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Abstract

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Contents

1 Beilinson's regulator

In previous lectures, we have seen differential extensions of topological K-theory, $\widehat{\mathbf{KU}}^0$, or algebraic K-theory of number rings, $\widehat{\mathbf{K}R}^0$. The aim for this and the next talk is to construct a differential extension of the cohomology theory represented by the algebraic K-theory spectrum KX of more general schemes X. Motivated by Beilinson's conjectures, the idea is that in this case the realification of KX is modeled by the real absolute Hodge cohomology and the realification map by Beilinson's regulator. The first step is to give a particular model for the function spectrum $\mathbf{Sm}(KX)$ and use it to construct (a version of) the Beilinson regulator as a map of sheaves of $\mathcal{E}_\infty$-ring spectra.

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1.1 Bundles and K-theory

We denote by **Mf** the category of smooth manifolds and by **Reg** the category of regular schemes which are separated and of finite type over $\mathbb{Z}$, or over $\mathbb{Q}$. Let $M \in \mathbf{Mf}$ and $X \in \mathbf{Reg}$. Then we can consider $M \times X$ as a topological space. We equip $M \times X$ with the structure sheaf $\mathcal{O}_{M\times X} := \mathtt{pr}_X^{-1}\mathcal{O}_X$ where $\mathtt{pr}_X \colon M \times X \to X$ is the projection and $\mathtt{pr}_X^{-1}$ denotes the inverse image in the sense of abelian sheaves.

Definition 1.1. *A* bundle *on $M \times X$ is a locally free sheaf of $\mathcal{O}_{M\times X}$-modules of finite rank.*

A bundle on $M \times X$ can be viewed as a locally constant family of algebraic vector bundles on X, parameterized by M. If we take $X := \mathtt{Spec}(R)$ for some number ring R a bundle on $M \times X$ is "the same" as a locally constant sheaf of finitely generated projective R-modules on M.

We denote by $\mathtt{Vect}(M \times X)$ the category of bundles on $M \times X$. Under direct sum and tensor product, $\mathtt{Vect}(M \times X)$ can be considered as a semiring (aka rig) object in the $(2,1)$-category **Cat** of categories. In this situation, we can define K-theory via group completion:

Definition 1.2. *The K-theory functor K is given by the following composition of functors between ∞-categories:*

$$
\begin{aligned}
K \colon \mathbf{Rig}(\mathbf{Cat}) &\to \mathbf{Rig}(\mathcal{S}) && \textit{(associated space)} \\
&\to \mathbf{Ring}(\mathcal{S}) && \textit{(group completion)} \\
&\overset{\cong}{\leftarrow} \mathbf{CAlg}(\mathbf{Sp}^{\geq 0}) && \Omega^\infty \\
&\to \mathbf{CAlg}(\mathbf{Sp}) && \textit{(forget connectivity)}
\end{aligned}
$$

Here $\mathcal{S}$ is the symmetric monoidal ∞-category of spaces, **Sp** (resp. $\mathbf{Sp}^{\geq 0}$) that of (connective) spectra, and $\mathbf{CAlg}(\mathbf{Sp}) = \mathbf{CommMon}(\mathbf{Sp})$ denotes commutative monoid, or algebra objects in **Sp**, i.e. $\mathcal{E}_\infty$-ring spectra. The first map sends a category to its underlying ∞-groupoid, i.e. the category with the same objects but only isomorphisms between them.

In general, algebraic K-theory can be defined via group completion only for affine schemes. To extend this to all schemes in **Reg** we have to sheafify: We equip **Mf** with the usual topology, **Reg** with the Zariski topology, and the product category $\mathbf{Mf} \times \mathbf{Reg}$ with the induced topology. We consider $K(\mathtt{Vect})$ as a presheaf of ring spectra. Since bundles are locally constant in the manifold direction this is in fact a homotopy invariant presheaf,

$$K(\mathtt{Vect}) \in \mathbf{Fun}^I((\mathbf{Mf} \times \mathbf{Reg})^{op}, \mathbf{CAlg}(\mathbf{Sp})).$$

Definition 1.3. *We define the sheaf of K-theory spectra*

$$\mathbf{K} \in \mathbf{Fun}^{desc,I}((\mathbf{Mf} \times \mathbf{Reg})^{op}, \mathbf{CAlg}(\mathbf{Sp}))$$

as the sheafification of $K(\mathtt{Vect})$.

