Exercises for the lecture
"Commutative Algebra and Algebraic Geometry"
SS 2019 Sheet 4,
Submission Date: 28.05.2019

Remark 1 *See the correction in Exercise 1.2.* □

Exercise 1.

1. Show that $A/I \otimes_A A/J = 0$ if I and J are ideals of A such that $I + J = 1$.

2. Let A be a ring. Let M and N be free A-modules of rank n and m, respectively. Then show that $M \otimes_A N$ is free A-module of rank nm.

3. Let A be a ring, I an ideal of A and M an A-module. Show that $A/I \otimes_A M$ is isomorphic to M/IM.

(6 Points)

Exercise 2.
Let A be a local ring. Let M and N be finitely generated A-module. If $M \otimes_A N = 0$, then show that $M = 0$ or $N = 0$.
(hint: use Nakayama Lemma.)

(3 Points)

Exercise 3.
Let $A \neq 0$ be a ring.

1. If $A^m \cong A^n$, then show that $m = n$.

2. If $\phi : A^m \to A^n$ is surjective, then show that $m \geq n$.

3. If $\phi : A^m \to A$ is injective, then show that $m \leq 1$ (see Exercise 6 for a generalization).

(6 Points)

Exercise 4.

Let M be a finitely generated A-module and let $\phi : M \to A^n$ be a surjective homomorphism. Then show that $\ker(\phi)$ is also finitely generated (see Exercise 7).

(3 Points)

Exercise 5.

Let

$$0 \to M \to N \to P \to 0$$
$$\downarrow \alpha \quad \downarrow \beta \quad \downarrow \gamma$$
$$0 \to M' \to N' \to P' \to 0$$

be a commutative diagram of A-modules with exact rows. Show that if any two of the homomorphisms α, β and γ are isomorphisms, then so is the third.

(4 Points)

Class Exercises(no points)

Remark 2 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 6.

If $\phi : A^m \to A^n$ is injective, then show that $m \leq n$.
(hint: Use Proposition 2.4 of Atiyah and MacDonald's book.)

Exercise 7.

1. Let $A = \mathbb{C}[x_1, x_2, x_3, \dots]$ be the ring of polynomials in infinite indeterminates. Then show that the the ideal $\langle x_1, x_2, x_3, \dots \rangle$ generated by all inderterminates is not finitely generated as an A-module (this shows that a submodule of a finitely generated module may not be finitely generated).

2. Let $B = A/\langle x_1 x_2, x_1 x_3, \dots \rangle$ and let $\phi : B \to B$ be the B-module homomorphism such that $\phi(1) = \overline{x_1}$. Then show that the $\ker(\phi)$ is not finitely generated as an B-module (this implies that the surjectivity of ϕ in Exercise 4 is necessary).