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Abstract

1 Bundles, connections and K-theory

1.1 Connections and characteristic forms

We consider a smooth manifold M with its sheaf $\mathcal{C}^\infty$ of smooth complex valued functions. The datum of a complex vector bundle $V \to M$ on M is equivalent to its sheaf of $\mathcal{C}^\infty$-modules of sections $\mathcal{V}$. Vice versa, by Swan's theorem, if $\mathcal{V}$ is projective and finitely generated, then it is the sheaf of smooth sections of a complex vector bundle.

Example 1.1. Consider the sheaf $\mathcal{L}$ on $\mathbb{CP}^n$ of smooth functions f with values in $\mathbb{C}^{n+1}$ such that $f(H) \in H$ for every line $H \subset \mathbb{C}^{n+1}$. It is the sheaf of sections of the tautological line bundle $L \to \mathbb{CP}^n$.

A vector bundle is called trivial if

$$\mathcal{V} \cong \underbrace{\mathcal{C}^\infty \oplus \cdots \oplus \mathcal{C}^\infty}_{n}$$

for an appropriate $n \in \mathbb{N}$.

Problem 1.2. *How can we descide whether a vector bundle is trivial.*

Characteristic classes provide necesseary conditions, see Corollary 1.6.

Let $\mathcal{A}$ be the sheaf of differential graded algebras of complex-valued differential forms on $\mathbf{Mf}$. Thus $\mathcal{A}(M)$ is the complexified de Rham complex of M. Note that $\mathcal{A}^i_{|M}$ is the sheaf

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of sections of the bundle $\Lambda^i T^*M \otimes \mathbb{C} \to M$, in particular $\mathcal{A}^0 \cong \mathcal{C}^\infty$. The cohomology of the complex of global sections

$$H^*_{dR}(M) := H^*(\mathcal{A}(M))$$

is called the de Rham cohomology of M.

A connection on V is a $\mathcal{A}$-derivation of degree one

$$\nabla : \mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A} \to \mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A} \ .$$

It is uniquely determined by its restriction $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A}^1$. If V is trivial, then we define a connection by

$$\begin{tikzcd} \mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A} \arrow[r, "\nabla^{triv}"] \arrow[d, "\cong"] & \mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A} \arrow[d, "\cong"] \\ \mathcal{A} \oplus \cdots \oplus \mathcal{A} \arrow[r, "\oplus d"] & \mathcal{A} \oplus \cdots \oplus \mathcal{A} \end{tikzcd} \ .$$

Local connections can be glued using a partition of unity. Since a bundle is locally trivial it admits connections locally, and therefore globally.

The map $R^\nabla := \nabla \circ \nabla$ is a map of $\mathcal{A}^\sharp$-modules ($\sharp$ forgets the differential) and called the curvature of ∇. In fact,

$$R^\nabla \in (\mathcal{A}^2 \otimes_{\mathcal{C}^\infty} \mathtt{End}(\mathcal{V}))(M) \ .$$

If V is trivial, then $R^{\nabla^{triv}} = 0$. If $R^\nabla = 0$, then ∇ is called flat.

Definition 1.3. *We define the Chern form by*

$$\mathbf{ch}(\nabla) := \mathtt{Tr}\exp(R^\nabla) \in \mathcal{A}^{ev}(M) \ .$$

Given two connections ∇_i on V we can form a connection $\tilde{\nabla}$ on $\mathtt{pr}_M^* V \to [0,1] \times M$ which restricts to ∇_i at the end points. We choose the affine path $\tilde{\nabla} := t\nabla_1 + (1-t)\nabla_0 + dt\partial_t$ for the next definition.

Definition 1.4. *We define the transgression Chern form by*

$$\tilde{\mathbf{ch}}(\nabla_1, \nabla_0) := \int_{[0,1]\times M/M} \mathbf{ch}(\tilde{\nabla}) \in \mathcal{A}^{odd}(M) \ .$$

Lemma 1.5.

1. $d\mathbf{ch}(\nabla) = 0$.
2. $d\mathbf{ch}(\tilde{\nabla}) = \mathbf{ch}(\nabla_1) - \mathbf{ch}(\nabla_0)$.
3. *The class* $\mathbf{ch}(V) := [\mathbf{ch}(\nabla)] \in H^{ev}_{dR}(M)$ *is well-defined.*
4. *If* V *is trivial, then* $\mathbf{ch}(V) = \dim(V)$.
5. *If* V *admits a flat connection, then* $\mathbf{ch}(V) = \dim(V)$.

Corollary 1.6. *If $\mathbf{ch}(V) \neq \dim(V)$, then V is not trivial and does not even admit a flat connection.*

Example 1.7. We continue example 1.1. We have an orthogonal projection $P : \mathbb{CP}^n \times \mathbb{C}^{n+1} \to L$. We define a connection on L by $\nabla^L := P\nabla^{triv}$. Since L is one-dimensional we can identify $\mathtt{End}(L) \cong \mathbb{CP}^n \times \mathbb{C}^{n+1}$ so that $R^\nabla \in \mathcal{A}^2(\mathbb{CP}^{n+1})$ is closed. The group $U(n+1)$ acts on L and preserves ∇. Hence R is $U(n+1)$-invariant. Since $\mathbb{CP}^{n+1} \cong U(n+1)/U(n)$ is a symmetric space its invariant differential form are its harmonic forms. Therefore R^∇ is proportional to the unique $\omega \in \mathcal{A}^2(\mathbb{CP}^n)$ which satisfies $\int_{\mathbb{CP}^1} \omega = 1$. The proportionality factor can be determined by a local calculation:

$$R^\nabla = 2\pi i \omega \ .$$

Consequently, we have

$$\mathbf{ch}(\nabla) = \exp(2\pi i \omega) \ . \tag{1}$$

We set $c := [\omega] \in H^2_{dR}(\mathbb{CP}^n)$. Then $H^*_{dR}(\mathbb{CP}^n) \cong \mathbb{C}[c]/(c^{n+1})$. For example, if $n = 3$, then

$$\mathbf{ch}(L) = 1 + 2\pi i c + \frac{(2\pi i)^2}{2} c^2 + \frac{(2\pi i)^3}{6} c^3 \ .$$

Let $\bar{V}$ denote V with the opposite complex structure. A hermitean metric on V is a bundle map $h : \bar{V} \otimes V \to M \times \mathbb{C}$ which is fibrewise a scalar product. It induces a map

$$h : \overline{(\mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A})} \otimes_{\mathcal{A}} (\mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A}) \to \mathcal{V} \otimes_{\mathcal{C}^\infty} \mathcal{A} \ .$$

Lemma 1.8. *Given a connection ∇ on V there exists an adjoint connection ∇^{*_h} such that*

$$h \circ (\nabla \otimes 1 + 1 \otimes \nabla^{*_h}) = d \circ h \ .$$

Its curvature satisfies

$$h \circ (1 \otimes R^{\nabla^{*_h}}) = h \circ (R^\nabla \otimes 1) \ .$$

We have

$$\mathbf{ch}_{2n}(\nabla^{*_h}) = (-1)^n \overline{\mathbf{ch}_{2n}(\nabla)} \ . \tag{2}$$

The last relation can explicitly be verified in (1).

