3. Exercises

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(1) Finite dimensional Hodge theory. Let (A^*, d) be a bounded complex of finite dimensional real vector spaces. Assume that each piece A^* is provided with an Euclidean product. Prove Theorem 1.2 and Corollary 1.3 in this case.

(2) Let k be a field and $\mathcal{C}$ the category of filtered k-vector spaces.
   - (a) Give an example of a morphism f in $\mathcal{C}$ such that $\mathrm{Img}(f) \neq \mathrm{Coimg}(f)$. Conclude that $\mathbb{C}$ is not an abelian category.
   - (b) Show that, if f is a morphism in $\mathcal{C}$ that is strictly compatible with the filtrations, then $\mathrm{Img}(f) = \mathrm{Coimg}(f)$.
   - (c) Show that, if $f \colon H \to H'$ is a morphism of pure Hodge structures of weight n, then it is strictly compatible with the Hodge filtration.
   - (d) Let (A^*, d, F) be a filtered bounded complex of finite dimensional vector spaces, with F a bounded decreasing filtration. Prove that the differential d is strictly compatible with the filtration F if and only if the spectral sequence associated to F degenerates at E_1.

(3) On the projective line $X = \mathbb{P}^1$, with absolute coordinate z, consider the differential form

$$\omega = \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{(1 + z\overline{z})^2}.$$

   - (a) Show that ω is smooth on the whole X and represents a generator of $H^1(X, \Omega^1_{X_{\mathbb{Q}}})$. (Hint: Use first, that in Čech cohomology, a generator of $H^1(X, \Omega^1_{X_{\mathbb{Q}}})$ is given by $\mathrm{d}\, z/z$ in $\mathbb{P}^1 \setminus \{0, 1\}$ and second, the Čech complex of the Dolbeault resolution of $\Omega^1_{X_{\mathbb{C}}}$.)
   - (b) Show that $\int_X \omega = 2\pi i$ and conclude that the Hodge structure of the cohomology group $H^2(X)$ is the Lefschetz Hodge structure.

(4) Compute $Ext^1(\mathbb{Z}(i), \mathbb{Z}(j))$.

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