# Exercises for the lecture
# “Commutative Algebra and Algebraic Geometry”
# SS 2019 Sheet 10,
# Submission Date: 16.07.2019

**Definition 1** *The complex projective space* $\mathbb{P}^n$ *of dimension* $n$ *is defined to be the set of* $(n+1)$*-tuples* $(a_1, \dots, a_{n+1})$ *of complex numbers, not all zero, up to the equivalence*

$$(a_1, \dots, a_{n+1}) = (\lambda a_1, \dots, \lambda a_{n+1}), \text{ for } \lambda \in \mathbb{C} \setminus \{0\}.$$

*An element of* $\mathbb{P}^n$ *is said to be a point. If* $P$ *is a point of* $\mathbb{P}^n$*, then any* $(n+1)$*-tuple in the equivalence class* $P$ *is said to be the homogeneous coordinate of* $P$*.*

**Exercise 1.**
Let $f \in \mathbb{C}[T_1, \dots, T_{n+1}]$ be a homogeneous polynomial of degree $d$, i.e., all monomials in $f$ are of degree $d$. Show that $f(a_1, \dots, a_{n+1}) = 0$ if and only if $f(\lambda a_1, \dots, \lambda a_{n+1}) = 0$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Conclude that for $P \in \mathbb{P}^n$, the expression “$f(P) = 0$” is well defined.

(2 Points)

**Definition 2** *For homogeneous polynomials* $f_1, \dots, f_r$ *in* $(n+1)$ *variables, let*

$$Z(f_1, \dots, f_r) = \{P \in \mathbb{P}^n : f_1(P) = f_2(P) = \dots = f_r(P) = 0\}.$$

*In general, for a set* $S$ *of homogeneous polynomials in* $(n+1)$ *variables, we define* $Z(S) := \{P \in \mathbb{P}^n : f(P) = 0, \forall f \in S\}$*.*

*An ideal* $I \subset \mathbb{C}[T_1, \dots, T_{n+1}]$ *is said to be a homogeneous ideal if* $I$ *can be generated by a set of homogeneous elements. Define* $Z(I) := Z(S)$ *where* $S$ *is a set of homogeneous polynomials generating* $I$*.*

**Exercise 2.**
Show that the set $Z(I)$ does not depend on the set of generators of $I$.

(2 Points)

**Remark 1** *In some of the literatures, the set* $Z(I)$ *is denoted by* $V(I)$*.* □

**Exercise 3.**

1. Let $\mathcal{I}$ be an indexing set (may be infinite) and for $j \in \mathcal{I}$, let $I_j$ be a homogeneous ideal of $\mathbb{C}[T_1, \dots, T_{n+1}]$. Then show that
$$Z(\sum_{j \in \mathcal{I}} I_j) = \bigcap_{j \in \mathcal{I}} Z(I_j).$$
2. For homogeneous ideals $I_1$ and $I_2$, show that $Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$.
3. Show that $Z(0) = \mathbb{P}^n$ and $Z(\langle T_1, \dots, T_{n+1} \rangle) = \varnothing$.
4. If $I_1 \subset I_2$, then $Z(I_2) \subset Z(I_1)$.

Conclude that the sets $Z(I)$ for homogeneous ideal $I$ of $\mathbb{C}[T_1, \dots, T_{n+1}]$ satisfy the axioms of the closed subsets of a topological space. The induced topology is said to be the Zariski topology on $\mathbb{P}^n$.

(8 Points)

**Exercise 4.**

Let $D(T_{n+1})$ denote the open subset $\mathbb{P}^n \setminus Z(T_{n+1})$. Then show that the map
$$\phi : \mathbb{A}^n \to \mathbb{P}^n$$
defined by $\phi(a_1, \dots, a_n) = (a_1, \dots, a_n, 1)$, yields an isomorphism $\phi : \mathbb{A}^n \xrightarrow{\cong} D(T_{n+1})$ of topological spaces (with Zariski topology on $\mathbb{A}^n$ and induced topology on the open subset $D(T_{n+1})$).

(8 Points)

## Class Exercises(no points)

**Remark 2** *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

**Exercise 5.**

Observe that projective plane $\mathbb{P}^2$ is a union of the affine plane $\mathbb{A}^2$ and a projective line $\mathbb{P}^1$. By a line $L$ in $\mathbb{P}^2$, we mean the closed subset $Z(a_1T_1 + a_2T_2 + a_3T_3) \subset \mathbb{P}^2$ where $(a_1, a_2, a_3) \in \mathbb{C}^3$, not all zero. Show that two lines in $\mathbb{P}^2$ meet exactly at a point.

**Exercise 6.**

Show that if $I$ is a homogeneous ideal then so is $\mathrm{rad}(I)$ and $Z(I) = Z(\mathrm{rad}(I))$.