Definition 1.9. *∇ is called hermitean (and h is called parallel), if $\nabla^{*_h} = \nabla$. We call ∇ unitarizable, if there exists a parallel hermitean metric.*

Problem 1.10. *How can we decide whether ∇ is unitarizable.*

If ∇ is trivial, then it admits a parallel metric. In general we can again define characteristic classes which provide obstructions against the existence of a parallel metric. We set $\tilde{\mathbf{ch}}(\nabla, h) := \mathbf{ch}(\nabla^{*_h}, \nabla)$.

Lemma 1.11. *1. We have $d\,\mathrm{Im}(i^p \tilde{\mathbf{ch}}_{2p-1}(\nabla, h)) = 0$.*

2. *The class* $\omega(\nabla) := \sum_{p\geq 1}[\mathrm{Im}(i^p\tilde{\mathbf{ch}}_{2p-1}(\nabla,h))] \in H^{odd}_{\mathbb{R},dR}(M)$ *does not depend on* h.

3. *If* ∇ *admits a parallel metric, then* $\omega(\nabla)=0$.

Proof. 1. follows from Lemma 1.5, 2. and (2). For 2. we consider two metrics h_0, h_1. We form the metric $\tilde{h} := th_1 + (1-t)h_0$ on $\mathtt{pr}^*V \to [0,1]\times M$. Then

$$\tilde{\mathbf{ch}}(\nabla^{*h_1},\nabla) - \tilde{\mathbf{ch}}(\nabla^{*h_0},\nabla) = d\int_{[0,1]\times M/M} \tilde{\mathbf{ch}}(\mathtt{pr}^*\nabla,\tilde{h}) + \int_{[0,1]\times M/M} d\tilde{\mathbf{ch}}(\mathtt{pr}^*\nabla,\tilde{h}) \ .$$

For 3. we use that $\tilde{\mathbf{ch}}(\nabla,\nabla)=0$. □

Corollary 1.12. *If* ∇ *is unitarizable, then* $\omega(\nabla)=0$

Example 1.13. The connection on $\mathcal{L}$ described in example 1.7 is unitarizable. Indeed, the metric on L induced by the embedding $L\to \mathbb{CP}^{n+1}\times\mathbb{C}^{n+1}$ is parallel. We have $\mathrm{Im}(i^p\tilde{\mathbf{ch}}_{2p-1}(\nabla,h))=0$ by the calculation above.

Problem 1.14. *We consider the trivial bundle* $\mathbb{R}^2\times\mathbb{C}^2\to\mathbb{R}^2$ *with connections*

$$\nabla := d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dx \ , \quad \nabla' := d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ydx$$

Show that ∇ *is unitarizable, while* ∇' *is not. The connection* ∇ *descends to the quotient* $\mathbb{R}^2/\mathbb{Z}^2$*, but this descent is not unitarizable.*

Example 1.15. We consider $\lambda\in\mathbb{C}^*$ and define the action of $\mathbb{Z}$ on the trivial bundle $\mathbb{R}\times\mathbb{C}\to\mathbb{R}$ by $n(t,z)\mapsto(t+n,\lambda^n z)$. This action preserves the trivial connection ∇^{triv}. We let $V(\lambda)\to S$ be the quotient with the induced flat connection $\nabla(\lambda)$. One can calculate that

$$\omega(\nabla(\lambda)) = -2\log|\lambda|\mathtt{or}_{S^1} \in H^{odd}_{\mathbb{R},dR}(S^1) \ .$$

Hence $\nabla(\lambda)$ admits a parallel metric if and only if $\lambda\in U(1)$.

We will see more examples of non-unitarizable connections later in this course.

1.2 Bundles and topological K-theory

We consider the functor

$$\overline{Bun} : \mathbf{Mf} \to \mathbf{Mon}$$

which associates to a manifold M the monoid of isomorphism classes of vector bundles on M under direct sum and to $f: M\to M'$ the pull-back f^*. This functor is homotopy invariant. In general it is a difficult problem to calculate $\overline{Bun}(M)$. In contrast, a functor $F^0:\mathbf{Mf}\to\mathbf{Ab}$ is amenable to calculations if it is part of a cohomology theory $F^*:\mathbf{Mf}\to\mathbf{Ab}_{\mathbb{Z}-graded}$. For $F^*(M)$ we have a Mayer-Vietoris sequence and an Atiyah-Hirzebruch spectral sequence with second page $E_2^{p,q}\cong H^p(M;F^q)$. A good way to study the functor $\overline{Bun}$ is to look first at its approximation by complex K-theory KU^0.

Complex K-theory is represented by a spectrum KU with homotopy groups

$$\pi_*(KU) \cong \mathbb{R}[b, b^{-1}]$$

with $\deg(b) = -2$. We have a natural transformation

$$\mathtt{cycl} : \overline{Bun} \to KU^0 \tag{3}$$

of monoid valued functors such that a vector bundle $V \to M$ gives rise to a class $\mathtt{cycl}(V) \in KU^0(M)$. Note that KU is a ring spectrum, and $\mathtt{cycl}$ is multiplicative, of one considers on $\overline{Bun}$ the monoid structure induced by the tensor product of bundles.

Remark 1.16. One can construct complex K-theory by the following procedure. Instead of $\overline{Bun}$ one considers the homotopy invariant functor (presheaf)

$$Bun \in \mathbf{Fun}^{const}(\mathbf{Mf}^{op}, \mathbf{CommMon}(\mathcal{S}))$$

which associates to a manifold M the monoid space of bundles. We group-complete and sheafify in order to get a sheaf

$$K(Bun) \in \mathbf{Fun}^{desc,const}(\mathbf{Mf}, \mathbf{CommGrp}(\mathcal{S}))$$

of commutative group spaces (E_∞-spaces). As any homotopy invariant sheaf of spaces it representable, in this case by a commutative group space which we can defined to be $\Omega^\infty KU$. If one takes the tensor product of bundles into account, then one gets the multiplicative structure on KU as well. In particular, we have the isomorphism

$$KU^0(M) \cong \pi_0(K(Bun)(M))$$

as rings. If we define topological K-theory in this way the cycle map (3) is tautological. We will develop the techniques for this procedure during this week.

Example 1.17. We can identify $KU^0(\mathbb{CP}^n) \cong \mathbb{Z}[z]/(z^{n+1})$ as rings, where $z := \mathtt{cycl}(L) - 1$. For this one can use the multiplicative version Atiyah-Hirzebruch spectral sequence.

We further consider the Eilenberg-MacLane spectrum $H\mathbb{R}[b, b^{-1}]$ of the ring $\mathbb{R}[b, b^{-1}]$. The de Rham isomorphism identifies $H\mathbb{R}[b, b^{-1}]^0(M) \cong H^{ev}_{dR}(M)$.

