Algebraic Number Theory 2018 — Discussion set

Date: Monday, the 23d of April, 2018

Exercise 0 (Generators of ideals)**.** Let R be a commutative ring, and let S be a subset of R. Write $\langle S \rangle$ for the subset of R consisting of elements $x \in R$ of the form $x = r_1 s_1 + r_2 s_2 + \ldots + r_n s_n$, with $n \in \mathbb{Z}_{\geq 0}$, $r_i \in R$, and $s_i \in S$.

Show that $\langle S \rangle$ is the smallest ideal of R that contains S. It is the ideal generated by S.

Exercise 1 (Operations on ideals)**.** Let R be a commutative ring, and let I and J be ideals of R. We define the ideal $I + J$ as the ideal generated by $\{i + j \mid i \in I, j \in J\}$. Similarly, we define the ideal $I \cdot J$ as the ideal generated by $\{i \cdot j \mid i \in I, j \in J\}$. The ideals I and J are called *coprime* if $I + J = R$.

(a) Show that $I \cap J$ is an ideal of R. Show with an example that $I \cup J$ is in general not an ideal.

(b) Let J_1 and J_2 be two ideals of R. Prove: $I \cdot (J_1 + J_2) = I \cdot J_1 + I \cdot J_2$

(c) Suppose that I and J are coprime. Prove: $I \cap J = I \cdot J$.

Exercise 2. Let m and n be two integers. Compute $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$.

Exercise 3. Show that every element of a Noetherian ring can be written as product of finitely many irreducible elements. (*Warning: this factorisation is in general not unique!*)

Exercise 4. Consider the ring $R = \mathbb{Z}[\sqrt{-19}]$. Inside this ring we can factor 20 as $2 \cdot 2 \cdot 5$ into a product of irreducible elements of R. Write 20 as product of two irreducible elements of R.