Exercises for the lecture
"Commutative Algebra and Algebraic Geometry"
SS 2019 Sheet 1,
Submission Date: 06.05.2019

We let A, B denote commutative rings with unity. We let $A[x]$ denote the ring of polynomials in an indeterminate x with coefficients in A.

Exercise 1.

1. Prove that a proper ideal $\mathfrak{p}$ of A is prime if and only if, for all ideals $\mathfrak{a}, \mathfrak{b}$ of A, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.
2. If $\mathfrak{p}$ is a prime ideal and $\mathfrak{a}^n \subset \mathfrak{p}$ for an ideal $\mathfrak{a}$ of A and for some $n \geq 0$, then show that $\mathfrak{a} \subset \mathfrak{p}$.

(4 Points)

Exercise 2.

1. Let $\phi : A \to B$ be a ring homomorphism and let $I \subset B$ be an ideal of B. Then prove that $\phi^{-1}(I)$ is an ideal of A.
2. Let $B \neq 0$ and let $\phi : A \to B$ be a surjective ring homomorphism. Prove that $\ker(\phi) := \{a \in A : \phi(a) = 0\}$ is a prime ideal of B if and only if B is an integral domain.
3. Prove that $\langle x^2 + 1 \rangle \subset \mathbb{R}[x]$ is a prime ideal of $\mathbb{R}[x]$.
4. Show that $\langle x^2 + 1 \rangle \subset \mathbb{C}[x]$ is not a prime ideal of $\mathbb{C}[x]$.

(8 Points)

Exercise 3.

1. Show that $x(\text{mod}\langle x^m\rangle) \in \frac{\mathbb{C}[x]}{\langle x^m\rangle}$ is a nilpotent element while $a + x(\text{mod}\langle x^m\rangle) \in \frac{\mathbb{C}[x]}{\langle x^m\rangle}$ is a unit for all $a \in \mathbb{C} \setminus \{0\}$.

2. In general, prove that for a nilpotent element $a \in A$ and for a unit $u \in A$, the element $u + a$ is a unit of A.

(4 Points)

Exercise 4.

Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Then

1. Let $g = b_0 + b_1 x + \cdots + b_m x^m \in A[x]$ such that $fg = 1 \in A[x]$. Show (by induction on r) that $a_n^{r+1} b_{m-r} = 0$ for all $r = 0, 1, \ldots, m$.

2. Using Exercise 3, show that f is a unit in $A[x]$ if and only if a_0 is a unit of A and $a_1, \ldots, a_n$ are nilpotent elements of A.

3. Prove that f is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent.

4. Show that if f is a zero-divisor then there exists $0 \neq a \in A$ such that $af = 0$.

(1 + 2 + 2 + 3 Points)