Proposition 1.18. *There is a unique map* $\mathbf{ch} : KU \to H\mathbb{R}[b, b^{-1}]$ *of spectra such that the following diagram commutes*

$$\begin{tikzcd}
\overline{Bun}(M) \arrow[r, "\mathtt{cycl}"] \arrow[d, "\mathbf{ch}"] & KU^0(M) \arrow[d, "\mathbf{ch}"] \\
H^{ev}_{dR}(M) \arrow[r, "\cong"] & H\mathbb{R}[b, b^{-1}]^0(M)
\end{tikzcd} .$$

During this week we will present a construction of regulators. If applied to topological the Chern character forms and the approach to topological K-theory 1.16 we get a simple proof of 1.18.

Example 1.19. *The map* $\mathbf{ch} : KU^0(\mathbb{CP}^n) \to H\mathbb{R}[b, b^{-1}]^0(\mathbb{CP}^n)$ *sends* z *to* $\exp(2\pi i bc) - 1$.

1.3 Flat bundles and algebraic K-theory

We now consider the functor

$$\overline{Bun}_{\nabla,flat} : \mathbf{Mf} \to \mathbf{Mon}$$

which associates to a manifold M the monoid of isomorphism classes of vector bundles with flat connection on M under direct sum and to $f : M \to M'$ the pull-back f^*. Its approximation

$$\overline{Bun}_{\nabla,flat} \xrightarrow{forget\,\nabla} \overline{Bun} \stackrel{cycl}{\to} KU^0$$

by complex K-theory loses most of the information. A better approximation is by the algebraic K-theory of $\mathbb{C}$.

The algebraic K-theory spectrum $K\mathbb{C}$ again represents a cohomology theory.

Remark 1.20. One can construct in analogy to 1.16 the algebraic K-theory spectrum $K\mathbb{C}$ as follows. One starts with the homotopy invariant presheaf

$$Bun_{\nabla,flat} \in \mathbf{Fun}^{const}(\mathbf{Mf}^{op}, \mathbf{CommMon}(\mathcal{S}))$$

which associates to a manifold M the monoid space of bundles with flat connection. We group-complete and sheafify in order to get a sheaf

$$K(Bun_{\nabla,flat}) \in \mathbf{Fun}^{desc,const}(\mathbf{Mf}, \mathbf{CommGrp}(\mathcal{S}))$$

of commutative group spaces (E_∞-spaces). The sheaf is again represented by a commutative group space which define to be $\Omega^\infty K\mathbb{C}$. If one takes the tensor product of bundles into account, then one even gets the multiplicative structure on $K\mathbb{C}$ as well. In particular, we have the isomorphism

$$KC^0(M) \cong \pi_0(K(Bun_{\nabla,flat})(M))$$

as rings.

Proposition 1.21. *There is a cycle map*

$$\mathtt{cycl} : \overline{Bun}_{\nabla,flat} \to K\mathbb{C}^0 \ .$$

Proof. This is either an immediate consequence of the construction 1.20 or shown using classifying spaces as follows. Note that $K\mathbb{C}^0(M) \cong [M, \Omega^\infty K\mathbb{C}]$, and that for every $n \in \mathbb{N}$ there is a natural map $BGL(\mathbb{C}^\delta, n) \to \Omega^\infty K\mathbb{C}$. If $(V, \nabla) \in \overline{Bun}_{\nabla,flat}(M)$, then the connection induces a reduction of the structure group of V from $GL(\mathbb{C}, \dim(V))$ to $GL(\mathbb{C}^\delta, \dim(V))$ and therefore a unique homotopy class of classifying maps in $[M, BGL(\mathbb{C}^\delta, \dim(V))]$ which induces the class $\mathtt{cycl}(V, \nabla)$ in the natural way. □

We consider the Eilenberg-MacLane spectrum

$$H\mathbb{R}\langle b_0, b_1, b_3, b_5 \dots \rangle \ ,$$

where b_j is an additive generator in degree j. The de Rahm isomorphism provides an identification

$$H\mathbb{R}\langle b_0, b_1, b_3, \dots\rangle^0(M) \cong H^0_{\mathbb{R},dR}(M) \oplus H^{odd}_{\mathbb{R},dR}(M) \ .$$

Proposition 1.22. *There exists a unique map of spectra $r_{\mathbb{C}} : K\mathbb{C} \to H\mathbb{R}\langle b_0, b_1, b_3, \dots\rangle$ such that the following diagram commutes.*

$$\begin{tikzcd}
\overline{Bun}_{\nabla,flat}(M) \arrow[r, "\mathtt{cycl}"] \arrow[d, "b_0 \dim \oplus \omega"] & K\mathbb{C}^0(M) \arrow[d, "r_{\mathbb{C}}"] \\
H^0_{\mathbb{R},dR}(M) \oplus H^{odd}_{\mathbb{R},dR}(M) \arrow[r, "\cong"] & H\mathbb{R}\langle b_0, b_1, b_3, \dots\rangle^0(M)
\end{tikzcd} \ .$$

Example 1.23. We have $K\mathbb{C}^0(S^1) \cong \pi_0(K\mathbb{C}) \oplus \pi_1(K\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{C}^*$. Under this identification $\mathtt{cycl}(V(\lambda), \nabla(\lambda)) = 1 \oplus \lambda$. Furthermore,

$$r_{\mathbb{C}}(n \oplus \lambda) = nb_0 - 2\log|\lambda| b_1 \mathtt{or}_{S^1} \in H\mathbb{R}\langle b_0, b_1, b_3, \dots\rangle^0(S^1) \ .$$

1.4 Differential cohomology

A bundle with connection $(V, \nabla) \in \overline{Bun}_\nabla(M)$ gives rise to a class $\mathtt{cycl}(V) \in KU^0(M)$ and a form

$$\mathbf{ch}(\nabla) := \mathtt{Tr}\exp(bR^\nabla) \in Z^0((\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[b, b^{-1}])(M))$$

such that $\mathtt{Rham}(\mathbf{ch}(\nabla)) = \mathbf{ch}(\mathtt{cycl}(V))$. It can happen that both invariants are trivial, but (V, ∇) is non-trivial.