This is our model for the smooth function spectrum:

Lemma 1.4. *We have equivalences*

$$\mathbf{K}(M \times X) \cong \mathbf{Sm}(KX)(M). \tag{1}$$

In particular, $\pi_i\left(\mathbf{K}(M \times X)\right) \cong KX^{-i}(M)$.

Proof. If we fix X, both sides of (1) are homotopy invariant sheaves in the manifold M. Hence we may assume that M is a point. Now we use that algebraic K-theory satisfies Zariski descent for regular, separated, noetherian schemes to further reduce to the case that X is affine. □

1.2 Differential forms

We now construct a sheaf of commutative dgas $\mathbf{IDR}$ on $\mathbf{Mf} \times \mathbf{Reg}$ which is built from differential forms and such that $\mathbf{IDR}(M \times X)$ computes the cohomology of M with coefficients in a weak version of the real absolute Hodge cohomology of X. This will be the target for the regulator map.

We consider $M \times X \in \mathbf{Mf} \times \mathbf{Reg}$. Because X is regular, the set of $\mathbb{C}$-valued points $X(\mathbb{C})$ is a complex manifold. Let $\mathcal{A}$ be the de Rham complex of smooth complex valued differential forms. On $\mathcal{A}(M \times X(\mathbb{C}))$ we have the Hodge filtration $\mathcal{F}$ where a form ω is in $\mathcal{F}^p\mathcal{A}(M \times X(\mathbb{C}))$ if, in local coordinates x of M and holomorphic local coordinates z of $X(\mathbb{C})$, it can be written in the form

$$\omega = \sum_{I,J,K,|J|\geq p} f_{I,J,K}\, dx^I dz^J d\bar{z}^K$$

with smooth functions $f_{I,J,K}$. We have an action of $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathcal{A}(M \times X(\mathbb{C}))$ which is induced by the action of $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$ on the complex manifold $X(\mathbb{C})$ and on the coefficients of the differential forms. Let I be the unit interval $[0, 1]$.

Definition 1.5. *For* $p \geq 0$ *we define the complex*

$$\mathbf{IDR}(p)(M \times X) := \left\{\omega \in \mathcal{A}(I \times M \times X(\mathbb{C}))[2p] \,|\, \omega|_0 \in (2\pi i)^p \mathcal{A}_{\mathbb{R}}, \omega|_1 \in \mathcal{F}^p\mathcal{A}\right\}^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

With the usual wedge product of forms

$$\mathbf{IDR} := \prod_{p\geq 0} \mathbf{IDR}(p)$$

becomes a sheaf of commutative differential graded algebras.

Denote by $\mathbf{Ch}$ the infinity category of chain complexes with quasi-isomorphisms inverted. Because we are working with fine sheaves, $\mathbf{IDR}$ still satisfies descent when considered as a presheaf with values in $\mathbf{Ch}$. By the Poincaré lemma, it is also homotopy invariant. Hence we can consider $\mathbf{IDR}$ as an object

$$\mathbf{IDR} \in \mathbf{Fun}^{desc,I}((\mathbf{Mf} \times \mathbf{Reg})^{op}, \mathbf{CAlg}(\mathbf{Ch})).$$

Lemma 1.6. *The map*

$$\begin{aligned}\mathbf{IDR}(p) \to \mathtt{Cone}((2\pi i)^p \mathcal{A}_{\mathbb{R}} \oplus \mathcal{F}^p\mathcal{A} \xrightarrow{(\alpha,\beta)\mapsto\alpha-\beta} \mathcal{A})[2p-1] \\ \omega \mapsto (\omega|_0 \oplus \omega|_1, -\int_{I\times(\_)/(\_)} \omega)\end{aligned}$$

is an equivalence in $\mathbf{Fun}^{desc,I}((\mathbf{Mf}\times\mathbf{Reg})^{op}, \mathbf{Ch})$. *In particular, if the generic fibre* $X_{\mathbb{Q}}$ *is proper over* $\mathbb{Q}$ *then for* $n \leq 0$

$$H^n(\mathbf{IDR}(p)(*\times X)) \cong H^{n+2p}_{\mathcal{AH}}(X_{\mathbb{R}}, \mathbb{R}(p))$$

where the right hand side is real absolute Hodge cohomology.

