Contents

1. Absolute Hodge cohomology of smooth complex varieties

In this section we will show how to construct a complex that computes the real absolute Hodge cohomology of a smooth complex variety that is not necessarily proper.

Let X be a smooth complex variety of dimension d. By Nagata [Nag62] and Hironaka [Hir64] we can find a proper complex variety $\overline{X}$ and a dense open embedding $j\colon X \to \overline{X}$ such that $D = \overline{X} \setminus X$ is a simple normal crossing divisor (NCD). This means that, each point $x \in \overline{X}$ has a coordinate neighbourhood U with coordinates $(z_1, \dots, z_d)$ on which x has coordinates $(0, \dots, 0)$, there is an integer k with $0 \le k \le d$ such that

$$D \cap U = \{(z_1, \dots, z_d) \mid z_1 \dots z_k = 0\},$$

and each irreducible component D_i of D is smooth (that is, the irreducible components of D do not have self intersections). Such coordinate neighborhood is called *adapted to D*.

Definition 1.1. The sheaf of holomorphic forms with logarithmic poles along D, denoted $\Omega^*_{\overline{X}}(\log D)$ is the subsheaf of algebras of $j_*\Omega^*_X$ generated locally in each adapted coordinate neighborhood as before by $\Omega^*_{\overline{X}}$ and the forms

$$\frac{\mathrm{d}\, z_i}{z_i},\ i = 1, \dots, k.$$

The weight filtration W of $\Omega^*_{\overline{X}}(\log D)$ is the multiplicative filtration that assigns weight zero to the sections of $\Omega^*_{\overline{X}}$ and weight one to the sections $\mathrm{d}\, z_i/z_i$. The decalé filtration of W is denoted by $\hat{W}$. The Hodge filtration F of $\Omega^*_{\overline{X}}(\log D)$ is the decreasing filtration

$$F^p\Omega^*_{\overline{X}}(\log D) = \bigoplus_{q \ge p} \Omega^q_{\overline{X}}(\log D).$$

Lemma 1.2. *The inclusion $\Omega^*_{\overline{X}}(\log D) \hookrightarrow j_*\Omega^*_X$ is a quasi-isomorphism. Therefore there are isomorphisms*

$$\mathbb{H}^*(\overline{X}, \Omega^*_{\overline{X}}(\log D)) \simeq \mathbb{H}^*(\overline{X}, j_*\Omega^*_X) \simeq H^*_{\mathrm{dR}}(X, \mathbb{C}).$$

Thus we can use differential forms with logarithmic poles to compute de Rham cohomology of X. The key point now is that this complex of forms allow us to relate the cohomology of X to the cohomology of smooth proper complex varieties of different dimensions.

To see this we define the residue map. Let $D = \bigcup_{i=1}^k D_i$ be the decomposition of D into irreducible components. For each $I \subset \{1, \dots, k\}$ we denote by $D_I = \cap_{i \in I} D_i$

and we write

$$\widetilde{D}^n = \coprod_{|I|=n} D_I,$$

where for a subset I as before we denote by $|I|$ its cardinal. Then $\widetilde{D}^n$ is a disjoint union of smooth proper complex varieties of dimension $d-n$. Let U be a coordinate neighborhood with coordinates $(z_1,\dots,z_k)$ adapted to D. For each $j \in \{1,\dots,k\}$ we will denote by i_j the index of the component of D satisfying $D_{i_j} = \{z_j = 0\}$. If

$$\eta = \alpha \wedge \frac{\mathrm{d}\, z_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}\, z_{j_m}}{z_{j_m}} \in W_n \Omega^r_{\overline{X}}(logD),\ \ m \leq n$$

then the residue is defined locally by

$$\operatorname{Res}(\eta) = \begin{cases} 0, & \text{if } m < n, \\ \alpha|_{D_{i_{j_1}} \cap \cdots \cap D_{i_{j_n}}}, & \text{if } m = n. \end{cases}$$

Lemma 1.3. *The residue of a differential forms with logarithmic poles defines an isomorphism*

$$\operatorname{Res}\colon \operatorname{Gr}^W_n \Omega^*_{\overline{X}}(\log D) \to \Omega^*_{\widetilde{D}^n}[-n].$$

Lemma 1.4. *The natural maps*

$$(\Omega^*_{\overline{X}}(\log D), W) \longleftarrow (\Omega^*_{\overline{X}}(\log D), \tau) \longrightarrow (j_*\Omega_X, \tau)$$

are filtered quasi-isomorphisms.