Example 1.24. *We consider the bundle $V(\lambda) \to S^1$ from Example 1.15. Then $1 = [V(\lambda)]$ and $\mathbf{ch}(\nabla(\lambda)) = 1$.*

Differential K-theory $\widehat{KU}^0$ can capture a refined invariant. It fits into an exact sequences

$$\begin{aligned}
&K^{-1}(M) \to H\mathbb{C}[b, b^{-1}]^{-1}(M) \xrightarrow{a} \widehat{KU}^0(M) \to \\
\xrightarrow{(I,R)} \quad & KU^0(M) \times_{H\mathbb{C}[b,b^{-1}]^0(M)} Z^0((\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[b, b^{-1}])(M)) \to 0 \ . \\
& (\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[b, b^{-1}])^{-1}(M) \xrightarrow{a} \widehat{KU}^0(M) \xrightarrow{I} KU^0(M) \to 0
\end{aligned}$$

such that $R \circ a = d$. Furthermore,

$$(KU \wedge M\mathbb{C}/\mathbb{Z})^0(M) \cong \widehat{KU}^0_{flat}(M) := \mathtt{ker}(R : \widehat{KU}^0(M) \to Z^0((\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[b, b^{-1}])(M))) \ . \tag{4}$$

We will explain the definition in later talks. We have a refined cycle map

$$\widehat{\mathtt{cycl}} : \overline{Bun}_\nabla(M) \to \widehat{KU}^0(M)$$

such that $R(\widehat{\mathtt{cycl}}(V, \nabla)) = \mathbf{ch}(\nabla)$ and $I(\widehat{\mathtt{cycl}}(V, \nabla)) = \mathtt{cycl}(V)$.

Example 1.25. We have $\widehat{\mathtt{cycl}}(V(\lambda), \nabla(\lambda)) = 1 - a(\log \lambda \mathtt{bor}_{S^1}) \in \widehat{KU}^0(S^1)$. Here we can choose any leaf of the logarithm.

Proposition 1.26. *There exists a lift*

$$\begin{tikzcd} & \widehat{KU}^0(M) \arrow[d] \\ K\mathbb{C}^0(M) \arrow[r] \arrow[ur, dotted, "\phi"] & KU^0(M) \end{tikzcd}$$

such that $R(\phi(x)) = \dim(x)$.

Using (4) we can define a map

$$\phi : K\mathbb{C} \to \Sigma^{-1}(KU \wedge M\mathbb{C}/\mathbb{Z}) \tag{5}$$

which in degree zero cohomology induces $\phi - \dim$. Work out the details. It detects torsion elements in $\pi_*(K\mathbb{C})$. By a theorem of Suslin [Sus84] we have

$$\pi_n(K\mathbb{C})_{tors} \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Is ϕ injective on $\pi_n(K\mathbb{C})_{tors}$? By a theorem of Quillen [Qui76] the unit $S \to K\mathbb{C}$ is injective on the image of the J-homomorphism. By a theorem of Jones-Westbury [JW95],

$$S \to K\mathbb{C} \to \Sigma^{-1}(KU \wedge M\mathbb{R}/\mathbb{Z})$$

detects the image of the J-homomorphism.

2 TIC

2.1 Differential algebraic K-theory of a number ring

We consider a number field k and its ring of integers R. Then we have the algebraic K-theory spectrum KR. We first explain real model of KR. The set $\mathrm{Spec}(R)(\mathbb{C})$ is the set of embeddings $R \hookrightarrow \mathbb{C}$ on which the group $\mathbb{Z}/2\mathbb{Z}$ acts by complex conjugation. We define the graded abelian group

$$A(R) := b_0\mathbb{R} \oplus \mathbb{R}\langle b_{2i+1,\sigma} | i \in \mathbb{N} \, , \sigma \in \mathrm{Spec}(R)(\mathbb{C})\rangle / \sim \, ,$$

with the relations $b_{2i+1,\bar{\sigma}} = (-1)^i b_{2i+1,\sigma}$. We define a map

$$c : KR \to \bigoplus_{\sigma \in \mathrm{Spec}(R)(\mathbb{C})} K\mathbb{C} \stackrel{\oplus r_{\mathbb{C}}}{\to} \bigoplus_{\sigma \in \mathrm{Spec}(R)(\mathbb{C})} H(\mathbb{R}b_0 \oplus \mathbb{R}\langle b_{2i+1} | i \in \mathbb{N}\rangle) \to HA(R) \, .$$

The component at σ of the first map is induced by the inclusion σ. The map $r_{\mathbb{C}}$ is a defined in Proposition 1.22. The third map is the obvious projection on the higher degree part and given by $\oplus_\sigma x_\sigma b_0 \mapsto \frac{1}{|\mathrm{Spec}(R)(\mathbb{C})|} x_\sigma b_0$ in degree 0.

Theorem 2.1 (Borel, [Bor74]). *The map c induces an isomorphism in homotopy groups $\pi_i(KR)\otimes\mathbb{R}\to\pi_i(HA(R))$ for $i\neq 1$. The map $\pi_1(KR)\otimes\mathbb{R}\to\pi_1(HA(R))$ is injective with one-dimensional cokernel generated by $\sum_{\sigma\in\mathrm{Spec}(R)(\mathbb{C})} b_{1,\sigma}$.*

Definition 2.2. *We define the spectrum $\overline{KR}$ as the homotopy cofibre of c.*

One can check using Borel's theorem that

$$\overline{KR}\cong KR\wedge M\mathbb{R}/\mathbb{Z}\oplus H\mathbb{R}\ ,$$

For a spectrum E we let $\mathbf{Sm}(E)\in\mathbf{Fun}^{desc}(\mathbf{Mf}^{op},\mathbf{Sp})$ be the function spectrum which evaluates as $\mathbf{Sm}(E)(M)=\mathtt{Map}(M,E)$. If C is a chain complex of real vector spaces, then we can form the sheaf of chain complexes $\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C$ and the sheaf of spectra $H(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)\in\mathbf{Fun}^{desc}(\mathbf{Mf}^{op},\mathbf{Sp})$. Its homotopy type is given by the de Rham Lemma.

Proposition 2.3 (De Rham Lemma). *There exists a canonical equivalence*

$$\mathtt{Rham}:H(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)\xrightarrow{\sim}\mathbf{Sm}(HC)\ .$$

If $\mathcal{C}\in\mathbf{Fun}^{desc}(\mathbf{Mf}^{op},\mathbf{Ch})$, then we get a sheaf of zero cycles $Z^0(\mathcal{C})\in\mathbf{Fun}^{desc}(\mathbf{Mf}^{op},\mathbf{Ab})$. If we consider the latter as a presheaf of chain complexes concentrated in degree zero in $\mathbf{Fun}(\mathbf{Mf}^{op},\mathbf{Ch}[W^{-1}])$, then it may not be a sheaf. We let $\tilde{Z}^0(\mathcal{C})\in\mathbf{Fun}^{desc}(\mathbf{Mf}^{op},\mathbf{Ch}[W^{-1}])$ denote its sheafification. We have a map $\tilde{Z}^0(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)\to\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C$ which finally induces

$$z:H(\tilde{Z}^0(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C))\to H(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)\stackrel{\mathtt{Rham}}{\to}\mathbf{Sm}(HC)\ .$$

Under the obvious identifications, the induced map

$$z:\pi_0(\tilde{Z}^0(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)(M))\to\pi_0(\mathbf{Sm}(HC)(M))$$

maps a cycle $\omega\in Z^0(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)(M)$ to its cohomology class $z(\omega)\in HC^0(M)$.
Assume that we have fixed a map of spectra $c:E\to HC$ and let $\overline{E}$ be its cofibre.