Proof. This is an exercise. □

1.3 Characteristic forms

We now introduce additional geometric data on bundles in order to construct characteristic forms which lie in the complex **IDR**. Later on, we will use these to construct the regulator.

Let $\mathcal{V}$ be a bundle on $M \times X$. Then

$$\mathcal{V}_{\mathbb{C}} := \mathcal{V}|_{M\times X(\mathbb{C})} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_X|_{M\times X(\mathbb{C})}} \mathcal{C}^{\infty}_{M\times X(\mathbb{C})}$$

The differential in the de Rham complex $\mathcal{A}$ of $M \times X(\mathbb{C})$ can be written as $d = d^M + \partial + \bar\partial$, where d^M is the partial differential in the M-direction, $\partial, \bar\partial$ are the differentials in the holomorphic and the antiholomorphic X-direction. Because d^M and $\bar\partial$ vanish on $\mathtt{pr}_X^{-1}\mathcal{O}_X\big|_{M\times X(\mathbb{C})}$, they induce

- a natural flat partial connection ∇^I on $\mathcal{V}_{\mathbb{C}}$ in the M-direction,
- a holomorphic structure $\bar\partial$ on $\mathcal{V}_{\mathbb{C}}$ in the X-direction.

Definition 1.7. *A* geometry *on* $\mathcal{V}$ *is a connection* $\widetilde{\nabla}$ *on* $\mathtt{pr}^*_{M\times X(\mathbb{C})}\mathcal{V}_{\mathbb{C}}/I \times M \times X$ *such that* $\widetilde{\nabla}|_0$ *is unitarizable and* $\widetilde{\nabla}|_1$ *extends the partial connection* $\nabla^I + \bar\partial$. *We furthermore require that* $\widetilde{\nabla}$ *is* $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$*-invariant in a suitable sense.*

Geometries exist locally and can be glued. Hence geometries always exist.

Given a bundle $\mathcal{V}$ with geometry $\widetilde{\nabla}$, we consider the Chern character form

$$\mathbf{ch}(\widetilde{\nabla}) = \mathtt{Tr}\exp(-R^{\widetilde{\nabla}}) \in \mathcal{A}(I \times M \times X(\mathbb{C})).$$

The conditions on $\widetilde{\nabla}$ are designed in such a way that, in fact,

$$\mathbf{ch}(\widetilde{\nabla}) \in Z^0(\mathbf{IDR}(M \times X)).$$

We denote the category of bundles on $M \times X$ with a geometry and geometry preserving maps by $\mathtt{Vect}^{geom}(M \times X)$. We denote the set of its isomorphism classes by $\overline{\mathtt{Vect}}^{geom}(M \times X)$. There is a natural notion of the direct sum and tensor product of two geometries. Thus we can consider $\mathtt{Vect}^{geom}(M \times X)$ as a semiring object in **Cat**,

$$\mathtt{Vect}^{geom}(M \times X) \in \mathbf{Rig}(\mathbf{Cat})$$

and $\overline{\mathtt{Vect}}^{geom}(M \times X)$ as a semiring. Since the Chern character forms are additive and multiplicative, we can view **ch** as a natural transformation between semiring valued functors

$$\mathbf{ch}\colon \overline{\mathtt{Vect}}^{geom} \to Z^0(\mathbf{IDR}). \tag{2}$$

1.4 The regulator

We now explain how one can use characteristic forms to construct (a weak version of) the Beilinson regulator as a map of sheaves of $\mathcal{E}_\infty$-ring spectra, i.e. as a map between objects in $\mathbf{Fun}^{desc,I}((\mathbf{Mf} \times \mathbf{Reg})^{op}, \mathbf{CAlg}(\mathbf{Sp}))$

$$\mathtt{r}^{\mathtt{Beil}}\colon \mathbf{K} \to H(\mathbf{IDR}).$$

The construction is a prototypical example. In the same way, one could construct the Chern character map from complex K-theory to real cohomology, or Borel's regulator for the K-theory of number rings.

The principal idea is to view a set as a discrete category. A semiring can then be viewed as a semiring in **Cat** and we can apply the K-theory functor to it.