Proof. The right arrow is a quasi-isomorphims because the inclusion $\Omega^*_{\overline{X}}(\log D) \longrightarrow j_*\Omega_X$ is a quasi-isomorphism. The right arrow is a filtered quasi-isomorphism because, by Lemma 1.3

$$\mathscr{H}^r(\operatorname{Gr}^W_n \Omega^*_{\overline{X}}(\log D)) = \begin{cases} 0, & \text{if } r \neq n, \\ \iota_{\widetilde{D}^n}\mathbb{C}_{\widetilde{D}^n}, & \text{if } r = n. \end{cases}$$

where $\iota_{\widetilde{D}^n}\colon \widetilde{D}^n \to \overline{X}$ is the map induced by the inclusion $D \subset \overline{X}$ and $\mathbb{C}_{\widetilde{D}^n}$ is the constant sheaf with fiber $\mathbb{C}$. Hence, the cohomology sheaf of $\operatorname{Gr}^W_n \Omega^*_{\overline{X}}(\log D)$ is concentrated in degree n. Therefore

$$\mathscr{H}^n(\operatorname{Gr}^W_n \Omega^*_{\overline{X}}(\log D)) = \mathscr{H}^n(\Omega^*_{\overline{X}}(\log D)) = \mathscr{H}^n(\operatorname{Gr}^\tau_n \Omega^*_{\overline{X}}(\log D)),$$

and, for $r \neq n$,

$$\mathscr{H}^r(\operatorname{Gr}^W_n \Omega^*_{\overline{X}}(\log D)) = 0 = \mathscr{H}^r(\operatorname{Gr}^\tau_n \Omega^*_{\overline{X}}(\log D)).$$

□

We now choose flasque resolutions of the sheaves A_X, $A_X \otimes Q$, $\mathbb{C}_X$ and Ω^*_X that fit in a commutative diagram

$$\begin{tikzcd} A_X^\sharp \arrow[r] & A_X \otimes Q^\sharp \arrow[r] & \mathbb{C}_X^\sharp \arrow[r] & \Omega^{*\,\sharp}_X \\ A_X \arrow[r] \arrow[u] & A_X \otimes Q \arrow[r] \arrow[u] & \mathbb{C}_X \arrow[r] \arrow[u] & \Omega^*_X \arrow[u] \end{tikzcd}$$

of complexes of sheaves in X. Consider the diagram

$$\begin{tikzcd}
(j_*\mathbb{C}_X^\sharp,\tau) \arrow[r] & (j_*\Omega^{*\sharp},\tau) & (\Omega^*_{\overline{X}}(\log D),\tau) \arrow[l, "i_1"'] \arrow[d, "i_2"] \\
 & & (\Omega^*_{\overline{X}}(\log D),W)
\end{tikzcd}$$

The arrows in the above diagram are filtered quasi-isomorphisms and we will denote $(\widetilde{\Omega}^*,W)=\operatorname{cone}(i_1-i_2)$. From the diagram we obtain a filtered quasi-isomorphism $(j_*\mathbb{C}_X^\sharp,\tau)\to(\widetilde{\Omega}^*,W)$ and $(\Omega^*_{\overline{X}}(\log D),W)\to(\widetilde{\Omega}^*,W)$. We now consider the diagram of complexes of sheaves on $\overline{X}$

$$\left(\begin{tikzcd}[column sep=small]
 & j_*(A_X\otimes\mathbb{Q}^\sharp) & & (\widetilde{\Omega}^*,W) & \\
j_*(A_X^\sharp) \arrow[ur] & & (j_*(A_X\otimes\mathbb{Q}^\sharp),\tau) \arrow[ul] \arrow[ur] & & (\Omega^*_{\overline{X}}(\log D),W,F) \arrow[ul]
\end{tikzcd}\right)$$

that we denote $\mathcal{D}(X,\overline{X})$. We choose compatible (filtered) flasque resolutions on the sheaves of such diagram and denote the corresponding diagram by $\mathcal{D}(X,\overline{X})^\sharp$.

We finally apply the functor of global sections Γ to the above complex to obtain a diagram of filtered complexes of A-modules that we denote by $\Gamma(\mathcal{D}(X,\overline{X})^\sharp)$.