Definition 2.4 (Differential cohomology). *We define the sheaf of differential cohomology spectra*

$$\mathtt{Diff}(E)\in\mathbf{Fun}^{desc}(\mathbf{Mf},\mathbf{Sp})$$

as the pull-back such that

$$\begin{tikzcd}
\mathtt{Diff}(E) \arrow[r, "R"] \arrow[d, "I"] & H(\tilde{Z}^0(\mathcal{A}_\mathbb{R}\otimes_\mathbb{R}C)) \arrow[d, "z"] \\
\mathbf{Sm}(E) \arrow[r, "c"] & \mathbf{Sm}(HC)
\end{tikzcd}$$

We define the differential cohomology group of M by

$$\widehat{E}^0(M):=\pi_0(\mathtt{Diff}(E)(M))\ .$$

It fits into the natural exact sequence

$$E^{-1}(M) \to HC^{-1}(M) \xrightarrow{a} \widehat{E}^{0}(M) \to \tag{6}$$

$$\xrightarrow{(I,R)} \quad E^{0}(M) \times_{HC^{0}(M)} Z^{0}((\mathcal{A}_{\mathbb{R}} \otimes_{\mathbb{R}} C)(M)) \to 0 \ . \tag{7}$$

$$(\mathcal{A}_{\mathbb{R}} \otimes_{\mathbb{R}} C)^{-1}(M) \xrightarrow{a} \widehat{W}^{0}(M) \xrightarrow{I} W^{0}(M) \to 0$$

Furthermore, we have a natural identification

$$\overline{E}^{0}(M) \cong \widehat{E}^{0}_{flat}(M) := \mathtt{ker}(R) \ .$$

Work out the argument.

We apply this construction to the map $c : KR \to HA(R)$ and obtain the differential algebraic K-theory of R.

A sheaf of finitely generated R-modules on M which will be called an R-bundle. If $\mathcal{V}$ is such a sheaf, then $\mathcal{V}_\sigma := \mathcal{V} \otimes_{R,\sigma} \mathcal{C}^\infty$ is a sheaf of sections of a complex vector bundle $(V_\sigma, \nabla_\sigma)$ with flat connection.

Definition 2.5. *A geometry on $\mathcal{V}$ is a collection of metrics $g := (h^{V_\sigma})$ which is invariant under complex conjugation.*

Let

$$\overline{\mathtt{Loc}}_R, \overline{\mathtt{Loc}}^{geom}_R : \mathbf{Mf}^{op} \to \mathbf{Mon}$$

be the functors which map a manifold to the monoids of R-bundles without and with geometry. We have a cycle map

$$\mathtt{cycl} : \overline{\mathtt{Loc}}_R \to KR^0 \ .$$

We define the characteristic form

$$\omega : \overline{\mathtt{Loc}}^{geom}_R \to Z^0(\mathcal{A} \otimes_{\mathbb{R}} A(R))$$

such that

$$\omega(g) := b_0 \dim(V) \oplus \sum_{i\in\mathbb{N},\ \sigma\in\mathrm{Spec}(R)(\mathbb{C})} \tilde{\mathbf{ch}}_{2i+1}(\nabla_\sigma, h^\sigma) b_{2i+1,\sigma} \ .$$

Proposition 2.6 (B.-Gepner, [BG13])**.** *There exists an additive natural transformation*

$$\widehat{\mathtt{cycl}} : \overline{\mathtt{Loc}}^{geom}_R \to \widehat{KR}^0$$

such that

$$\begin{tikzcd}
 & & Z^0(\mathcal{A}_{\mathbb{R}} \otimes_{\mathbb{R}} A) \\
\overline{\mathtt{Loc}}^{geom}_R \arrow[urr, "(\mathcal{V},g)\mapsto\omega(g)"] \arrow[r, "\widehat{\mathtt{cycl}}"] \arrow[drr, "(\mathcal{V},g)\mapsto\mathtt{cycl}(\mathcal{V})"'] & \widehat{KR}^0 \arrow[ur, "R"] \arrow[dr, "I"] & \\
 & & KR^0
\end{tikzcd}$$

commutes.

Note that this transformation is not unique.

Example 2.7. We fix a prime $p \in \mathbb{N}$ and consider the cyclotomic field $\mathbb{Q}(\xi)$ with $\xi^p = 1$ and let $R \subset \mathbb{Q}(\xi)$ be its ring of integer. Then $\xi \in R$. We consider a manifold M with base point m_0 and an identification $\pi_1(M, m_0) \cong \mathbb{Z}/p\mathbb{Z}$. For example, we can take the total space of the $U(1)$-bundle $L_p^{2n+1} \to \mathbb{CP}^n$ with Chern class pc. The representation $[1] \mapsto \xi$ of $\pi_1(M, m_0)$ on R induces an R-bundle $\mathcal{V}$ on M. It carries a canonical parallel geometry g. We therefore get a class

$$\widehat{\mathtt{cycl}}(\mathcal{V}, g) \in \widehat{KR}^0(M)$$

such that $R(\widehat{\mathtt{cycl}}(\mathcal{V}, g)) = b_0$. Calculate $\phi(\mathtt{cycl}(\sigma(\mathcal{V})) \in (KU \wedge M\mathbb{R}/\mathbb{Z})^0(L_p^{2n+1})$, where ϕ is as in (5).

2.2 Lott's relation

Let us consider an exact sequence of R-bundles on a smooth manifold M

$$\mathcal{V} \ : \ 0 \to \mathcal{V}_0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to 0 \ .$$

It is known that algebraic K-theory splits exact sequences.

Lemma 2.8. *We have*

$$\mathtt{cycl}(\mathcal{V}_0) + \mathtt{cycl}(\mathcal{V}_2) = \mathtt{cycl}(\mathcal{V}_1)$$

in $KR^0(M)$.

Proof. First reduce to the case bundles which are fibrewise free. Then $\mathcal{V}_1$ corresponds to a representation of $\pi_1(M, m_0) \to GL(\dim(V_1), R)$ of the block form

$$\left(\begin{array}{cc} \rho_0 & * \\ 0 & \rho_2 \end{array}\right) .$$

Show that one can deform the corresponding classifying map $M \to BGL(\dim(V_1), R)^+$ to the map given by the representation

$$\left(\begin{array}{cc} \rho_0 & 0 \\ 0 & \rho_2 \end{array}\right) .$$

This implies the result. □

We now choose geometries g_i on $\mathcal{V}_i$ for $i = 0, 1, 2$ and call $g := (g_i)_{i=0,1,2}$ a geometry on $\mathcal{V}$. If $\mathcal{V}$ splits in a way compatible with the geometry, then we have

$$\widehat{\mathtt{cycl}}(\mathcal{V}_0, g_0) + \widehat{\mathtt{cycl}}(\mathcal{V}_2, g_2) = \widehat{\mathtt{cycl}}(\mathcal{V}_1, g_1)$$

in $\widehat{KR}^0(M)$. In general, this does not hold true even on the level of curvatures, i.e. after applying R. To remedy this fact, Bismut-Lott [BL95] introduced a higher torsion form $\mathcal{T}(\mathcal{V}, g) \in \mathcal{A}_{\mathbb{R}} \otimes_{\mathbb{R}} A(M)/\mathtt{im}(d)$ which is uniquely characterized by the following properties:

1. $\omega(g_0) + \omega(g_2) - \omega(g_1) = d\mathcal{T}(\mathcal{V}, g)$,
2. $\mathcal{T}(\mathcal{V}, g) = 0$ if $\mathcal{V}$ splits metrically,
3. $\mathcal{T}(\mathcal{V}, g)$ depends smoothly on $(\mathcal{V}, g)$
4. $\mathcal{T}(\mathcal{V}, g)$ is natural w.r.t to pull-back.