We start with the Chern character from (2) and apply K to it:

$$K(\overline{\mathtt{Vect}}^{geom}) \xrightarrow{K(\mathbf{ch})} K(Z^0(\mathbf{IDR})) \tag{3}$$

Since $Z^0(\mathbf{IDR})$ is already a presheaf of rings, the group completion doesn't change anything, and one there is a natural equivalence

$$K(Z^0(\mathbf{IDR})) \cong H(Z^0(\mathbf{IDR})) \tag{4}$$

where H is the Eilenberg-MacLane functor from chain complexes to spectra and we view $Z^0(\mathbf{IDR})$ as a complex concentrated in degree 0 (Exercise!). The natural inclusion of complexes $Z^0(\mathbf{IDR}) \hookrightarrow \mathbf{IDR}$ induces

$$H(Z^0(\mathbf{IDR})) \to H(\mathbf{IDR}). \tag{5}$$

If we restrict from $\mathtt{Vect}^{geom}$ to its underlying groupoid $i\mathtt{Vect}^{geom}$ where we allow only isomorphisms as maps, we get a functor $i\mathtt{Vect}^{geom} \to \overline{\mathtt{Vect}}^{geom}$ sending each bundle to its isomorphism class. Since, by construction, K first restricts a category to its underlying groupoid we get a natural map

$$K(\mathtt{Vect}^{geom}) \to K(\overline{\mathtt{Vect}}^{geom}). \tag{6}$$

Composing (6), (3), (4), and (5) we have now constructed a map

$$K(\mathtt{Vect}^{geom}) \to H(\mathbf{IDR}). \tag{7}$$

We have a natural "forget the geometry" map $\mathtt{Vect}^{geom} \to \mathtt{Vect}$ and hence $K(\mathtt{Vect}^{geom}) \to K(\mathtt{Vect})$. However, this is not an equivalence. For instance, $K(\mathtt{Vect})$ is homotopy invariant, whereas $K(\mathtt{Vect}^{geom})$ is not. We can try to make $K(\mathtt{Vect})$ homotopy invariant by the following procedure:

Exercise 1.8 ($\bar{\mathbf{s}}$-construction). Let $\mathbf{C}$ be an ∞-category which has all colimits, and $F \in \mathbf{Fun}(\mathbf{Mf}^{op}, \mathbf{C})$. Let $\Delta^\bullet \in \mathbf{Fun}(\Delta, \mathbf{Mf})$ be the cosimplicial manifold of standard simplices. We can define an endofunctor $\bar{\mathbf{s}}$ of $\mathbf{Fun}(\mathbf{Mf}^{op}, \mathbf{C})$ which on objects is given by

$$\bar{\mathbf{s}}(F)(M) = \mathtt{colim}_{\Delta^{op}} F(\Delta^\bullet \times M).$$

If F was homotopy invariant then the natural map $F \to \bar{\mathbf{s}}(F)$ is an equivalence.

Lemma 1.9. *The natural map*

$$\bar{\mathbf{s}}K(\mathtt{Vect}^{geom}) \to \bar{\mathbf{s}}K(\mathtt{Vect}) \tag{8}$$

is an equivalence.

Proof. This boils down to show that the map of simplicial sets

$$\mathtt{N}_q(i\mathtt{Vect}^{geom}(\Delta^\bullet \times M \times X)) \to \mathtt{N}_q(i\mathtt{Vect}(\Delta^\bullet \times M \times X)),$$

where $\mathtt{N}$ denotes the nerve functor from categories to simplicial sets and $\mathtt{N}_q$ denotes the set in simplicial degree q, is a trivial Kan fibration. This in turn follows from the fact that for a bundle on $\Delta^p \times M \times X$, any geometry given on $\partial\Delta^p \times M \times X$ can be extended to a geometry on $\Delta^p \times M \times X$. $\square$

To finish the construction of the regulator, we apply $\bar{\mathbf{s}}$ to (7) and define $\mathtt{r}^{\mathtt{Beil}} \colon \mathbf{K} \to H(\mathbf{IDR})$ through the following diagram

$$\begin{tikzcd}
K(\mathtt{Vect}) \arrow[r, "\cong", "1.8"'] \arrow[d] & \bar{\mathbf{s}}K(\mathtt{Vect}) & \bar{\mathbf{s}}K(\mathtt{Vect}^{geom}) \arrow[l, "\cong"', "1.9"] \arrow[r, "\bar{\mathbf{s}}((7))"] & \bar{\mathbf{s}}H(\mathbf{IDR}) \\
\mathbf{K} \arrow[rrr, dotted, "\mathtt{r}^{\mathtt{Beil}}"] & & & H(\mathbf{IDR}) \arrow[u, "\cong", "1.8"']
\end{tikzcd}$$

The dotted arrow exists since $H(\mathbf{IDR})$ is a sheaf and $\mathbf{K}$ was defined as the sheafification of $K(\mathtt{Vect})$.

