sep=small] & \mathcal{F}'_1 & & \mathcal{F}'_2, & & \cdots & & \mathcal{F}'_k & \\ \mathcal{F}_0 \arrow[ur, "\varphi_1"] & & \mathcal{F}_1 \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & \mathcal{F}_2 \arrow[ul, "\psi_2"'] & & \cdots & & \mathcal{F}_k \arrow[ul, "\psi_k"'] \end{tikzcd} \right)$$

we denote $\widetilde{\Gamma}\mathcal{D} = \mathrm{cone}(\sum \varphi_i - \sum \psi_i)[-1]$. If $\mathbf{H}$ is an A-Hodge complex as in Definition 1.2, then we write

$$\mathcal{D}(\mathbf{H}) = \left(\begin{tikzcd}[column sep=small] & \mathcal{F}'_{\mathbb{Q}} & & \hat{W}_0\mathcal{F}'_{\mathbb{C}}, & \\ \mathcal{F}_A \arrow[ur, "\varphi_1"] & & \hat{W}_0\mathcal{F}_{\mathbb{Q}} \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\hat{W}_0 \cap F^0)\mathcal{F}_{\mathbb{C}} \arrow[ul, "\psi_2"'] \end{tikzcd} \right)$$

Theorem 2.2. *Let* $\mathbf{H}$ *be an* A*-Hodge complex. Then, in* $D^b(A\text{-}\mathcal{MOD})$*, there is an isomorphism*

$$\mathrm{RHom}(A(0), H) \xrightarrow{\simeq} \widetilde{\Gamma}(\mathcal{D}(\mathbf{H}))$$

From this result we obtain a concrete complex that computes the absolute Hodge cohomology of a complex variety

Assume that we have an A-Hodge complex as in Definition 1.2 that represents $\mathbf{H}(X)$ for some complex variety X. Then the diagram $\mathcal{D}(\mathbf{H}(X)(j))$ is given by

$$\left(\begin{tikzcd}[column sep=small] & \mathcal{F}'_{\mathbb{Q}} & & \hat{W}_{2j}\mathcal{F}'_{\mathbb{C}}, & \\ \mathcal{F}_A \arrow[ur, "\varphi_1"] & & \hat{W}_{2j}\mathcal{F}_{\mathbb{Q}} \arrow[ul, "\psi_1"'] \arrow[ur, "(2\pi i)^j\varphi_2"] & & (\hat{W}_{2j} \cap F^j)\mathcal{F}_{\mathbb{C}} \arrow[ul, "\psi_2"'] \end{tikzcd} \right)$$

and the absolute Hodge cohomology of X is

$$H^n_{\mathcal{AH}}(X, A(j)) = H^n(\widetilde{\Gamma}(\mathcal{D}(\mathbf{H}(X)(j)))). \tag{2.3}$$

This expression of absolute Hodge cohomology gives us some long exact sequences that relate it with the usual cohomology groups. In the exercises we will see one of these exact sequences.

3. Absolute Hodge Homology and the cycle class map

As an example of the use of absolute Hodge cohomology, in this section we will construct the class of a cycle, that integrates at the same time, the usual cohomology class and the Abel-Jacobi map. The description of the cycle map is easier using Homology and Poincaré duality.

Let X be a smooth complex variety. The space of currents on X of degree n, denoted ${}'E^n(X)$, is the topological dual of the group of differential forms with compact support $E_c^{-n}(X)$. The differential

$$\mathrm{d} : {}'E^n(X) \longrightarrow {}'E^{n+1}(X)$$

is defined by

$$\mathrm{d}\, T(\varphi) = (-1)^n T(\mathrm{d}\, \varphi);$$

here T is a current and φ a test form. The bigrading of $E_c^*(X)$ induces a bigrading on ${}'E^*(X)$.

There is a pairing

$$E^n(X) \otimes {}'E^m(X) \longrightarrow {}'E^{n+m}(X), \quad \omega \otimes T \longmapsto \omega \wedge T,$$

where the current $\omega \wedge T$ is defined by

$$(\omega \wedge T)(\eta) = T(\eta \wedge \omega).$$

There a bigrading ${}'E^n_{\mathbb{C}}(X) = {}'E^n(X) \otimes \mathbb{C} = \bigoplus_{p+q=n} {}'E^{p,q}(X)$ and an associated Hodge filtration F.

If X is equidimensional of dimension d, there is a morphism of complexes $E^*(X) \to {}'E^*(X)[-2d]$ that sends ω to the current

$$\eta \mapsto \int \eta \wedge \omega.$$

Let now $(C_*(X, \mathbb{Z}), \partial)$ be the homological complex of smooth singular chains and write

$$C^n(X, \mathbb{Z}) = C_{-n}(X, \mathbb{Z}), \qquad \mathrm{d} = (-1)^n \partial \colon C^n(X, \mathbb{Z}) \to C^{n+1}(X, \mathbb{Z}).$$