Theorem 2.9 (B.-Tamme, [BT12]). *We have the relation*

$$\widehat{\mathtt{cycl}}(\mathcal{V}_0, g_0) + \widehat{\mathtt{cycl}}(\mathcal{V}_2, g_2) - \widehat{\mathtt{cycl}}(\mathcal{V}_1, g_1) = a(\mathcal{T}(\mathcal{V}, g))$$

in $\widehat{KR}^0(M)$.

The proof is a consequence of a generalization of differential algebraic K-theory from number rings to higher-dimensional schemes. It is a first step towards Lott's challenge [Lot00] relating a secondary index for flat bundles with the Becker-Gottlieb transfer. The transfer index conjecture formulated below would then imply Lott's challenge.
As an application of Lott's relation we have the following:

Corollary 2.10. *Assume that we have two exact sequences* $\mathcal{V}, \mathcal{V}'$ *with the same bundles* $\mathcal{V}_i$, $i = 0, 1, 2$. *Then* $[\mathcal{T}(\mathcal{V}, g) - \mathcal{T}(\mathcal{V}', g)] \in HA^{-1}(M)$ *is a class in the image of the regulator* $KR^{-1}(M) \to HA^{-1}(M)$ *and independent of the geometry* g.

2.3 The analytic index

We consider a proper submersion $\pi : W \to B$ between smooth manifolds. We choose a metric $g^{T^v\pi}$ on the vertical tangent bundle $T^v\pi := \mathtt{ker}(d\pi)$ and a connection $T^h\pi$. One way to do this is to choose a Riemannian metric on the manifold W and define $g^{T^v\pi}$ as the induced metric on the fibres and $T^h\pi$ as the orthogonal complement of the vertical bundle. The triple $(\pi, g^{T^v\pi}, T^h\pi)$ is called a geometric proper submersion.

Let $(\mathcal{V}, g) \in \overline{\mathtt{Loc}}_R^{geom}$ be a geometric bundle. The main goal of this subsection is to define the differential analytic index $\widehat{\mathtt{index}}^{an}(\mathcal{V}, g) \in \widehat{KR}^0(B)$. It comes as an additive map

$$\widehat{\mathtt{index}}^{an} : \overline{\mathtt{Loc}}_R^{geom}(W) \to \widehat{KR}^0(B)$$

which is natural under pull-back squares for geometric proper submersions

For every $i \in \mathbb{N}$ we consider the bundle $R^i\pi_*\mathcal{V}$ on B.

Definition 2.11. *We define the analytic index*

$$\mathtt{index}^{an} : \overline{\mathtt{Loc}}_R(W) \to KR^0(B)$$

by

$$\mathtt{index}^{an}(\mathcal{V}) := \sum_{i=0}^{\infty} (-1)^i \mathtt{cycl}(R^i\pi_*\mathcal{V}) \in KR^0(B) \ .$$

We use fibrewise Hodge theory in order to define a geometry on these bundles which will be used to define the differential refinement of the analytic index.

We let d^v be the vertical part of the differential of the sheaf of complexes $\mathcal{A}_{|W} \otimes_{\sigma,R} \mathcal{V}$. The complex $\pi_*(\mathcal{A}_{|W} \otimes_{R,\sigma} \mathcal{V}, d^v)$ is a complex of sheaves of $\mathcal{C}^\infty_{|B}$-modules whose cohomology groups are naturally identified with $R^i\pi_*\mathcal{V} \otimes_{R,\sigma} \mathcal{C}^\infty_{|B}$. The metric $g^{T^v\pi}$ together with g induce a fibrewise L^2-metric on $\pi_*(\mathcal{A}_{|W} \otimes_{R,\sigma} \mathcal{V})$, and by fibrewise Hodge theory we can identify $R^i\pi_*\mathcal{V} \otimes_{R,\sigma} \mathcal{C}^\infty_{|B}$ with the sheaf of fibrewise harmonic forms. In particular we get an induced metric $h^{R^i\pi_*\mathcal{V}}_\sigma$. The collection $(h^{R^i\pi_*\mathcal{V}}_\sigma)_{\sigma\in \mathrm{Spec}(R)(\mathbb{C})}$ is a geometry $g^{R^i\pi_*\mathcal{V}}$ on $R^i\pi_{\mathcal{V}}$. We define the bare analytic index by

$$\widehat{\mathtt{index}}^{an}_0(\mathcal{V}, g) := \sum_{i=0}^{\infty} (-1)^i \widehat{\mathtt{cycl}}(R^i\pi_*\mathcal{V}, g^{R^i\pi_{\mathcal{V}}}) \in \widehat{KR}^0(B) \ .$$

The theory of Bismut-Lott solves the problem of the calculation of the class

$$[R(\widehat{\mathtt{index}}^{an}_0(\mathcal{V}, g))] \in HA(R)^0(B) \ .$$

Theorem 2.12 (Bismut-Lott index theorem [BL95]). *We have*

$$[R(\widehat{\mathtt{index}}^{an}_0(\mathcal{V}, g))] = \int_{W/B} \chi(T^v\pi) \cup [R(\mathcal{V}, g)] \ ,$$

where $\chi(T^v\pi)$ is the Euler class of $T^v\pi$.

Example 2.13. We continue example 2.7. The circle bundle $\pi : L^{2n+1}_p \to \mathbb{CP}^n$ has a canonical vertical metric and connection. One checks that $R^i_*\pi\mathcal{V} = 0$ for all $i \geq 0$ and therefore $\widehat{\mathtt{index}}^{an}_0(\mathcal{V}, g) = 0$ and $\mathtt{index}^{an}(\mathcal{V}) = 0$. This is consistent with the Bismut-Lott index theorem by the fact that $\chi(T^v\pi) = 0$.

For our purpose we need the local version of the Bismut-Lott index theorem. First of all, the vertical metric and the connection on π induce a connection $\nabla^{T^v\pi}$, a version of the Levi-Civita connection. Therefore we get an Euler form

$$\chi(\nabla^{T^v\pi}) \in (\mathcal{A}^{\dim(W/B)}_{\mathbb{R}|W} \otimes_{\mathbb{R}} \Lambda_{W/B})(W)$$

representing the class $\chi(T^v\pi)$, where $\Lambda_{W/B}$ is the relative orientation bundle. The main ingredient of the local Bismut-Lott index theorem is the higher analytic torsion form

$$\mathcal{T} := \mathcal{T}(\mathcal{V}, g, T^h\pi, g^{T^v\pi}) \in (\mathcal{A}_{\mathbb{R}} \otimes_{\mathbb{R}} A(R))(B)^{-1}$$

Its detailed definition is complicated, but its main properties are naturality under pull-back alongs maps $B' \to B$ and the following theorem.