2 Multiplicative differential algebraic K-theory

Here we give the construction of differential algebraic K-theory for schemes $X \in \mathbf{Reg}$. Again, in the same way one can construct differential extensions of any other generalized cohomology theory. We will then show how, using these constructions, the definition of topological and geometric cycle maps becomes a tautology. Finally, we will give an application of the theory presented here to the construction of classes in K_3 of a number ring.

2.1 The construction

For a chain complex C and an integer k, we denote by $\sigma^{\geq k}C$ its stupid truncation in degree k.

Definition 2.1. *For each integer k, we define a sheaf of spectra*

$$\widehat{\mathbf{K}}^{(k)} \in \mathbf{Fun}^{desc}((\mathbf{Mf} \times \mathbf{Reg})^{op}, \mathbf{Sp})$$

via the pull-back

$$\begin{tikzcd}
\widehat{\mathbf{K}}^{(k)} \arrow[r, "R"] \arrow[d, "I"'] & H(\sigma^{\geq k}\mathbf{IDR}) \arrow[d] \\
\mathbf{K} \arrow[r, "\mathtt{r}^{\mathtt{Beil}}"] & H(\mathbf{IDR}).
\end{tikzcd}$$

We define the abelian group valued functors of differential algebraic K-theory

$$\widehat{\mathbf{K}}^{k} \in \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg})^{op}, \mathbf{Ab})$$

by $\widehat{\mathbf{K}}^{k}(M \times X) := \pi_{-k}\left(\widehat{\mathbf{K}}^{(k)}(M \times X)\right)$. We have induced maps

$$\begin{aligned}
&R\colon \widehat{\mathbf{K}}^{k} \to Z^k(\mathbf{IDR}) && \textit{(curvature)} \\
&I\colon \widehat{\mathbf{K}}^{k} \to \mathbf{K}^k && \textit{(underlying K-theory class)}
\end{aligned}$$

In fact, one can refine $\bigvee_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^{(k)}$ to a sheaf of $\mathcal{E}_\infty$-ring spectra. Thus $\bigoplus_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^{k}$ becomes a functor with values in graded commutative graded rings, and I and R become homomorphisms of graded rings.

From the definition of $\widehat{\mathbf{K}}^{(k)}$ as a pull-back, we immediately get the fundamental exact sequences of a differential cohomology:

$$\begin{gathered}
\mathbf{K}^{k-1} \xrightarrow{\mathtt{r}^{\mathtt{Beil}}} \mathbf{IDR}^{k-1}/\mathtt{im}(d) \xrightarrow{a} \widehat{\mathbf{K}}^{k} \xrightarrow{I} \mathbf{K}^k \to 0 \\
\mathbf{K}^{k-1} \xrightarrow{\mathtt{r}^{\mathtt{Beil}}} H^{k-1}(\mathbf{IDR}) \xrightarrow{a} \widehat{\mathbf{K}}^{k} \xrightarrow{(I,R)} \mathbf{K}^k \times_{H^k(\mathbf{IDR})} Z^k(\mathbf{IDR}) \to 0
\end{gathered} \tag{9}$$

Moreover, we have the relation

$$R \circ a = d.$$

2.2 Cycle maps

Let $\mathcal{V}$ be a bundle on $M \times X$. Its isomorphism class $[\mathcal{V}]$ can be considered as an element in $\pi_0(i\mathtt{Vect}(M \times X))$. Since $\Omega^\infty K(\mathtt{Vect}(M \times X))$ is the group completion of $i\mathtt{Vect}(M \times X)$ we have natural maps

$$\pi_0(i\mathtt{Vect}(M \times X)) \to \pi_0\left(\Omega^\infty K(\mathtt{Vect}(M \times X))\right) \cong \pi_0\left(K(\mathtt{Vect}(M \times X))\right) \to \pi_0\left(\mathbf{K}(M \times X)\right)$$

This composition defines the topological cycle map.