Theorem 1.5. *Let X be a smooth complex variety and $\overline{X}$ a compactification of X with $D=\overline{X}\setminus X$ a NCD. The complex $\Gamma(\mathcal{D}(X,\overline{X})^\sharp)$ is a mixed Hodge complex and its class in $A\mathcal{HC}$ does not depend on $\overline{X}$ nor on the choices of flasque resolutions.*

We now want to particularize to the case when $A=\mathbb{R}$. In this case we can simplify the construction of a mixed $\mathbb{R}$-Hodge complex by considering an acyclic resolution of the sheaf $\Omega^*_{\overline{X}}(\log D)$ that has a real structure. In this way we will be able to use the same complex to recover also the real piece of the cohomology, avoiding most of the technicalities of the previous construction.

Definition 1.6. The sheaf of differential forms with logarithmic singularities along D[BG94], denoted $\mathscr{A}^*_{\overline{X}}(\log D)$ is the subsheaf of algebras of $j_*\mathscr{A}^*_X$ generated locally in each adapted coordinate neighborhood as before by $\mathscr{A}^*_{\overline{X}}$ and the forms

$$\frac{\mathrm{d}\,z_i}{z_i},\frac{\mathrm{d}\,\bar{z}_i}{\bar{z}_i},\log(z_i\bar{z}_i)\ i=1,\dots,k.$$

The weight filtration W of $\mathscr{A}^*_{\overline{X}}(\log D)$ is the multiplicative filtration that assigns weight zero to the sections of $\mathscr{A}^*_{\overline{X}}$ and weight one to the sections $\mathrm{d}\,z_i/z_i$, $\mathrm{d}\,\bar{z}_i/\bar{z}_i$, and $\log(z_i\bar{z}_i)$. The decalé filtration of W is denoted by $\hat{W}$. The Hodge filtration F of $\mathscr{A}^*_{\overline{X}}(\log D)$ is the decreasing filtration

$$F^p\mathscr{A}^*_{\overline{X}}(\log D)=\bigoplus_{q\geq p}\mathscr{A}^q_{\overline{X}}(\log D).$$

Clearly, $\mathscr{A}^*_{\overline{X}}(\log D)$ is stable under complex conjugation. We will denote by $\mathscr{A}^*_{\overline{X},\mathbb{R}}(\log D)$ the subsheaf of algebras consisting of forms invariant under complex conjugation. Since the weight filtration is invariant under complex conjugation, it induces a filtration $\hat{W}$ on $\mathscr{A}^*_{\overline{X},\mathbb{R}}(\log D)$.

Since the complex $\mathscr{A}^*_{\overline{X}}(\log D)$ is a complex of $\mathscr{A}^*_{\overline{X}}$-algebras, it is a complex of fine sheaves and hence of acyclic sheaves.

We will denote by $A^*_{\overline{X}}(\log D)$ (respectively $A^*_{\overline{X},\mathbb{R}}(\log D)$) the complex of global sections of $\mathscr{A}^*_{\overline{X}}(\log D)$ (respectively $\mathscr{A}^*_{\overline{X},\mathbb{R}}(\log D)$)

The main properties we need of the complex of differential forms with logarithmic singularities are summarized in the following result.

Theorem 1.7. *With the hypothesis of Theorem 1.5*

(1) *The natural inclusion* $(\Omega^*_{\overline{X}}(\log D), W, F) \to (\mathscr{A}^*_{\overline{X}}(\log D), W, F)$ *is a bifiltered quasi-isomorphism of complexes of sheaves.*
(2) *The map* $A^*_{\overline{X},\mathbb{R}}(\log D) \to S^*(X, \mathbb{R})$ *given by integration is a quasi-isomomorphims.*

Joining together theorems 1.5 and 1.7 we obtain

Corollary 1.8. *Let X be a smooth complex variety and $\overline{X}$ a compactification of X with $D = \overline{X} \setminus X$ a NCD. Then the diagram*

$$\begin{tikzcd}
 & (A^*_{\overline{X}}(\log D), \hat{W}) & \\
(A^*_{\overline{X},\mathbb{R}}(\log D), \hat{W}) \arrow[ur, "\iota"] & & (A^*_{\overline{X}}(\log D), \hat{W}, F) \arrow[ul, "\mathrm{Id}"']
\end{tikzcd}$$

is an $\mathbb{R}$-Hodge complex that agrees with $\mathbf{H}(X)$.

The next task is to get rid of a particular compactification of X to this end we consider the category $\mathcal{C}_X$ of compactifications of X. That is, the objects of $\mathcal{C}_X$ are triples $(\overline{X}, D, \iota)$ where X is a proper complex variety, $D \subset X$ is a NCD and $\iota\colon X \to \overline{X} \setminus D$ is an isomorphism. A morphisms between $(\overline{X}, D, \iota)$ and $(\overline{X}', D', \iota')$ is a morphism of varieties $f\colon \overline{X} \to \overline{X}'$ such that $f \circ \iota = \iota'$. The opposed category $\mathcal{C}_X^\circ$ is directed.

