Algebraic Number Theory 2018 — Set 11 (Bonus)

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Deadline: 12.00 on Thursday, the 12th of July, 2018

Remark. This set consists entirely of bonus exercises.

Exercise 1. Let p be a prime number, and fix $x \in \mathbb{Q}_p$. Show that the p-adic series expansion of x is eventually periodic if and only if $x \in \mathbb{Q} \subset \mathbb{Q}_p$.

Exercise 2. Let $K, |\cdot|$ be a non-Archimedean valued field. Prove that balls in K are either disjoint or contained in each other.

Exercise 3. Show that $\mathbb{Z}_2$ is homeomorphic to the Cantor set.

Exercise 4. Does every number field embed into $\mathbb{Q}_p$ for some p? (Hint: Cebotarev)

Exercise 5. Prove that $\overline{\mathbb{Q}_p}$ is not complete.

Exercise 6*. Prove that every finite extension of $\mathbb{Q}_p$ is the completion of some number field with respect to a norm $|\cdot|_{\mathfrak{p}}$.

Hint. Prove Krasner's lemma: Let $\alpha, \beta \in \overline{\mathbb{Q}_p}$ be two elements. Assume that $|\alpha - \beta| < |\sigma(\alpha) - \beta|$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ with $\sigma(\alpha) \neq \alpha$. Then $\mathbb{Q}_p(\alpha) \subset \mathbb{Q}_p(\beta)$. To prove this, consider $\sigma \in \mathrm{Hom}_{\mathbb{Q}_p(\beta)}(\mathbb{Q}_p(\alpha, \beta), \overline{\mathbb{Q}_p})$.