To construct the geometric cycle map

$$\widehat{\mathtt{cycl}}\colon \overline{\mathtt{Vect}}^{geom} \to \widehat{\mathbf{K}}^0$$

we use that $\widehat{\mathbf{K}}^{(0)}$ was defined as a pull-back in (pre)sheaves of spectra. It follows immediately from the construction of $\mathtt{r}^{\mathtt{Beil}}$ that the solid part of the following diagram commutes:

$$\begin{tikzcd}
K(\mathtt{Vect}^{geom}) \arrow[d] \arrow[dr, dotted] \arrow[drr, bend left, "K(\mathbf{ch})"] & & \\
K(\mathtt{Vect}) \arrow[dr, bend right] & \widehat{\mathbf{K}}^{(0)} \arrow[r, "R"] \arrow[d, "I"] & H(\sigma^{\geq 0}\mathbf{IDR}) \arrow[d] \\
& \mathbf{K} \arrow[r, "\mathtt{r}^{\mathtt{Beil}}"] & H(\mathbf{IDR})
\end{tikzcd}$$

Hence the dotted arrow exists, and we define $\widehat{\mathtt{cycl}}$ as the composition

$$\overline{\mathtt{Vect}}^{geom} = \pi_0(i\mathtt{Vect}^{geom}) \to \pi_0(K(\mathtt{Vect}^{geom})) \to \pi_0(\widehat{\mathbf{K}}^{(0)}) = \widehat{\mathbf{K}}^0.$$

By construction, for a bundle $\mathcal{V}$ with geometry $\tilde{\tilde{\nabla}}$ on $M \times X$ we have

$$\begin{aligned}
R(\widehat{\mathtt{cycl}}(\mathcal{V}, \tilde{\tilde{\nabla}})) &= \mathbf{ch}(\tilde{\tilde{\nabla}}) \in Z^0(\mathbf{IDR}(M \times X)) \\
I(\widehat{\mathtt{cycl}}(\mathcal{V}, \tilde{\tilde{\nabla}})) &= \mathtt{cycl}(\mathcal{V}) \in \mathbf{K}^0(M \times X).
\end{aligned}$$

2.3 Application: a secondary Steinberg relation

As an application of the theory presented so far, we explain how a differential version of the Steinberg relation leads to the construction of K_3-classes for number rings from elements in the Bloch group, and their relation to the dilogarithm. The result itself is not new, it goes back to Bloch. However, differential algebraic K-theory provides the framework to give an elegant conceptual proof.

We first state the result. The second polylogarithm function is defined for $|z| < 1$ by

$$\mathtt{Li}_2(z) := \sum_{n \geq 1} \frac{z^n}{n^2}.$$

It extends meromorphically to a covering of $\mathbb{C}$.

Definition 2.2. *The* Bloch-Wigner dilogarithm *is the real valued function on* $\mathbb{C}$ *given by*

$$D^{BW}(z) := \log(|z|)\arg(1-z) + \mathrm{Im}(\mathtt{Li}_2(z)).$$

For any ring R we write $R^\circ := \{\lambda \in R^\times \,|\, 1 - \lambda \in R^\times\}$.

Definition 2.3. *The third* Bloch group $\mathcal{B}_3(R)$ *is defined as the kernel*

$$\mathcal{B}_3(R) := \mathtt{ker}\left(\mathbb{Z}[R^\circ] \xrightarrow{\lambda \mapsto \lambda \wedge (1-\lambda)} R^\times \wedge R^\times\right).$$

Now let R be the ring of integers in a number field and $X := \mathtt{Spec}(R)$. The target of the regulator $\mathtt{r}^{\mathtt{Beil}}$ on $\mathbf{K}^{-3}(X)$ is $H^{-3}(\mathbf{IDR}(X))$. Since $X(\mathbb{C}) \cong \{\sigma\colon R \hookrightarrow \mathbb{C}\}$ is zero dimensional we have

$$H^{-3}(\mathbf{IDR}(X)) \cong H^{-3}(\mathbf{IDR}(2)(X)) \cong \mathbf{IDR}(2)(X)/\mathtt{im}(d) \cong \left[2\pi i \mathbb{R}^{X(\mathbb{C})}\right]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}. \quad (10)$$