Definition 1.9. The complex of differential forms on X with logarithmic singularities along infinity is

$$A^*_{\log}(X) = \lim_{\substack{\longrightarrow \\ \mathcal{C}_X^\circ}} A^*_{\overline{X}}(\log D).$$

This complex inherits a weight filtration, a Hodge filtration and a complex conjugation, hence a real structure $A^*_{\log,\mathbb{R}}(X)$.

For any $(X, D, \iota) \in \mathcal{C}_X$, the map $(A^*_{\log}(X), \hat{W}, F) \to (A^*_{\overline{X}}(\log D), \hat{W}, F)$ is a bifiltered quasi-isomorphism and the map $(A^*_{\log,\mathbb{R}}(X), \hat{W}) \to (A^*_{\overline{X},\mathbb{R}}(\log D), \hat{W})$ is a filtered quasi-isomorphism. Therefore the diagram

$$\mathbf{A}_{\log}(X) = \left(\begin{tikzcd}
 & (A^*_{\log}(X), \hat{W}) & \\
(A^*_{\log,\mathbb{R}}(X), \hat{W}) \arrow[ur, "\iota"] & & (A^*_{\log}(X), \hat{W}, F) \arrow[ul, "\mathrm{Id}"']
\end{tikzcd}\right)$$

is an $\mathbb{R}$-Hodge complex that again agrees with $\mathbf{H}(X)$.

We can now apply Theorem 2.2 of the previous lecture to this complex.

Definition 1.10. Let X be a smooth complex variety. Then the *absolute Hodge cohomology* complex of X is the complex of sheaves $\mathbf{RAH}_{\mathbb{C}}(X)$ is defined as $\mathbf{RAH}_{\mathbb{C}}(X) = \widetilde{\Gamma}(\mathcal{D}(\mathbf{A}_{\log}(X)))$. Unraveling this definition, we have

$$\mathbf{RAH}_{\mathbb{C}}(X) = \operatorname{cone}(\hat{W}_0 A_{\log,\mathbb{R}}(X) \oplus \hat{W}_0 \cap F^0 A_{\log}(X) \xrightarrow{\varphi} \hat{W}_0 A_{\log}(X))[-1],$$

where $\varphi(r, f) = r - f$.

More generally the *twisted absolute Hodge cohomology* complex of X is

$$\mathbf{RAH}_{\mathbb{C}}(X, p) = \widetilde{\Gamma}(\mathcal{D}(\mathbf{A}_{\log}(X)(p))).$$

In this case we obtain

$$\mathbf{RAH}_{\mathbb{C}}(p)(X) = \operatorname{cone}((2\pi i)^p \hat{W}_{2p} A_{\log,\mathbb{R}}(X) \oplus \hat{W}_{2p} \cap F^p A_{\log}(X) \xrightarrow{\varphi} \hat{W}_{2p} A_{\log}(X))[-1].$$

One of the main advantages of this construction is that it is functorial on the nose and we do not need to choose any particular flasque resolution.

Lemma 1.11. *Let $\mathbf{Sm}_{\mathbb{C}}$ be the category of smooth complex varieties. The assignment $X \mapsto \mathbf{A}_{\log}(X)$ is a presheaf of $\mathbb{R}$-Hodge complexes in $\mathbf{Sm}_{\mathbb{C}}$. The assignments $X \mapsto \mathbf{RAH}_{\mathbb{C}}(p)(X)$, $p \in \mathbb{Z}$ are presheaves of complexes of real vector spaces.*

There is a variant of the absolute Hodge cohomology called *weak Hodge cohomology* or *Deligne-Beilinson* cohomology that is obtained by ignoring the effect of the weight filtration. The real Deligne-Beilinson cohomology of X is

$$H^*_{\mathcal{D}}(X, \mathbb{R}(p)) = H^*(\operatorname{cone}((2\pi i)^p A_{\log,\mathbb{R}}(X) \oplus F^p A_{\log}(X) \xrightarrow{\varphi_p} A_{\log}(X))[-1])).$$

We end this section discussing the case of real varieties. Let X be a real variety. To it we can associate a complex variety $X_{\mathbb{C}}$ together with an conjugate-linear involution F_∞ of $X_{\mathbb{C}}$. In fact the category of real varieties is equivalent to the category of complex varieties with a conjugate linear involution. On the space of differential forms $A^*(X_{\mathbb{C}})$ we define an involution σ given by

$$\sigma(\omega) = \overline{F^*_\infty \omega}.$$

that is, σ acts as complex conjugation on both, the space and the coefficients of the differential form. It is easy to see that σ induces an involution in the complexes $\mathbf{RAH}_{\mathbb{C}}(p)(X_{\mathbb{C}})$.

