4. Exercises

Exercises

(1) Let (A^*, W) be a complex with a filtration. Denote the Decalé filtration of W as $\mathrm{Dec}(W)$.

  (a) Show that the term E_2 of the spectral sequence associated to W agrees (up to renumbering) with the term E_1 of the spectral sequence of $\mathrm{Dec}(W)$.

  (b) Given a filtration W construct a new filtration $\mathrm{Und}(W)$ with the property that $\mathrm{Dec}(\mathrm{Und}(W)) = W$, and although

$$(A, \mathrm{Und}(\mathrm{Dec}(W))) \xrightarrow{\mathrm{Id}} (A, W)$$

  is not a filtered quasi-isomorphism, the associated spectral sequences agree from the term E_2 on.

(2) Prove that, if H is a Hodge complex, then the spectral sequence associated to the filtration $\hat{W}$ degenerates at the term E_1 (Hint: the spectral sequence associated to the weight filtration is a spectral sequence of Hodge structures).

(3) Let X be a complex variety.

  (a) Assume that A is a field. Prove that the absolute A-Hodge cohomology fits in a long exact sequence

$$\begin{aligned}\dots \longrightarrow W_{2j}H^{n-1}(X, \mathbb{C}) \longrightarrow H^n_{\mathcal{AH}}(X, A(j))& \\ \longrightarrow (2\pi i)^j W_{2j}H^n(X, A) \oplus F^j W_{2j} H^n(X, \mathbb{C}) &\longrightarrow \dots\end{aligned}$$

  (b) Write down the corresponding long exact sequence when A is not a field and particularize it to the case when X is smooth and projective.

(4) Compute the absolute $\mathbb{Z}$-Hodge cohomology of $\mathrm{Spec}\,\mathbb{C}$.

(5) Given a smooth projective variety X, compute the absolute $\mathbb{R}$-Hodge cohomology of X in terms of the usual Hodge structure of the cohomology.

References

[Bei83] A.A. Beilinson, *Notes on absolute Hodge cohomology*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory (S. Bloch, ed.), Contemporary Mathematics, vol. 55, AMS, 1983, pp. 35–68.