Theorem 2.14 (local Bismut-Lott index theorem [BL95]). *We have the equality*

$$R(\widehat{\mathtt{index}}^{an}_0(\mathcal{V}, g)) = \int_{W/B} \chi(\nabla^{T^v\pi}) \wedge R(\mathcal{V}, g) + a(\mathcal{T}) \ .$$

Definition 2.15. *We define the analytic index* $\widehat{\mathtt{index}}^{an} : \overline{\mathtt{Loc}}_R^{geom}(W) \to \widehat{KR}^0(B)$ *by*

$$\widehat{\mathtt{index}}^{an}(\mathcal{V}, g) = \mathtt{index}_0^{an}(\mathcal{V}, g) - a(\mathcal{T}) \ .$$

Example 2.16. We continue example 2.13. We get

$$\widehat{\mathtt{index}}^{an}(\mathcal{V}, g) = -a(\mathcal{T}) \ .$$

In this case the higher analytic torsion form has been calculated by Bismut-Lott [BL95] explicitly:

$$\begin{aligned} \mathcal{T} \ &= \ b_0\mathcal{T}_0 + \sum_{\sigma \in \mathrm{Spec}(R)(\mathbb{C})} \frac{2}{|\mathtt{Stab}(\sigma)|} \left(\sum_{j even} (-1)^{j/2} \frac{1}{(2\pi)^j} \frac{(2j+1)!}{2^{2j}(j!)^2} \mathtt{Re}(\mathtt{Li}_{j+1}(\sigma(\xi)))\, p^j \omega^j \ \ b_{2j+1,\sigma} \right. \\ &\left. + \sum_{j odd} (-1)^{(j-1)/2} \frac{1}{(2\pi)^j} \frac{(2j+1)!}{2^{2j}(j!)^2} \mathtt{Im}(\mathtt{Li}_{j+1}(\sigma(\xi)))\, p^j \omega^j \ \ b_{2j+1,\sigma} \right) . \end{aligned}$$

2.4 The topological index

We continue to consider a geometric proper submersion $(\pi : W \to B, g^{T^v\pi}, T^h\pi)$. The main goal of this subsection is to describe a differential Becker-Gottlieb transfer

$$\hat{\mathtt{tr}} : \widehat{KR}^0(W) \to \widehat{KR}^0(B) \ .$$

It will be used in the definition of the topological index.

Definition 2.17. *We define the topological index*

$$\widehat{\mathtt{index}}^{top} : \overline{\mathtt{Loc}}_R^{geom}(W) \to \widehat{KR}^0(B)$$

by

$$\widehat{\mathtt{index}}^{top}(\mathcal{V}, g) := \hat{\mathtt{tr}}(\widehat{\mathtt{cycl}}(\mathcal{V}, g)) \ .$$

Consider a compact manifold M. We can choose an embedding $M \hookrightarrow S^n \setminus \{\infty\}$ whose normal bundle is denoted by ν. We get a canonical map

$$S^n \to S^n/_h(S^n \setminus M) \cong M^\nu \ ,$$

where $/_h$ denotes the homotopy quotient and M^ν is the Thom space of ν. The inclusion $\nu \hookrightarrow TM \oplus \nu \cong M \times \mathbb{R}^n$ induces a map $M^\nu \to \Sigma_+^n M$. The composition of these two maps gives after stabilization a map of spectra $S \to \Sigma_+^\infty M$. Applying this construction fibrewise to π we get the transfer map

$$\mathtt{tr} : \Sigma_+^\infty B \to \Sigma_+^\infty W$$

which induces a Becker-Gottlieb transfer [BG75] in every cohomology theory, in particular

$$\mathtt{tr} : KR^0(W) \to KR^0(B) \ .$$

Definition 2.18. *We define the topological index* $\mathtt{index}^{top} : \overline{\mathtt{Loc}}_R \to KR^0(B)$ *by*

$$\mathtt{index}^{top}(\mathcal{V}) := \mathtt{tr}(\mathtt{cycl}(\mathcal{V})) \ .$$

Theorem 2.19 (Dwyer-Weiss-Williams index theorem [DWW03])**.** *We have*

$$\mathtt{index}^{an} = \mathtt{index}^{top} \ .$$

One can check that the Dwyer-Weiss-Williams index theorem is consistent with the Bismut-Lott index theorem. This follows from

$$\mathtt{tr}([R(\widehat{\mathtt{cycl}}(\mathcal{V},g))]) = \int_{W/B} \chi(T^v\pi) \cup [R(\widehat{\mathtt{cycl}}(\mathcal{V},g))] \ .$$

Let $c : E \to HC$ and $\overline{E}$ be as above.

Theorem 2.20 (B.-Gepner [BG13])**.** *There exists a natural differential refinement of the Becker-Gottlieb transfer*

$$\hat{\mathtt{tr}} : \widehat{E}^0(W) \to \widehat{E}^0(B)$$

such that

$$R(\hat{\mathtt{tr}}(x)) = \int_{W/B} \chi(\nabla^{T^v\pi}) \wedge R(x) \ , \quad I(\hat{\mathtt{tr}}(x)) = \mathtt{tr}(I(x))$$

for all $x \in \widehat{E}^0(W)$*. Furthermore, its restriction to* $\widehat{E}^0_{flat}$ *coincides with* $\mathtt{tr}$ *under the identification* $\widehat{E}^0_{flat} \cong \overline{E}^0$*.*

Example 2.21. Assume that there exists a nowhere vanishing vertical vector field $X \in C^\infty(W, T^v\pi)$. Then

$$\hat{\mathtt{tr}}(x) = a(\int_{W/B} \Psi_X \wedge R(x)) \ ,$$

where $\Psi_X \in (A^{\dim(W/B)-1} \otimes \Lambda_{W/B})(W)$ is the Mathai-Quillen form of X such that $d\Psi_X = \chi(\nabla^{T^v\pi})$.

Example 2.22. We continue the example 2.16 and calculate $\widehat{\mathtt{index}}^{top}(\mathcal{V},g)$. Since $T^v\pi$ is trivialized we have $\mathtt{tr} = 0$. Further note that $R(\widehat{\mathtt{cycl}}(\mathcal{V},g)) = b_0$. Example 2.21 and the fact that $\int_{W/B} \Psi_X \wedge b_0 = 0$ now implies that $\hat{\mathtt{tr}}(\widehat{\mathtt{cycl}}(\mathcal{V},g)) = \widehat{\mathtt{index}}^{top}(\mathcal{V},g) = 0$.