Definition 1.12. Let X be a smooth real variety. Then the *absolute Hodge cohomology* complex of X is

$$\mathbf{RAH}(p)(X) = (\mathbf{RAH}_{\mathbb{C}}(p)(X_{\mathbb{C}}))^{\sigma = \mathrm{Id}}.$$

That is, the complex formed by the elements of $\mathbf{RAH}_{\mathbb{C}}(p)(X_{\mathbb{C}})$ that are invariant under σ.

2. Zariski descent

One of the basic results in topology concerning singular cohomology is the Mayer-Vietoris sequence, that relates the cohomology of two open subsets with that of its union and intersection. This result is what makes singular cohomology "easily" computable. The Mayer-Vietoris theorem can be generalized as the Čech spectral sequence in sheaf cohomology and in Deligne's theory of homological descent, that allow us to compute the cohomology of a variety in terms of hyper-resolutions. The fact that the Čech spectral sequence associated to a Zariski open cover converges to the cohomology is a particular case of homological descent called Zariski descent.

We denote by $\mathbf{Sm}_{\mathbb{C}}$ the category of smooth complex varieties. Then $\mathbf{RAH}_{\mathbb{C}}(p)$ is a pre-sheaf on $\mathbf{Sm}_{\mathbb{C}}$. That is, a contravariant functor. Let $X \in \mathrm{Ob}(\mathbf{Sm}_{\mathbb{C}})$ and let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be a Zariski open covering of X. The nerve of $\mathfrak{U}$, denoted $\mathcal{N}(\mathfrak{U})$ is the simplicial smooth variety with

$$\mathcal{N}(\mathfrak{U})_n = \coprod_{(\alpha_0,\dots,\alpha_n)\in\Lambda^n} U_{\alpha_0}\cap\cdots\cap U_{\alpha_n},$$

the degeneracy map $\sigma_i\colon \mathcal{N}(\mathfrak{U})_n \to \mathcal{N}(\mathfrak{U})_{n+1}$ sends the component $U_{\alpha_0}\cap\cdots\cap U_{\alpha_n}$ identically to the component $U_{\alpha_0}\cap\cdots\cap U_{\alpha_i}\cap U_{\alpha_i}\cap\cdots\cap U_{\alpha_n}$, while the face map $\delta_i\colon \mathcal{N}(\mathfrak{U})_n \to \mathcal{N}(\mathfrak{U})_{n+1}$ includes the component $U_{\alpha_0}\cap\cdots\cap U_{\alpha_i}\cap\cdots\cap U_{\alpha_n}$ into the component $U_{\alpha_0}\cap\cdots\cap\widehat{U_{\alpha_i}}\cap\cdots\cap U_{\alpha_n}$, where the simbol $\widehat{\ }$ means that the subset is omitted.

If we apply the functor $\mathbf{RAH}_{\mathbb{C}}(p)$ to it we obtain a cosimplicial complex $\mathbf{RAH}_{\mathbb{C}}(p)(\mathfrak{U})$. To this cosimplicial complex we can associate the total complex $\mathrm{tot}(\mathbf{RAH}_{\mathbb{C}}(p)(\mathfrak{U}))$. Since $\mathbf{RAH}_{\mathbb{C}}(p)$ is a functor, there is a canonical map $\mathbf{RAH}_{\mathbb{C}}(p)(X) \to \mathrm{tot}(\mathbf{RAH}_{\mathbb{C}}(p)(\mathfrak{U}))$.

Definition 2.1. Let $\mathcal{F}$ be a pre-sheaf of complexes on $\mathbf{Sm}_{\mathbb{C}}$. We says that $\mathcal{F}$ satisfies Zariski descent if the canonical map

$$\mathcal{F}(X) \to \mathcal{F}(\mathfrak{U})$$

is a quasi-isomorphims.