By Stokes theorem, there is a morphism of complexes

$$\alpha \colon C^*(X, \mathbb{Z}) \to {}'E^*(X)$$

given, for $c \in C^n(X, \mathbb{Z})$, by

$$\alpha(c)(\eta) = \int_c \eta.$$

Thus, the complex of currents contains, at the same time, differential forms and smooth singular chains. One of the interesting properties of the complex of currents is that it is *covariant with respect to proper morphisms*: Let $f \colon Y \to X$ be a proper morphism of smooth complex varieties, then there is a morphism of complexes $f_* \colon {}'E^*(Y) \to {}'E^*(X)$ given by

$$f_* T(\eta) = T(f^* \eta).$$

If X is a smooth proper complex variety, The homology A-Hodge complex of X, denoted $'\mathbf{H}(X)$ is the diagram

$$
\begin{tikzcd}
 & C^*(X, A\otimes\mathbb{Q}) & & ('E^*_{\mathbb{C}}(X), \tau) & \\
C^*(X,A) \arrow[ur] & & (C^*(X, A\otimes\mathbb{Q}), \tau) \arrow[ul] \arrow[ur] & & ('E^*_{\mathbb{C}}(X), \tau, F), \arrow[ul]
\end{tikzcd}
$$

where τ is the canonical filtration and F is the Hodge filtration associated to the bigrading of $'E^*_{\mathbb{C}}(X)$.

Thanks to the covariance properties of the complex of smooth singular chains and of the complex of currents, if $f\colon Y \to X$ is a morphism of smooth proper complex varieties, then there is an induced map

$$f_* \in \operatorname{Hom}_{D(\text{A-}\mathcal{HC})}('\mathbf{H}(Y), '\mathbf{H}(X)).$$

The last piece of information that we need to construct the class of a cycle is Poincaré duality.

Theorem 3.1 (Poincaré duality)**.** *Let X be a smooth proper complex variety of dimension d. Then there is a canonical isomorphism in $D(\text{A-}\mathcal{HC})$*

$$\mathbf{H}(X) \longrightarrow {}'\mathbf{H}(X)[-2d](-d)$$

Let now Y be a codimension p irreducible subvariety of X. Let $\widetilde{Y}$ be a resolution of singularities of Y. Then we have a composition of maps

$$A(-p)[-2p] \to \mathbf{H}(\widetilde{Y})[-2p](-p) \to {}'\mathbf{H}(\widetilde{Y})[-2d](-d) \to {}'\mathbf{H}(X)[-2d](-d) \to \mathbf{H}(X).$$

Thus, we have an element

$$\operatorname{cl}(Y) \in \operatorname{Hom}_{D(\text{A-}\mathcal{HC})}(A(-p)[-2p], \mathbf{H}(X)) = H^{2p}_{\mathcal{AH}}(X, A(p))$$

A codimension p-cycle is a finite formal linear combination of irreducible subvarieties. Thus, once we have defined the class of a subvariety, by linearity, we can define the class of any cycle.

We now use the exact sequence (2.1). Let Y be a codimension p cycle then the class $\operatorname{cl}(Y) \in H^{2p}_{\mathcal{AH}}(X, \mathbb{Z}(p))$ is mapped to the group of Hodge cycles $\alpha(H^{2p}(X,\mathbb{Z}))\cap H^{p,p}(X,\mathbb{C})$, which is the usual cohomology class of the cycle. If this class is zero, then $\operatorname{cl}(Y)$ can be lifted to the intermediate Jacobian $J(H^{2p-1}(X))$. This is the Abel-Jacobi map.

4. Exercises

Exercises

(1) Let (A^*, W) be a complex with a filtration. Denote the Decalé filtration of W as $\mathrm{Dec}(W)$.
  (a) Show that the term E_2 of the spectral sequence associated to W agrees (up to renumbering) with the term E_1 of the spectral sequence of $\mathrm{Dec}(W)$.
  (b) Given a filtration W construct a new filtration $\mathrm{Und}(W)$ with the property that $\mathrm{Dec}(\mathrm{Und}(W)) = W$, and although
$$(A, \mathrm{Und}(\mathrm{Dec}(W))) \xrightarrow{\mathrm{Id}} (A, W)$$
is not a filtered quasi-isomorphism, the associated spectral sequences agree from the term E_2 on.

(2) Prove that, if H is a Hodge complex, then the spectral sequence associated to the filtration $\hat{W}$ degenerates at the term E_1 (Hint: the spectral sequence associated to the weight filtration is a spectral sequence of Hodge structures).

(3) Let X be a complex variety.
  (a) Assume that A is a field. Prove that the absolute A-Hodge cohomology fits in a long exact sequence
$$\dots \longrightarrow W_{2j}H^{n-1}(X, \mathbb{C}) \longrightarrow H^n_{\mathcal{AH}}(X, A(j)) \\ \longrightarrow (2\pi i)^j W_{2j}H^n(X, A) \oplus F^j W_{2j}H^n(X, \mathbb{C}) \longrightarrow \dots$$
  (b) Write down the corresponding long exact sequence when A is not a field and particularize it to the case when X is smooth and projective.

(4) Compute the absolute $\mathbb{Z}$-Hodge cohomology of $\operatorname{Spec}\mathbb{C}$.

(5) Given a smooth projective variety X, compute the absolute $\mathbb{R}$-Hodge cohomology of X in terms of the usual Hodge structure of the cohomology.

References

[Bei83] A.A. Beilinson, *Notes on absolute Hodge cohomology*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory (S. Bloch, ed.), Contemporary Mathematics, vol. 55, AMS, 1983, pp. 35–68.