Exercises for the lecture "Commutative Algebra and Algebraic Geometry" SS 2019 Sheet 2, Submission Date: 14.05.2019

Exercise 1.
An element $x \in A$ is said to be idempotent if $x^2 = x$. Prove that a local ring does not have any idempotent except 0 and 1.

(2 Points)

Exercise 2.
Show that the following conditions on A are equivalent.

1. A has exactly one prime ideal.
2. $A \neq 0$ and every element of A is either a unit or nilpotent.
3. $\mathrm{nil}(A) = \{a \in A : a^n = 0 \text{ for some } n \geq 1\}$ is a maximal ideal.

(6 Points)

Exercise 3.
Let $\psi : \mathbb{Q}[x] \to \mathbb{Q}[y]$ be a ring homomorphism.

1. Prove that $\psi(a) = a$ for all $a \in \mathbb{Q}$.
2. State (with prove) the conditions under which ψ is an isomorphism.

(2 + 4 Points)

Exercise 4.
For ideals I and J of A, let $(I : J) = \{a \in A : aJ \subset I\}$.

1. Prove that $(I : J)$ is an ideal of A.
2. $I \subset (I : J)$.
3. $(I : J)J \subset I$.
4. $((I : J) : K) = (I : JK) = ((I : K) : J)$

(1 + 1 + 1 + 2 Points)

Class Exercises(no points)

Remark 1 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.
Generalize Exercise 4 of sheet 1 for $A[[x]]$, the ring of formal power series $f = \sum_{i=0}^{\infty} a_i x^i$ with coefficients in A, i.e., show that

1. f is a unit in $A[[x]]$ if and only if a_0 is a unit in A.
2. If f is nilpotent then a_i is nilpotent for all $i \geq 0$. See that the converse is not true in general.
3. f belongs to the Jacobson radical of $A[[x]]$ if and only if a_0 belong to the Jacobson radical of A.

Exercise 6.
Let $I = (m)$ and $J = (n)$ be the ideals of $\mathbb{Z}$ generated by the integers m and n respectively. Then show that $(I : J) = (q)$, where $q = \frac{m}{(m,n)}$ and (m, n) denotes the highest common factor of m and n.