Theorem 2.2. *The pre-sheaf* $\mathbf{RAH}_{\mathbb{C}}(p)$ *satisfies Zariski descent.*

Proof. Since de Rham cohomology satisfies the Mayer-Vietoris theorem, the complexes A^* and $A^*_{\mathbb{R}}$ satisfy descent for the ordinary topology, hence for the Zariski topology. Since $A_{\log}(X)^* \to A^*(X)$ and $A_{\log,\mathbb{R}}(X)^* \to A^*_{\mathbb{R}}(X)$ are quasi-isomorphisms, then $A^*_{\log}$ and $A^*_{\log,\mathbb{R}}$ satisfy Zariski descent.

Since $\mathbf{A}_{\log}$ is a presheaf, to the simplicial variety $\mathcal{N}\mathfrak{U}$ we can associate a cosimplicial $\mathbb{R}$-Hodge complex $\mathbf{A}_{\log}(\mathcal{N}\mathfrak{U})$.

We now use the following result by Deligne.

Theorem 2.3. *Let* $\mathbf{H}^{\cdot}$ *be a cosimplicial A-Hodge complex,*

$$\mathbf{H}^{\cdot} = \left(\begin{tikzcd}[column sep=small] & \mathcal{F}'^{\cdot}_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}},\hat{W})^{\cdot} & \\ \mathcal{F}_A^{\cdot} \arrow[ur, "\varphi_1"] & & (\mathcal{F}_{\mathbb{Q}},\hat{W})^{\cdot} \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & (\mathcal{F}_{\mathbb{C}},\hat{W},F)^{\cdot} \arrow[ul, "\psi_2"'] \end{tikzcd}\right)$$

Then the diagram $\mathrm{tot}(\mathbf{H})$ *given by*

$$\begin{tikzcd}[column sep=small] & \mathrm{tot}(\mathcal{F}'_{\mathbb{Q}}) & & \mathrm{tot}(\mathcal{F}'_{\mathbb{C}},\hat{W}) & \\ \mathrm{tot}(\mathcal{F}_A) \arrow[ur, "\varphi_1"] & & \mathrm{tot}(\mathcal{F}_{\mathbb{Q}},\hat{W}) \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & \mathrm{tot}(\mathcal{F}_{\mathbb{C}},\hat{W},F) \arrow[ul, "\psi_2"'] \end{tikzcd}$$

By this theorem, the total complex of $\mathbf{A}_{\log}(\mathcal{N}\mathfrak{U})$ is again an $\mathbb{R}$-Hodge complex. There is a map of $\mathbb{R}$-Hodge complexes $\mathbf{H}(X) \to \mathrm{tot}\,\mathbf{A}_{\log}(\mathcal{N}\mathfrak{U})$. This induces a map of mixed $\mathbb{R}$-Hodge structures in cohomology. By Zariski descent of $A^*_{\mathbb{R}}$, this morphism induces an isomorphism of the corresponding real vector spaces. By Theorem 2.8 of the first lecture, it is an isomorphism of mixed Hodge structures.

Therefore, for all $p \in \mathbb{Z}$ $\hat{W}_{2p}A^*_{\log,\mathbb{R}}$, $\hat{W}_{2p}A^*_{\log}$ and $\hat{W}_{2p} \cap F^p A^*_{\log}$ satisfy Zariski descent. Hence $\mathbf{RAH}_{\mathbb{C}}(p)$ satisfies Zariski descent. □

3. Multiplicative properties

Since absolute Hodge cohomology is the cohomology of a diagram of differential graded commutative algebras it comes naturally with a graded commutative product that we will discuss in this section.

Assume that we have a diagram

$$\mathcal{D} = \left(\begin{tikzcd}[column sep=small] & \mathcal{F}'_1 & & \mathcal{F}'_2, & \dots & \mathcal{F}'_k & \\ \mathcal{F}_0 \arrow[ur, "\varphi_1"] & & \mathcal{F}_1 \arrow[ul, "\psi_1"'] \arrow[ur, "\varphi_2"] & & \mathcal{F}_2 \arrow[ul, "\psi_2"'] & \dots & \mathcal{F}_k \arrow[ul, "\psi_k"'] \end{tikzcd} \right) \tag{3.1}$$

such that all the entries $\mathcal{F}_i$ and $\mathcal{F}'_i$ are differential graded A-algebras and such that all the morphisms φ_i and ψ_i are morphisms of differential graded A-algebras. Then the complex $\widetilde{\Gamma}(\mathcal{D})$ has several structures of graded algebras. Let $\alpha \in A$ and let $\omega = (\omega_0, \dots, \omega_k, \omega'_1, \dots, \omega'_k) \in \widetilde{\Gamma}^n(\mathcal{D})$ and $\eta = (\eta_0, \dots, \eta_k, \eta'_1, \dots, \eta'_k) \in \widetilde{\Gamma}^m(\mathcal{D})$. Then we define

$$\begin{aligned} \omega \cup_\alpha \eta = (&\omega_0\eta_0, \dots, \omega_k\eta_k, \\ &\omega'_1((1-\alpha)\varphi_1(\eta_0) + \alpha\psi(\eta_1)) + (-1)^n(\alpha\varphi_1(\omega_0) + (1-\alpha)\psi_1(\omega_1))\eta'_1, \dots) \end{aligned}$$