2.5 The transfer index conjecture

We consider a geometric proper submersion $(\pi : W \to B, g^{T^v\pi}, T^h\pi)$ and a geometric R-module $(\mathcal{V},g) \in \overline{\mathtt{Loc}}^{geom}_R(W)$.

Conjecture 2.23 (Transfer index conjecture)**.** *We have the equality*

$$\widehat{\mathtt{index}}^{an}(\mathcal{V},g) = \widehat{\mathtt{index}}^{top}(\mathcal{V},g) \ .$$

One checks in a straightforward manner that

$$R(\widehat{\mathtt{index}}^{an}(\mathcal{V},g)) = R(\widehat{\mathtt{index}}^{top}(\mathcal{V},g))$$

is equivalent to the local Bismut-Lott index theorem, and that

$$I(\widehat{\mathtt{index}}^{an}(\mathcal{V},g)) = I(\widehat{\mathtt{index}}^{top}(\mathcal{V},g))$$

is equivalent to the Dwyer-Weiss-Williams index theorem. Hence, apriori in view of (6)

$$\widehat{\mathtt{index}}^{an}(\mathcal{V},g) - \widehat{\mathtt{index}}^{top}(\mathcal{V},g) \in HA(R)^{-1}(M)/\mathtt{im}(KR^{-1}(M) \to HA(R)^{-1}(M)) \ .$$

Example 2.24. We continue example 2.22. The TIC predicts that $a(\mathcal{T}) = 0$. This is equivalent the assertion that there exists an element $x \in KR^{-1}(\mathbb{CP}^n)$ such that $c(x) = [\mathcal{T}]$. There exists an identification (use multiplicative structure of KR and Atiyah-Hirzeburch spectral sequence)

$$KR^{-1}(\mathbb{CP}^n) \otimes \mathbb{Q} \cong K_*(R) \otimes \mathbb{Q}[c]/(c^{n+1}) \ .$$

In this identification we can write

$$x = \sum_{j \geq 0} x_{2j+1} \otimes c^j \ ,$$

where $x_j \in K_{2j+1}(R) \otimes \mathbb{Q}$ are uniquely determined. Therefore the TIC predicts:

Fact 2.25. *For every $j \in \mathbb{N} \setminus \{0\}$ there exists $x_j \in K_{2j+1}(R) \otimes \mathbb{Q}$ such that for every $\sigma \in \mathrm{Spec}(R)(\mathbb{C})$ w have*

$$r_{Borel,\sigma}(x_j) = \left\{ \begin{array}{cc} (-1)^{j/2} \frac{1}{(2\pi)^j} \frac{(2j+1)!}{2^{2j}(j!)^2} \mathtt{Re}(\mathtt{Li}_{j+1}(\sigma(\xi)))\, p^j & j \ even \\ (-1)^{(j-1)/2} \frac{1}{(2\pi)^j} \frac{(2j+1)!}{2^{2j}(j!)^2} \mathtt{Im}(\mathtt{Li}_{j+1}(\sigma(\xi)))\, p^j & j \ odd \end{array} \right\}$$

This fact is a true statement, since the existence of such elements has been shown by Beilinson [Beĭ86].

3 Problems

Problem 3.1. *Let $L \to \mathbb{CP}^n$ be the tautological line bundle and ∇ the connection on L on the tautological line bundle as in the lecture. Let $\omega \in \mathcal{A}^2(\mathbb{CP}^n)$ be the unique harmonic form satisfying $\int_{\mathbb{CP}^1} \omega = 1$. Show that the curvature of ∇ satisfies*

$$R^{\nabla} = 2\pi i \omega.$$

Problem 3.2. *We consider $\lambda \in \mathbb{C}^*$ and define the action of $\mathbb{Z}$ on the trivial bundle $\mathbb{R} \times \mathbb{C} \to \mathbb{R}$ by $n(t, z) \mapsto (t + n, \lambda^n z)$. This action preserves the trivial connection ∇^{triv}. We let $V(\lambda) \to S^1$ be the quotient with the induced flat connection $\nabla(\lambda)$. Show that*

$$\omega(\nabla(\lambda)) = -2 \log |\lambda| \mathtt{or}_{S^1} \in H^{odd}_{dR}(S^1) \ .$$

Problem 3.3. *Show that one can identify $KU^0(\mathbb{CP}^n) \cong \mathbb{Z}[z]/(z^{n+1})$ as rings, where $z := \mathtt{cycl}(L) - 1$.*

Problem 3.4. *Let M be a manifold, I the interval $[0, 1]$ and $i_0, i_1 \colon M \hookrightarrow I \times M$ be the inclusions at the endpoints. Prove the following homotopy formula: If $\hat{x} \in \widehat{KU}^0(I \times M)$, then*

$$i_0^*(\hat{x}) - i_1^*(\hat{x}) = a(\int_{I \times M/M} R(\hat{x}))$$

where $\int_{I \times M/M} \colon \mathcal{A} \otimes \mathbb{R}[b, b^{-1}](I \times M) \to \mathcal{A} \otimes \mathbb{R}[b, b^{-1}](M)[-1]$ is given by integration along the interval I.

Problem 3.5. *Let $(V(\lambda), \nabla(\lambda))$ be as in Problem 3.2. Calculate $\widehat{KU}^0(S^1)$ and characterize the element $\widehat{\mathtt{cycl}}(V(\lambda), \nabla(\lambda))$.*

4 Problems II

Problem 4.1. *Show the de Rham Lemma* $H(A \otimes_{\mathbb{R}} C) \xrightarrow{\sim} \mathbf{Sm}(HC)$. *Hint: Construct equivalences* $\underline{HC} \xrightarrow{\sim} H(A \otimes_{\mathbb{R}} C)$ *and* $\underline{HC} \xrightarrow{\sim} \mathbf{Sm}(HC)$

Problem 4.2. *Show that* $Z^n(\mathcal{A}) \to \sigma^{\geq n}\mathcal{A}$ *represents the sheafification of* $Z^n(\mathcal{A})$ *in* $\mathbf{Fun}(\mathbf{Mf}, \mathbf{Ch}[W^{-1}])$.

Problem 4.3. *Verify the basic exact sequences of differential cohomology.*

Problem 4.4. *Show that algebraic* K*-theory splits exact sequences of bundles (give details for the proof of Lemma 2.8).*

Problem 4.5. *Calculate the class predicted in Corollary 2.10 in the case of complexes on* S^1

$$\mathcal{V} \;:\; 0 \to S^1 \times R \xrightarrow{\mathtt{id}} S^1 \times R \to 0 \to 0\;,$$

$$\mathcal{V}' \;:\; 0 \to S^1 \times R \xrightarrow{\lambda} S^1 \times R \to 0 \to 0\;,$$

where $\lambda \in R^*$ *is a unit.*

Problem 4.6. *Show that the Becker-Gottlieb transfer satisfies* $\mathtt{tr} = 0$ *if* $T^v\pi$ *admits a nowhere vanishing section.*

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