For $\lambda \in R^\circ$ we write

$$D_R^{BW}(\lambda) := \left(-iD^{BW}(\sigma(\lambda))\right)_{\sigma \in X(\mathbb{C})} \in \left[2\pi i \mathbb{R}^{X(\mathbb{C})}\right]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

The result we want to prove is:

Theorem 2.4 (Bloch)**.** *For any $x = \sum_{\lambda \in R^\circ} n_\lambda[\lambda] \in \mathcal{B}_3(R)$, there exists an element $bl(x) \in \mathbf{K}^{-3}(X)$ such that*

$$\mathtt{r}^{\mathtt{Beil}}(bl(x)) = \sum_\lambda n_\lambda D_R^{BW}(\lambda).$$

We start with some general considerations. Let R be a ring such that $X := \mathtt{Spec}(R) \in$ **Reg**. There is a natural map

$$c\colon R^\times \to \mathbf{K}^{-1}(X)$$

given as follows: To a unit $\lambda \in R^\times$ one associates the rank-1 bundle $\mathcal{V}(\lambda)$ on $S^1 \times X$ which is trivial in the X-direction and has holonomy λ along S^1. Then $\mathtt{cycl}(\mathcal{V}(\lambda)) \in \mathbf{K}^0(S^1 \times X) \cong \mathbf{K}^{-1}(X) \oplus \mathbf{K}^0(X)$ and $c(\lambda)$ denotes the first component. The Steinberg relation says that for $\lambda \in R^\circ$ we have

$$c(\lambda) \cdot c(1-\lambda) = 0 \text{ in } \mathbf{K}^{-2}(X).$$

From the exact sequence (9) we have that $I\colon \widehat{\mathbf{K}}^{-1}(X) \to \mathbf{K}^{-1}(X)$ is surjective and its kernel is a divisible abelian group. Hence we can lift c to a map $\hat{c}$

$$\begin{tikzcd} & \widehat{\mathbf{K}}^{-1}(X) \arrow[d, "I"] \\ R^\times \arrow[r, "c"] \arrow[ur, dotted, "\hat{c}"] & \mathbf{K}^{-1}(X). \end{tikzcd}$$

We get an induced map $R^\times \wedge R^\times \to \widehat{\mathbf{K}}^{-2}(X), \lambda \wedge \mu \mapsto \hat{c}(\lambda) \cdot \hat{c}(\mu)$. We consider the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B}_3(R) \arrow[r] \arrow[d, dotted, "bl"] & \mathbb{Z}[R^\circ] \arrow[r, "\lambda \mapsto \lambda\wedge(1-\lambda)"] \arrow[d, dotted, "\mathcal{D}"] & R^\times \wedge R^\times \arrow[d] & \\ 0 \arrow[r] & \mathbf{K}^{-3}(X)/\mathtt{ker}(\mathtt{r}^{\mathtt{Beil}}) \arrow[r, "\mathtt{r}^{\mathtt{Beil}}"] & \mathbf{IDR}^{-3}(X)/\mathtt{im}(d) \arrow[r, "a"] & \widehat{\mathbf{K}}^{-2}(X) \arrow[r, "I"] & \mathbf{K}^{-2}(X) \end{tikzcd}$$

Here the map $\mathcal{D}$ exists by the Steinberg relation and since $\mathbb{Z}[R^\circ]$ is a free abelian group. The arrow bl is the induced map on kernels. Of course, it depends on the choice of $\hat{c}$ and $\mathcal{D}$.