The basic properties of this product are summarized in the following theorem.

Lemma 3.1. (1) *For every α, the map $\cup_\alpha$ is a morphism of complexes.*

(2) *If $\alpha, \alpha' \in A$ then the morphisms $\cup_\alpha$ and $\cup_{\alpha'}$ are homotopic. A homotopy is given, for ω and η as before, by*

$$h(\omega \cup_\alpha \eta) = (0, \dots, 0, (\alpha - \alpha')\omega'_1\eta'_1, \dots, (\alpha - \alpha')\omega'_k\eta'_k).$$

Under the canonical automorphism T of $\widetilde{\Gamma}(\mathcal{D}) \otimes \widetilde{\Gamma}(\mathcal{D})$ given by $T(\omega \otimes \eta) = (-1)^{\deg\omega\deg\eta}(\eta \otimes \omega)$, the product $\cup_\alpha$ is transformed into $\cup_{1-\alpha}$. The product $\cup_0$ and $\cup_1$ are associative.

As a direct consequence of this lemma we deduce that

$$H^*_{\mathcal{AH}}(X, A(*)) = \bigoplus_{n,j \in \mathbb{Z}} H^n_{\mathcal{AH}}(X, A(j))$$

has a structure of bigraded associative algebra whose product is graded-commutative with respect to the first degree. Nevertheless, at the level of complexes we do not have a product that is at the same time associative and graded commutative. For $\alpha = 0, 1$ we have an associative product and if $\alpha = 1/2 \in A$ we have a graded commutative product, but none of the products $\cup_\alpha$ is at the same time associative and commutative.

For the applications it will be useful to have a complex that computes real absolute Hodge cohomology and that is, at the same time, associative and graded commutative. Thus, we now will construct a pre-sheaf that is quasi-isomorphic to $\mathbf{RAH}_{\mathbb{C}}(*)$ but that has a product that is at the same time associative and graded commutative.

Let L^* denote the de Rham complex of algebraic differential forms on $A^1_{\mathbb{R}}$. That is $L^0 = \mathbb{R}[t]$ and $L^1 = \mathbb{R}[t]\,\mathrm{d}\,t$. Since $\hat{W}_{2p}A^*_{\log}(X)$ and L^* are differential graded commutative algebras, the tensor product $L^*\otimes\hat{W}_{2p}A^*_{\log}(X)$ (or rather the associated total complex) has a natural structure of differential graded complex. Be aware of the usual sign convention for the tensor products. We will denote this product by $\wedge$.

Definition 3.2. Let X be a smooth complex variety. The complex $\mathbf{IRAH}^*_{\mathbb{C}}(p)(X)$ is the subcomplex of $L^*\otimes\hat{W}_{2p}A^*_{\log}(X)$ composed by forms that satisfy

$$\begin{aligned}\omega\mid_{t=0}&\in(2\pi i)^p\hat{W}_{2p}A_{\log,\mathbb{R}}(X),\\ \omega\mid_{t=1}&\in\hat{W}_{2p}\cap F^pA_{\log}(X).\end{aligned}$$

Clearly $\mathbf{IRAH}^*_{\mathbb{C}}(p)(X)$ is not just a subcomplex but also a subalgebra. Thus, it is a differential graded commutative algebra.

The elements of $\mathbf{IRAH}^*_{\mathbb{C}}(p)(X)$ are one parameter deformations between real forms and forms in a certain space of the Hodge filtration. Note that, instead of L^* we could consider differential forms in the manifold $\mathbb{R}$ or just the interval $[0,1]$. The important property of L^* is that it is a "one-dimensional contractible" complex.

There are maps

$$\mathbf{RAH}_{\mathbb{C}}(p)(X)\underset{I}{\overset{E}{\rightleftarrows}}\mathbf{IRAH}_{\mathbb{C}}(p)(X)$$

given by

$$\begin{aligned}E(r,f,\eta)&=t\otimes f+(1-t)\otimes r+\mathrm{d}\,t\otimes\eta,\\ I(\omega)&=(\omega\mid_{t=0},\omega\mid_{t=1},\int_{t=0}^{t=1}\omega),\end{aligned}$$

where

$$\int_{t=0}^{t=1}(p(t)+q(t)\,\mathrm{d}\,t)\otimes\eta=\left(\int_0^1 q(t)\,\mathrm{d}\,t\right)\eta.$$

Theorem 3.3. *Let X be a smooth complex variety.*

1. *The composition $I\circ E=\mathrm{Id}_{\mathbf{RAH}}$.*
2. *There is a homotopy h in $\mathbf{IRAH}$ such that $E\circ I-\mathrm{Id}_{\mathbf{IRAH}}=\mathrm{d}\circ h+h\circ\mathrm{d}$.*
3. *If $\omega\in\mathbf{RAH}^n_{\mathbb{C}}(p)(X)$ and $\eta\in\mathbf{RAH}^m_{\mathbb{C}}(q)(X)$, then*

$$I(E(\omega)\wedge E(\eta))=\omega\cup_{1/2}\eta.$$

Proof. We check the first statement. Let $(r,f,\eta)\in\mathbf{RAH}_{\mathbb{C}}(X)$. Then $E(r,f,\eta)=t\otimes f+(1-t)\otimes r+\mathrm{d}\,t\otimes\eta$. Since $\mathrm{d}\,t\mid_{t=0}=\mathrm{d}\,t\mid_{t=1}=0$, we deduce that $E(r,f,\eta)\mid_{t=0}=r$ and $E(r,f,\eta)\mid_{t=1}=f$. Since $\int_{t=0}^{t=1}E(r,f,\eta)=\eta$, the first statement is clear.

For each pair of integers n,p, we define $h\colon\mathbf{IRAH}^n_{\mathbb{C}}(p)(X)\to\mathbf{IRAH}^{n-1}_{\mathbb{C}}(p)(X)$ by

$$h(\omega)=t\otimes\int_{t=0}^{t=1}\omega-\int_0^t\omega,$$

where

$$\int_0^t(p(t)+q(t)\,\mathrm{d}\,t)\otimes\eta=\left(\int_0^t q(s)\,\mathrm{d}\,s\right)\otimes\eta.$$

Then one can check easily that

$$h(\mathrm{d}\,\omega)+\mathrm{d}\,h(\omega)=E\circ I(\omega)-\omega.$$

The third statement can also be checked easily. □

This theorem has the following immediate consequence.

Corollary 3.4. *Let X be a smooth complex variety. then*

$$H^n(\mathbf{IRAH}_{\mathbb{C}}(p)(X)) = H^n_{\mathcal{AH}}(X, \mathbb{R}(p)).$$

Moreover the product induced in $H^n_{\mathcal{AH}}(X, \mathbb{R}(p))$ agrees with the product induced by Lemma 3.1.

If X is a smooth real variety, the involution σ determines an involution, also denoted σ, on $\mathbf{IRAH}_{\mathbb{C}}(p)(X_{\mathbb{C}})$ that is compatible with the product. We define

$$\mathbf{IRAH}(p)(X) = (\mathbf{IRAH}_{\mathbb{C}}(p)(X_{\mathbb{C}}))^{\sigma = \mathrm{Id}}.$$

Corollary 3.5. *Let X be a smooth real variety. then*

$$H^n(\mathbf{IRAH}(p)(X)) = H^n_{\mathcal{AH}}(X, \mathbb{R}(p)).$$

Moreover the product of $\mathbf{IRAH}(p)(X)$ induces the usual product in $H^n_{\mathcal{AH}}(X, \mathbb{R}(p))$.

4. Exercises

Exercises

(1) Let K be a number field. Consider $X_{\mathbb{Q}} = \operatorname{Spec} K$ as a variety over $\mathbb{Q}$. Let $X_{\mathbb{R}}$ be the corresponding real variety. Compute the absolute absolute Hodge cohomology of $X_{\mathbb{R}}$. Compare the obtained dimensions with the rank of the groups $K_i(K)$.

(2) Compute the mixed Hodge structure of $\mathbb{P}^1 \setminus \{0, 1\}$.

References

[BG94] J. I. Burgos Gil, *A C^∞-logarithmic Dolbeault complex*, Compositio Math. **92** (1994), 61–86.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. **79** (1964), 109–326.

[Nag62] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10.