To fix these choices, we consider the universal situation. Let

$$\mathbb{X} := \mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\} \cong \mathtt{Spec}(\mathbb{Z}[\lambda, \lambda^{-1}, (1-\lambda)^{-1}]).$$

We first construct $\hat{c}$. Let $\mathcal{V}(\lambda)$ be the bundle on $S^1 \times \mathbb{X}$ described above. We want to construct a geometry on $\mathcal{V}(\lambda)$. Let t be a parameter on S^1 and log a local choice of a branch of the logarithm on $\mathbb{X}(\mathbb{C}) = \mathbb{C}^\times \setminus \{1\}$. Then $\phi = \lambda^t$ is a local section of $\mathcal{V}(\lambda)_{\mathbb{C}}$ which depends on the choice of logarithm. We first define a hermitian metric h and a connection ∇ on $\mathcal{V}(\lambda)_{\mathbb{C}}$. Since this is a line bundle they are determined by their value on the local sections ϕ. We set $h(\phi) = 1$ and $\nabla(\phi) = \log(\lambda)\phi dt$. These are well defined. Moreover, $\nabla^{(\lambda)}$ has holonomy λ along S^1 and ∇ is compatible with the holomorphic structure in the $\mathbb{X}$-direction. For the geometry $\widetilde{\nabla}$ we take the linear path between the associated unitary connection ∇^u and ∇. Explicitly, if u is the coordinate on the interval I, we have

$$\widetilde{\nabla}\phi = \left(\frac{1-u}{2}(\log(\lambda) - \log(\bar{\lambda})) + u\log(\lambda)\right)\phi dt.$$

We get

$$\widehat{\mathtt{cycl}}(\mathcal{V}(\lambda), \widetilde{\nabla}) \in \widehat{\mathbf{K}}^0(S^1 \times \mathbb{X}).$$

We integrate this along S^1 to get

$$\hat{c}(\lambda) \in \widehat{\mathbf{K}}^{-1}(\mathbb{X}).$$

Now we look for $\mathcal{D}(\lambda) \in \mathbf{IDR}^{-3}(\mathbb{X})/\mathtt{im}(d)$ such that $a(\mathcal{D}(\lambda)) = \hat{c}(\lambda) \cdot \hat{c}(1-\lambda)$. Since $R \circ a = d$, we must have

$$d(\mathcal{D}(\lambda)) = R(\hat{c}(\lambda)) \cdot R(\hat{c}(1-\lambda)) \in \mathbf{IDR}^{-2}(\mathbb{X}). \tag{11}$$

Because we want to specialize to number rings later on, we are only interested in the component $\mathcal{D}(\lambda)(2) \in \mathbf{IDR}(2)^{-3}(\mathbb{X})/\mathtt{im}(d)$ (cf. (10)) This is determined by (11) up to elements in $H^{-3}(\mathbf{IDR}(2)(\mathbb{X}))$. An easy computation shows that this group vanishes.

We now compute the right hand side of (11). We first compute

$$R\left(\widehat{\mathtt{cycl}}(\mathcal{V}(\lambda), \widetilde{\nabla})\right) = \mathbf{ch}(\widetilde{\nabla}) = 1 - R^{\widetilde{\nabla}} = 1 + idt \wedge d\arg(\lambda) + dt \wedge d\log(|\lambda|^u).$$

Hence

$$R(\hat{c}(\lambda)) = id\arg(\lambda) + d\log(|\lambda|^u).$$

To simplify further we integrate over the interval I as in Lemma 1.6. In view of (10), we are only interested in the imaginary part of (11). We get

$$\begin{aligned} i\,\mathrm{Im}\left(-\int_{I\times\mathbb{X}(\mathbb{C})/\mathbb{X}(\mathbb{C})} R(\hat{c}(\lambda)) \cdot R(\hat{c}(1-\lambda))\right) &= -i\log(|\lambda|)d\arg(1-\lambda) + i\log(|1-\lambda|)d\arg(\lambda) \\ &= d\left(-iD^{BW}(\lambda)\right) \end{aligned} \tag{12}$$

The last identity is an easy computation.

Now we return to the case of a number ring R. We have $R^\circ = \mathbb{X}(R)$, hence $\lambda \in R^\circ$ gives a morphism $\lambda\colon \mathtt{Spec}(R) \to \mathbb{X}$ which on $\mathbb{C}$-valued points is given by $\sigma \mapsto \sigma(\lambda)$. To define $\mathcal{D}(\lambda)(2) \in \mathbf{IDR}(2)(\mathtt{Spec}(R))/\mathtt{im}(d) \cong \left[2\pi i\mathbb{R}^{X(\mathbb{C})}\right]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}$ we can pull-back from the universal case via λ. It follows from (12) and (11) that this gives

$$\mathcal{D}(\lambda)(2) = \left(-iD^{BW}(\sigma(\lambda))\right) = D_R^{BW}(\lambda).$$